SHIFT-COMPACTNESS IN ALMOST ANALYTIC SUBMETRIZABLE BAIRE GROUPS AND SPACES

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Abstract. We survey the harmonious interplay between the concepts of normed group, almost completeness, shift-compactness, and certain refinement topologies of a metrizable topology (the 'ground topology'), characterized by the existence of a weak base consisting of analytic sets in the ground topology. This leads to a generalized Gandy-Harrington Theorem. The property of shift-compactness leads to simplifications and unifications.

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1. Introduction and overview

This survey aims to draw attention to a 'topological harmony of ideas' in the category of metrizable and submetrizable spaces. First in this harmony is a natural context, provided by *normed groups* (a step beyond metrizable topological groups) with norm topologies and their refinements (which are thus *submetrizable*, as in the title). These include groups of autohomeomorphisms. The second idea is the assumption that the topology be either *analytic* or *analytically generated.* There are several reasons for giving these pride

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of place in this discussion. The primary reason is *Nikodym's Theorem* (cf. [Jay-Rog], §2.9), which asserts that the Souslin operation preserves the Baire property. Closed sets have the Baire property (they differ from their interiors by a nowhere dense set), and so likewise do the analytic sets, being Souslin-$\mathcal{F}$ for $\mathcal{F}$ the family of closed sets. The existence of a 'heavy part' in a group, in the form of a (norm) non-meagre analytic subset, is equivalent to the *almost completeness* of the group (being an absolute-$\mathcal{G}_\delta$ modulo a meagre set) – hence the *almost analyticity* of the title. Almost completeness works almost as well as topological completeness. When the underlying space is almost complete the notions of Baire subspace and Baire property are almost indistinguishable (cf. Concluding Remarks §11.3); our third idea, a characteristic property of such subspaces, is *shift-compactness.* This is a relative of a notion earlier introduced by Parthasarathy [Par] in probability theory (in the context of convolution semi-groups of measures) and is inherently connected also to the notion of 'amenability at 1' studied recently by Solecki [Sol]. Shift-compactness manifests itself in Baire subspaces in two contexts:

(a) the action of translation in a group, i.e. *algebraic shifts* – here one is concerned with subsequences that converge 'after translation' (i.e. tz_n converges to a point t in a specified 'target' Baire subset T, for z_n converging to the identity);

(b) a transitive action of an algebraic group G of autohomeomorphisms, i.e. *topological shifts* – here one is concerned with subsequences that converge 'after homeomorphic embedding'. There are two interpretations. A primal one asserts that down a subsequence $\tau(z_n)$ converges to a point t in a Baire subset T for some homeomorphism $\tau \in G$ (which includes the case of the left translations of a left topological group); a dual one asserts that a subsequence of a sequence of autohomeomorphisms ζ_n converging to the identity embeds in T under the evaluation map $\zeta_n(t)$ for some point t in T.

A key example of this harmony is the Effros Open Mapping Theorem (§6.3) which is valid for almost-complete normed groups and is a consequence of shift-compactness.

Shift-compactness is a characteristic of Baire spaces (§3); indeed a shift-compact space is Baire, and a separable Baire normed group has shift-compactness.

As motivation we note an old result in real analysis, first studied by Kestelman [Kes-1], later by Borwein-Ditor [BoDi], and rediscovered by Trautner [Trau]. Early generalizations are due to Harry I. Miller, e.g. [MilH].

Theorem 1.1 (Kestelman-Borwein-Ditor Theorem). *Let* $\{z_n\} \to 0$ *be a null sequence in* $\mathbb{R}$. *If* T *is a measurable/Baire subset of* $\mathbb{R}$, *then for generically all (= almost all/quasi all)* $t \in T$ *there is an infinite set* $\mathbb{M}_t$ *(or subsequence) such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T. \tag{sub}$$

Here a proprty holds for almost all, resp. quasi all, t if it holds for all t off a null, resp. meagre, set.

It will be useful to recall that A is Lebesgue *measurable* if it takes the form $A = H\backslash N$ with H a $\mathcal{G}_\delta$ and N of measure zero. It is thus *almost a* $\mathcal{G}_\delta$. A set A is *Baire* (has the Baire property) if $A = (G\backslash M_1) \cup M_2$ with M_i meagre (A is quasi-open) and so $A = H \cup M$ with H a $\mathcal{G}_\delta$ and M meagre. Again it is *almost* a $\mathcal{G}_\delta$. The non-meagre Baire subsets are thus 'almost complete' – see below for a definition. Non-null measurable sets are also 'almost complete': either by passing to the *density* topology (for which see Section 4 or 9), or by adopting a different concept of negligibles, e.g. sets of some suitably small positive measure. Almost-completeness may often be used in lieu of completeness, proofs using the former take advantage of completeness arguments, but need to avoid certain forbidden negligible sets.

We recall that a *metric* space is *almost complete* if it is the union of a topologically complete subspace and a meagre subspace, so is a union of an absolute-$\mathcal{G}_\delta$ and a meagre set. (*Absolute*-$\mathcal{G}_\delta$ here means 'embeddable as a $\mathcal{G}_\delta$ subset in its completion'.) It is necessarily a Baire space. For general spaces, the notion originates with Frolík [Frol-60], is named 'almost Čech-complete' by Aarts and Lutzer [AL]. Its current abridged name was introduced by E. Michael in [Mich].

We call generalized versions of the KBD Theorem **shift theorems**. Generalized versions have many applications which include: Steinhaus-Pettis Theorems (e.g. that $A - A$ contains an interval around 0 for A non-null and measurable), continuity of homomorphisms (automatic continuity), Uniform Boundedness Theorem

(§6.1) and the Uniform Convergence Theorem of Regular Variation (the original source of inspiration, now the centrepiece of topological regular variation, for which see [BOst-TRI], [BOst-TRII], [BOst-TRIII], [BOst-Ind], [Ost-knit]).

We will see that the Baire case of the Kestelman-Borwein-Ditor Theorem extends to normed groups. One may deduce the measure case in the context of a locally compact topological group from the Baire case by employing the Haar-density topology. The moral of this is clear: category beats measure, making the Baire case of Th. KBD primary. This is particularly true in infinite-dimensional spaces, where Haar measure is not available, but category arguments are.

The basic message of the paper is that "shift-compactness is what groups like best" and a consequence that the *analytically-heavy* topologies, which we meet later in §7.1, form an important topological category.

The paper is structured around the three principal themes of: normed groups, shift-compactness, analyticity and almost-completeness. First in §2 we define normed groups, indicating a 'canonical' example, and place them historically. Next in §3 we define shift-compactness, and in passing explain why this is a notion of compactness; we sketch a proof of one version of the (primal) shift-compactness theorem based on a strong separation property (discussed in §4). We show how to prove the Banach-Steinhaus Theorem (§6.1) directly from shift-compactness. We give a simple proof of the Steinhaus-Picard-Pettis Theorem (§6.2) in a simple topological group context, and later (§9.2) identify a topological generalization. Next in §7 we introduce the ABC of analyticity: the analytic Baire and analytic Cantor Theorems, and note in passing a useful characterization of almost-completeness; our viewpoint is informed by a recent generalization (in [Ost-AH]) of the Gandy-Harrington Theorem as unifying the "Baire recognition" literature, cf. [AL], [HMc], (and we hope bridge-building between forcing and fine topology methods). We identify a dual shift-compactness theorem and by reference to the density topology recapture the measure case of the Kestelman-Borwein-Ditor Theorem.

As an application of some of these ideas we give two versions of the Steinhaus Subgroup Theorem and derive from it a theorem of

Loy [Loy] and Hoffmann-Jørgensen [HJ] concerning analytic topological groups (that when non-meagre they are Polish). We contrast this with a related result that under comparable circumstances a ‘semi-analytic’ normed group is Polish (§8).

We close by indicating applications in infinite combinatorics (§11.8) – not the first time such a connection has been made, cf. [FW] and [HS], and noting that normed groups are either topological or pathological (§11.2).

The presentation is based on a series papers, many of them with Nick Bingham, and one with Harry Miller, cited in the bibliography. It is a pleasure to thank them both for their intellectual stimulus. I am grateful also to Roman Pol and to Henryk Toruńczyk for some key conversations. There have been two other sources of inspiration, of which I am conscious despite the long germination time; I thank Anatole Beck, for introducing me to topological dynamics (flows on planes) and Ralph Kopperman, for his advocacy of a bitopological viewpoint. I am also grateful for the invitation of the organizers of the Kielce 25th Summer Topology Conference not only to give an overview of this corpus of work, but also to put down in writing in a survey article an updated version of the July 2010 talk.

2. Normed groups

Motivated by normed *vector* spaces and topological regular variation, where asymptotics (actually ‘divergence’) may be defined via $||x|| := d^X(x, e_X) \to \infty$, we have:

Definition 2.1. (a) For T an *algebraic* group with neutral element e, say that $||\cdot|| : T \to \mathbb{R}_+$ is a *group-norm* ([BOst-N]) if the following properties hold:
(i) *Subadditivity* (Triangle inequality): $||st|| \leq ||s|| + ||t||$;
(ii) *Positivity*: $||t|| > 0$ for $t \neq e$ and $||e|| = 0$;
(iii) *Inversion* (Symmetry): $||t^{-1}|| = ||t||$.
Then $(T, ||.||)$ is called a *normed-group.*
(b) The group-norm generates a right and a left *norm topology* via the right-invariant and left-invariant metrics $d_R^T(s,t) := ||st^{-1}||$ and $d_L^T(s,t) := ||s^{-1}t|| = d_R^T(s^{-1}, t^{-1})$. In the right norm topology the right shift $\rho_t(s) := st$ is a uniformly continuous homeomorphism, since $d_R(sy, ty) = d_R(s,t)$; likewise in the left norm topology the

left shift $\lambda_s(t) = st$ is a uniformly continuous homeomorphism. Since $d_L^T(t,e) = d_L^T(e,t^{-1}) = d_R^T(e,t)$, convergence at e is identical under either topology. In the absence of a qualifier, the 'right' norm topology is to be understood.
(c) Note that under the right norm topology $B(x,r) = \{t : d_R(t,x) < r\} = B(e_T,r)x$.
(d) Note the converse: if d^X is a one-sidedly invariant metric, then $||x|| := d^X(x,e_X)$ is a norm.
(e) Under either norm topology, there is continuity of operations at e. At further distances the topology may force the group operations to be increasingly 'less' continuous. See later: §10.
(f) The *symmetrization metric* $d_S^T := \max\{d_R^T, d_L^T\}$ is also of interest below and more particularly in §8.

The *Birkhoff-Kakutani Theorem* ([HR] Th. 8.3; cf. [Ru] Th. 1.24, albeit in a topological-vector-space setting) asserts that a metrizable topological group has an equivalent right-invariant metric. In fact, a close inspection of Kakutani's metrizability proof for topological groups yields the following characterization of normed groups.

Theorem 2.2 (Birkhoff-Kakutani Normability Theorem, [Bir], [Kak]). *A first-countable right topological group X is a normed group iff inversion and multiplication are continuous at the identity.*

Some history. Early use of group-norms occurs in work of A. D. Michal and his collaborators and was in providing a canonical setting for differential calculus (starting in the 1940s); a noteworthy example is a generalization of the implicit function theorem by Bartle (1955). In name the group-norm makes an explicit appearance in 1950 in a paper of Pettis [Pet] in the course of his classic closed-graph theorem (in connection with Banach's closed-graph theorem and the Banach-Kuratowski category dichotomy for groups). The notion reappears in the group context in 1963 under the name 'length function', motivated by word length, in the work of R. C. Lyndon on Nielsen's Subgroup Theorem. Gromov theory has a normed group context. See the Introduction of [BOst-N] for a wider discussion.

Key (and principal) example. For a metric space (X, d^X) consider $Auth(X)$, the *algebraic* group (i.e. not equipped with a topology) of homeomorphisms $h : X \to X$ (under composition) with identity $e_X(x) = id_X(x) = x$. A candidate metric, when finite, is the *supremum metric*

$$\hat{d}(h, h') := \sup\nolimits_x d^X(h(x), h'(x)). \tag{sup}$$

One thus restricts attention to the subgroup $\mathcal{H}(X)$ comprising those $h(x)$ such that $\sup_x d^X(h(x), e_X(x)) < \infty$; these are the 'bounded elements' (compare with h being *limited* by an open cover of X as in [vM2], p.5422). We note the following properties.

1. $\hat{d}$ is right invariant, so may be denoted $d_R^{\mathcal{H}}$:

$$\hat{d}(hg, h'g) = \sup\nolimits_x d^X(h(g(x)), h'(g(x))) = \sup\nolimits_y d^X(h(y), h'(y)) = \hat{d}(h, h').$$

2. As a result the equation

$$||h|| = ||h||_{\mathcal{H}} := \hat{d}(h, e_X) \text{ defines a norm on } \mathcal{H}(X),$$

so that

$$d_R^{\mathcal{H}}(g, h) = ||g^{-1}h|| \text{ and } d_L^{\mathcal{H}}(g, h) = ||gh^{-1}||.$$

3. The norm topology is topologically complete if X is compact. This follows from the next property.

4. The symmetrized topology provides a natural refinement topology, via the symmetrized metric

$$d_S^{\mathcal{H}}(g, h) = \max\{d_R^{\mathcal{H}}, d_L^{\mathcal{H}}\} = \max\{\hat{d}(g, h), \hat{d}(g^{-1}, h^{-1})\},$$

which is complete provided (X, d^X) is complete. When (X, d^X) is compact, $d_L^{\mathcal{H}}$ is equivalent to $d_R^{\mathcal{H}}$ (see e.g. [vM0]).

5. Since $d_R^{\mathcal{H}} \leq d_S^{\mathcal{H}}$ the symmetrized topology is indeed a finer topology, and while the latter need not be invariant, one may pursue a bi-topological study.

6. The $\hat{d}$ topology is Baire iff non-meagre, see e.g. [HJ] Prop. 2.2.3, cf. §3.2.

7. The norm topology enables continuity of action when $\mathcal{H}(X)$ acts on X via the evaluation $(h, x) \to h(x)$. This action is continuous as a map from $(\mathcal{H}(X), \hat{d}) \times (X, d^X) \to (X, d^X)$, so one can use this to develop topological dynamics (cf. [Ost-knit]).

Lemma 2.3 ([Dug] XII.8.3, p. 271)**.** *Under* $\hat{d}$ *on* $\mathcal{H}(X)$ *and* d^X *on* X, *the evaluation map* $(h, t) \to h(t)$ *from* $\mathcal{H} \times X$ *to* X *is continuous.*

Proof. Fix h_0 and x_0. The result follows from continuity of h_0 at x_0 via

$$\begin{aligned} d^X(h_0(x_0), h(x)) &\leq d^X(h_0(x_0), h_0(x)) + d^X(h_0(x), h(x)) \\ &\leq d^X(h_0(t_0), h_0(t)) + d^{\mathcal{H}}(h, h_0). \end{aligned}$$

∎

8. For our final property we need a definition.

Definition 2.4. Say that h is *bi-uniformly continuous* if both h and h^{-1} are uniformly continuous wrt d^X. Write

$$\mathcal{H}_u(X) = \{h \in \mathcal{H}_{unif}(X) : h^{-1} \in \mathcal{H}_{unif}\} \subseteq \mathcal{H}(X).$$

Theorem 2.5 (Dieudonné [Dieu], cf. [BOst-N]). *$\mathcal{H}_u(X)$ is complete, provided X is complete under d^X .*

The viewpoint that $\mathcal{H}_u(X)$ is a topological dual of X is pursued in [BOst-N].

2.1. Normed versus topological: Equivalence Theorem.

We will later need to refer to the following result.

Theorem 2.6 (Equivalence Theorem, [BOst-N]). *A normed group is topological iff the d_R^X topology is equivalent to the d_L^X topology. Furthermore, either of the following is equivalent to this condition:*
(i) each conjugacy $\gamma_t(x) := txt^{-1}$ is continuous at e in norm,
(ii) inversion is continuous in either d_R^X or d_L^X.

This motivates the following definition to which we refer in §10.

Definition 2.7. A point z lies in the *topological centre* $Z_\Gamma(X)$ of the normed group X, if $\gamma_z(x)$ is continuous (at e in norm).

We note also the immediate corollary:

Proposition 2.8. *An abelian group equipped with a group norm is topological under the norm topology.*

We mention two other results which rely on the Equivalence Theorem.

Theorem 2.9 ([BOst-N])**.** *For X a normed group which is separable, topologically complete, if each $\gamma_g(x) = gxg^{-1}$ is Baire, then X is topological.*

Theorem 2.10 (Borel/analytic inversion, [BOst-N])**.** *For X a normed group which is separable and topologically complete, if inversion $x \to x^{-1}$ from (X, d_R) to (X, d_R) is Borel (or has analytic graph), then X is topological.*

See the Concluding Remarks (§11.2) for a brief summary of when normed groups are topological.

3. Shift-compactness

In a normed topological group G, say that a set A is *(properly) right-shift compact* if, for any sequence of points a_n in A, (resp. in G) there is a point t and a subsequence $\{a_n : n \in \mathbb{M}_t\}$ such that $a_n t$ lies entirely in A and converges through $\mathbb{M}_t$ to a point $a_0 t$ in A; similarly for *left-shift compact* (cf. [BOst-StOstr], for the real line case). Evidently, finite Cartesian products of shift-compact sets are shift-compact (cf. §11.4). Thus a right-shift compact set A is precompact. (If the subsequence $a_m t$ converges to $a_0 t$, for m in $\mathbb{M}_t$, then likewise a_m converges to a_0, for m in $\mathbb{M}_t$.) Say that a set is *strongly right-shift compact* (or right-shift compact for arbitrarily small shifts) if the conditions just given hold and in addition the point t may be selected with $||t||$ arbitrarily small.

Remark 3.1. The compactness terminology is justified on two counts. Firstly suppose $T \subseteq \mathbb{R}$ is as in Theorem KBD, and a_n is a bounded sequence of points in T. Assume without loss of generality that $a_n \to a_0$; then $z_n := a_n - a_0 \to 0$. Now for some $t \in T$, $t + z_m \in T$ for m in some infinite set $\mathbb{M}_t$. Take $s := t - a_0$; then

$$s + a_n = (t - a_0) + a_n = t + z_n \in T$$

and $s + a_m$ converges through $\mathbb{M}_t$ to $s + a_0 = t \in T$. Thus after translation a subsequence of a_n converges to a point of T.
Secondly, this shift-compactness implies an (open) finite sub-covering theorem (covering after small shifts), as follows.

Definition 3.2. Say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *shift-covers* a subset X of G, or is a *shifted-cover* of X if, for some $d_1, ..., d_h$ in G,

$$(D_1 d_1) \cup ... \cup (D_h d_h) = X.$$

Say that X is *compactly shift-covered* if every open cover $\mathcal{U}$ of X contains a finite subfamily $\mathcal{D}$ which shift-covers X.

Definition 3.3. For N a neighbourhood of e_G say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *N-strongly shift-covers A*, or is an *N-strong shifted-cover* of A, if there are $d_1, ..., d_h$ in N such that

$$(D_1 d_1) \cup ... \cup (D_h d_h) \supseteq A.$$

Say that A is *compactly strongly shift-covered*, or *compactly shift-covered with arbitrarily small shifts,* if every open cover $\mathcal{U}$ of A contains for each neighbourhood N of e_G a finite subfamily $\mathcal{D}$ which N-strongly shift-covers A.

Correspondingly one has two results with almost identical proofs.

Theorem 3.4 (Compactness Theorem – modulo shift [BOst-StOstr])**.** *Let A be a right-shift compact subset of a separable normed topological group G. Then A is compactly shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, and for each member of $\mathcal{V}$ a translator, such that the corresponding translates of $\mathcal{V}$ cover A.*

Theorem 3.5 (Strong Compactness Theorem – modulo shift, cf. [BOst-StOstr])**.** *Let A be a strongly right-shift compact subset of a separable normed topological group G. Then A is compactly strongly shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A, and any neighbourhood of e_G there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, and for each member of $\mathcal{V}$ a translator in N such that the corresponding translates of $\mathcal{V}$ cover A.*

3.1. **Basis of generic behaviour.** Imagine a construction aiming to produce a set of points $F(T)$ by reference to a set T. For example:

$$t \in F(T) := \bigcap\nolimits_n \bigcup\nolimits_{m>n} (T - z_m) \text{ iff } t + z_n \in T \text{ infinitely often.}$$

The following result formalizes that if one can get at least one point of $F(T)$ in T itself, then 'most' points of T will be in $F(T)$. This explains the 'almost all/quasi all' aspect of the Kestelman-Borwein-Ditor Theorem.

Theorem 3.6 (Generic Dichotomy (Completeness) Principle, [BOst-KCC]). *For $F : \mathcal{B}a \to \mathcal{B}a$ monotonic,if $W \cap F(W) \neq \varnothing$ for all non-meagre $W \in \mathcal{G}_\delta$, then, for each non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi all of T.*
That is, either
(i) there is a non-meagre $S \in \mathcal{G}_\delta$ with $S \cap F(S) = \varnothing$, or,
(ii) for every non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi almost all of T.

3.2. **Shift-compact spaces.** For a subgroup $\mathcal{G} \subseteq Auth(X)$ say that X is *$\mathcal{G}$-shift-compact* if for any convergent sequence $x_n \to x_0$, any open subset U in X and any Baire set T co-meagre in U, there is $g \in \mathcal{G}$ such that $g(x_n) \in T \cap U$ along a subsequence. Say that the space is *shift-compact* if it is $\mathcal{H}(X)$-shift-compact. In such a space, any Baire non-meagre set is locally co-meagre (co-meagre on open sets) in view of the following.

Proposition 3.7. *For any subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, if X is $\mathcal{G}$-shift compact, then X is Baire.*

Proof. We argue as in [vM2] Prop 3.1 (1). Suppose otherwise; then X contains a non-empty meagre open set. By Banach's localization principle (for which see [Jay-Rog] p. 42, or [Kel] Th. 6.35), the union of all such sets is a largest open meagre set M, and is non-empty. Thus $X \backslash M$ is a co-meagre Baire set. For any $x \in M$ the sequence $x_n \equiv x$ is convergent and, since $X \backslash M$ is co-meagre in X, there is $g \in G$ with $g(x) \in X \backslash M$. But, as g is a homeomorphism, $g(M)$ is a non-empty open meagre set, so is contained in M, implying $g(x) \in M$, a contradiction. ∎

Proposition 3.8. *For any subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, if $\mathcal{G}$ acts transitively on a non-meagre X, then X is Baire.*

Proof. This is a similar argument. Refer again to M, the union of all meagre open sets, which, being meagre, has non-empty complement. For x_0 in this complement and any non-empty open U pick $u \in U$ and $g \in G$ such that $g(x_0) = u$. Now, as g is continuous, $g^{-1}(U)$ is a nhd of x_0, so is non-meagre, since every nhd of x_0 is non-meagre. But g is a homeomorphism, so $U = g(g^{-1}(U)$ is non-meagre. (For a generalization see [HJ, Prop. 2.2.3].) So X is Baire, as every non-empty open set is non-meagre.

4. Separation properties and shift-compactness

The following definition is inspired by recent work of van Mill, which allows the interpretation of shift-compactness via a separation property closely connected to that considered in [vM2]. The underlying theme is homeomorphic shifting of points, sequences and eventually nowhere dense compact sets into disjointness.

Definitions. 1. Say that a subgroup $G \subseteq \mathcal{H}(X)$ *separates points and closed nowhere dense sets* in $(X, \mathcal{T}_X)$ if for each $p \in X$ and F closed and nowhere dense in $\mathcal{T}_X$ there is in each neighbourhood of the identity e_G an element $g \in G$ such that $g(p) \notin F$. Here we assume that G is given either a norm topology, or some refinement of it.

2. Say that the separation of p from F, as in Definition 1 above, is *strong* if in each neighbourhood of the identity there is an non-empty open set H such that $h(p) \notin F$ for every $h \in H$.

Equivalently (when the group is right-topological), in each open nhd U of e_G there is $g \in U$ and an open nhd V of e_G such that $Vg \subseteq U$ and $Vg(p)$ is disjoint from F.

3. Denote by $Tr(\mathbb{R}^d)$ the group of c-translations $x \to x + c$ in $\mathbb{R}^d$. Under the sup-norm defined above in equation (sup) this group is isometric with $\mathbb{R}^d$. Thus any refinement of the Euclidean topology can be used as a topology also on $Tr(\mathbb{R}^d)$, as in the proposition below. Particularly useful refinements are provided by *density topologies*, as they permit measure properties to be handled topologically. Recall that density open sets are measurable sets W all of whose members are *density points*, that is $1 = \lim_{\varepsilon \to 0} |W \cap B_\varepsilon(w)|/|B_\varepsilon(w)|$

for every $w \in W$. Here $|\cdot|$ denotes Lebesgue measure and $B_\varepsilon(w)$ is the open ball of radius ε. For other density topologies in $\mathbb{R}^d$ (e.g. using density bases other than these balls) in particular, and refinement topologies in general see [LMZ]; for the locally compact case see [BOst-N] for topological groups, and [Ost-LBIII] for normed groups. We recall that in the density topology a set is nowhere dense iff it is null (has measure zero). The following result relies on a Lemma which we cite in Section 6.3.

Lemma 4.1 (Separation Lemma)**.** *If G is a separable normed group, acting separately continuously and transitively on a non-meagre space X, then for any fixed point x and F closed nowhere dense the set $W_{x,F} := \{\alpha : \alpha(x) \notin F\}$ is dense open. In particular, G separates points from nowhere dense closed sets.*

Proof. The set $W_{x,F}$ is open, as it takes the form $\varphi_x^{-1}(X \backslash F)$ and φ_x is continuous (by assumption). By the Lemma of Section 6.3, for U any non-empty open set in G, the set Ux is non-meagre, and so $Ux \backslash F$ is non-empty, as F is meagre. But, then for some $u \in U$ we have $u(x) \notin F$. ∎

Proposition 4.2 (Finitary Strong Separation)**.** *Suppose the subgroup G of $\mathcal{H}(X)$ strongly separates points from closed nowhere dense sets of X. Let U be open, $u_i \in U$ for $i \leq n$ and F closed and nowhere dense. Then, for each $\varepsilon > 0$, in $B_\varepsilon(e)$ there is a non-empty open set V of homeomorphisms η such that $\eta(u_i) \in U$ and $\eta(u_i) \notin F$ for each $i \leq n$.*

Proof. Let $\varepsilon > 0$. By assumption $\delta := \min\{\varepsilon, \min_i\{d(u_i, X\backslash U)\}/(n+1)\} > 0$. Let $B_0 := B_\delta(e)$. By induction on $i \leq n$, we select $\tau_1, ..., \tau_n$ and open nhds $B_1, ..., B_n$ of e such that for $\eta_i := \tau_i \circ ... \circ \tau_1$

(i) $\tau_i \in B_{i-1}$, $B_i\tau_i \subseteq B_{i-1}$,

(ii) $\tau\eta_i(u_j) \in U\backslash F$ for $\tau \in B_i$ for $j \leq i-1$, and

(iii) $\tau\eta_i(u_j) \in U$ for $\tau \in B_i$ for $j \leq n$.

It will follow that $\tau\eta_n(u_i) \in U\backslash F$ for all $i \leq n$ and each $\tau \in B_n$.

Choose $\tau_1 \in B_0 = B_\delta(e)$ and B_1 an open nhd of e such that $B_1\tau_1 \subseteq B_\delta(e)$ so that $\tau\tau_1(x_1) \in U\backslash F$ for each $\tau \in B_1$. For each such τ and each i one has $\tau\tau_1(u_i) \in U$, since $||\tau\tau_1|| \leq ||\tau|| + ||\tau_1|| < 2\delta \leq \varepsilon$.

Now choose τ_2 in B_1 and B_2 a nhd of e such that $B_2\tau_2 \subseteq B_1$ so that $\tau\eta_2(u_2) \in U\backslash F$ for each $\tau \in B_2$. For any such τ and each i one has $\tau\eta_2(u_i) \in U$ as $||\tau_1|| + ||\tau_2|| + ||\tau|| < 3\delta \leq \varepsilon$ and $\tau\eta_2(u_1) \in U\backslash F$ as $\tau\eta_2 \in B_2\tau_2\tau_1 \subseteq B_1\tau_1$.

Proceed similarly for any $i < n$, by selecting τ_i in B_{i-1} and B_i a nhd of e such that $B_i\tau_i \subseteq B_{i-1}$ so that $\tau\tau_i\eta_{i-1}(u_i) \in U\backslash F$ for each $\tau \in B_i$.

For any such τ and each $j < i$ one has $\tau\eta_i(u_j) \in U$ as $||\tau_1|| + ||\tau_2|| + ... + ||\tau_i|| + ||\tau|| < (i+1)\delta \leq n\delta < \varepsilon$ and $\tau\tau_i\eta_{i-1}(u_1) \in U\backslash F$ as $\tau\tau_i \in B_{i-1}$. Likewise for each $j < i$ one has $\tau\tau_i\eta_{i-1}(u_j) \in U$ as $\tau\tau_i \in B_{i-1}$. This completes the inductive step from $i-1$ to i.

Taking $V = B_n$, one has for $\tau'_n \in B_n$ that the shift $\eta := \tau'_n\eta_n$ has $||\eta|| < \min_i\{\varepsilon, d(u_i, X\backslash U)\}$, so $\eta(u_i) \in U$ and $\eta(u_i) \notin F$, as asserted. □

One may re-interpret the preceding result as saying that any finite number of points may be shifted locally into the complement of a closed nowhere dense set; in the semigroup setting a set into which any finite set may be left-shifted was termed by Mitchell [Mitch] *left thick*; for further connections see [Day].We now improve the last result from finite sets to convergent sequences, a matter we return to in the final remark of this section. This result is obvious in the case of the compact-open topology.

Lemma 4.3. *For* $K = \{x_n : n = 0, 1, 2, ..\}$ *with* $x_n \to x_0$ *the set* $W_{K,F} := \{\alpha : \alpha(K) \notin F\} = \{\alpha : \alpha(K) \subseteq X\backslash F\}$ *is dense open in the norm topology.*

Proof. As to open-ness, for $u \in W_{K,F}$ one has $u(K) \subseteq X\backslash F$, so $\varepsilon := \min_{k\in K}\{d(u(k), F\} > 0$, as K is compact. Then $Bu \subseteq W_{K,F}$ for $B = B_\varepsilon(e)$.

As to density, fix u and write $u_n := u(x_n)$. By the Separation Lemma we may assume $u(x_0) \notin F$. Now for some $\varepsilon > 0$ and integer N one has $B_\varepsilon(u(x_0)) \subseteq X\backslash F$ and $u(x_n) \in B_{\varepsilon/2}(u(x_0))$ for $n > N$. As in the Proposition find η with $||\eta|| < \varepsilon/2$ such that $\eta(u_i) \notin F$ for $i \leq N$. But for $n > N$ one has $\eta(u(x_n)) \in B_{\varepsilon/2}(u(x_n)) \subseteq$

$B_\varepsilon(u(x_0)) \subseteq X\backslash F$. Thus for all n one has $\eta(u(x_n)) \notin F$, as required. ∎

We now obtain a first generalization of the KBD Theorem. Here shift-compactness is a consequence of separation properties. A dual version in Section 9 turns out to be broader, but has other underpinnings.

Proposition 4.4. *For T a Baire non-meagre subset of a metric space X and G a separable normed group, Baire in its right norm topology (e.g. almost complete and so non-meagre in the norm topology), acting separately continuously and transitively on X : for every convergent sequence x_n with limit x there is $\tau \in G$ and integer N such that $\tau x \in T$ and*

$$\{\tau(x_n) : n > N\} \subseteq T.$$

Proof. Write $T := M \cup U\backslash \bigcup_n F_n$ with U open, M meagre and each F_n closed and nowhere dense in X. Let $u_0 \in T \cap U$. By transitivity there is $\sigma \in G$ with $\sigma x_0 = u_0$. Put $u_n := \sigma x_n$. Then $u_n \to u_0$. Take $K = \{u_n : n = 0, 1, 2, ..\}$.
As G is Baire the set

$\{\alpha : \alpha(u_0) \in U\} \cap C$, where $C := \bigcap_n \{\alpha : \alpha(K) \notin F_n\}$ is a dense $\mathcal{G}_\delta$,

is non-empty. For α in the above set we have: $\alpha(u_0) \in U\backslash \bigcup_n F_n$. Now $\alpha(u_n) \to \alpha(u_0)$, by continuity of α, and U is open. So for some N we have for $n > N$ that $\alpha(u_n) \in U$. Since $\{\alpha(u_m) : m = 1, 2, ..\} \in X\backslash \bigcup_n F_n$, we have for $n > N$ that $\alpha(u_n) \in U\backslash \bigcup_n F_n \subseteq T$.
Finally put $\tau := \alpha\sigma$ then $\tau(x) = \alpha\sigma(x) \in T$ and $\{\tau(x_n) : n > N\} \subseteq T$. ∎

A refinement of this argument (for which see [Ost-E]) yields:

Theorem 4.5 (Theorem S – Baire Shift Theorem)**.** *For T a Baire non-meagre subset of a metric space X and G a separable normed group, Baire in its right norm topology (e.g. almost complete and non-meagre in the norm topology), acting separately continuously and transitively on X :*

for every convergent sequence x_n with limit x and any Baire non-meagre $A \subseteq G$ with $e_G \in A$ such that $Ax \subseteq T$, there are $\alpha \in A$ and an integer N such that $\alpha x \in T$ and

$$\{\alpha(x_n) : n > N\} \subseteq T.$$

Remark 4.6. **(Generalizing shift-compactness via 'shifting into disjointness'.)** A metric space (X, d^X) is *strongly locally homogeneous* if there is a base consisting of open sets U with the property that for any $x, y \in U$ there is $h \in \mathcal{H}(X)$ taking x to y, equal to the identity outside U. (Denoting d^X-diameters also by d^X, evidently $||h|| < d^X(U)$.) In such a space any compact nowhere dense set has a disjoint homeomorphic image under a homeomorphism in $\mathcal{H}(X)$. Thus a compact nowhere dense set may be shifted into disjointness from itself. Indeed one might consider shift-compactness in the broader context of k-spaces, since closure of a set may be tested by reference to whether the trace on nowhere dense compact sets is closed (cf. [KuSz]). Then a complete group which shifts a zero-dimensional compact nowhere dense set into disjointness from another nowhere dense set exhibits a more general form of shift-compactness. (As to complications in regard to zero-dimensional subspaces, see e.g. [Cie].)

5. Properties of normed groups

Unless otherwise stated, the default norm topology is the right norm topology.

Theorem 5.1 (Squared Pettis Theorem, [BOst-N], Th. 5.8)**.** *Let X be an almost complete normed group and A Baire non-meagre (under the right norm topology). Then e_X is an interior point of $(AA^{-1})^2$.*

Squaring and higher powers of AA^{-1} were studied by R. Henstock [Hen] and E. Følner [Fol].

When X is a locally compact normed group there exists an invariant Haar measure on X ([Ost-LBIII]), and so a Haar-measure variant of Theorem KBD holds as does also a measure version of the Pettis Theorem (in fact without any squaring).

Theorem 5.2 (Baire Homomorphism Theorem, [BOst-N], Th. 11.11, and [Ost-LBIII])**.** *Let X and Y be normed groups analytic in the right-norm topology with X non-meagre. If $f : X \to Y$ is a Baire homomorphism (i.e. preimages of open sets havethe Baire property), then f is continuous.*

5.1. **Application: Subgroup Theorem.**

There are two well-known dichotomies (here 'small or large', rather than 'nice or nasty', as later) which assert that a Baire subset is *either meagre or clopen.* From our current perspective they are 'duals' (as with the generalization of Theorem KBD). There is the Banach-Kuratowski dichotomy where the context is a group G and the subset is a subgroup H (evidently invariant under the translation action of H), and there is the Kuratowski-McShane dichotomy, where the context is a topological space and the premise relies on a transitive action of a subgroup of autohomeomorphsims such that each action either leaves the subset invariant or shifts it into disjointness (for bibliography, see §11.5). The latter result is highly thematic here.

The dichotomies below are in keeping with this though they interpret large as 'total'. We give an application below – see [BOst-StOstr] for others related to additivity, sub-additivity, and convexity (or for more detailed analyses: [BOst-GenSub], [BOst-Aeq], [BOst-Thin]). As may be expected from the Banach-Kuratowski dichotomy, for totality one relies either on density or connectedness. The following direct proofs, based on the Squared Pettis Theorem, are inspired by a close reading of work by Hoffman-Jørgensen ([HJ] page 255), where the subgroup theorem is implictly used for a topological group. These assume less than completeness.

Theorem 5.3 (Subgroup Theorem - density version)**.** *In an almost complete normed group G, if H is a dense non-meagre subgroup with the Baire property, then $H = G$.*

Proof. We interpret the statement in the right norm topology. By the Squared Pettis Theorem, as H is Baire non-meagre $(H^{-1}H)(H^{-1}H) = H$ is an open nhd of e_G in G. For any $g \in G\backslash H$ one has $H \cap Hg = \emptyset$ (as otherwise $h_1 = h_2 g$ for $h_1, h_2 \in H$ implies

$g = h_2^{-1}h_1 \in H$) and so Hg is a nhd of g avoiding H. So H is closed in G; so, being dense in G, it is the whole of G. $\square$

Our own earlier normed-group approach in [BOst-N] relied on a weak Archimedean property in lieu of density to derive a similar result, whereas in [BOst-StOstr] we used Kronecker's Theorem to show that in the additive group $\mathbb{R}$ a non-meagre subgroup is dense. In the absence of density the argument above still goes through when the group is connected, as then the Archimedean property holds in regard to H, as we show below.

Theorem 5.4 (Subgroup Theorem – connected group version)**.** *In a connected almost complete normed group* G*, if* H *is a non-meagre subgroup with the Baire property, then* $G = \bigcup_{n\in\mathbb{N}} H^n$ *and so* $H = G$*.*

Proof. Working as before, by the Squared Pettis Theorem, since H is Baire non-meagre, $(H^{-1}H)(H^{-1}H) = H$ is an open nhd of e_G in G. Suppose that $B := B_\varepsilon(e) \subseteq H$. As B is symmetric and $BS = \bigcup_{s\in S} Bs$ is open for any set S, the set $C := \bigcup_{n\in\mathbb{N}} B^n$ is an open subgroup with $C \subseteq H$. Now for any $g \in G\backslash C$ one has $C \cap Cg = \emptyset$ (as before, since otherwise $c_1 = c_2 g$ for $c_1, c_2 \in C$ implies $g = c_2^{-1}c_1 \in C$), and so Cg is a nhd of g avoiding C. So the non-empty set C is both closed and open in G, and, by the connectedness of G, is the whole of G, i.e. $G = C = H$. ▮

Theorem 5.5 (Loy [Loy], Hoffman-Jørgensen [HJ])**.** *A non-meagre analytic topological group is Polish.*

Proof. An analytic topological group H, being separable, may be densely embedded (by completion) in a complete separable topological group G, but now H is a non-meagre subgroup with the Baire property (being analytic, by Nikodym's theorem, cf. [Jay-Rog], §2.9), so is all of G by the density version of the Subgroup Theorem. ▮

The argument above does not embrace general normed groups, since a normed group that can be extended to a complete normed group is necessarily a topological group (cf. [BOst-N] Th. 3.38). Nonetheless an informative analogue does exist in the form of the Semi-Polish Theorem, which refers to the symmetrized topology – see Section 8.

6. Applications of shift-compactness

In this section we illustrate how easy it is to derive results using shift-compactness. A deeper result is the Semi-Polish Theorem of Section 8.

6.1. **Uniform Boundedness Theorem.**

Theorem 6.1. *For X a non-meagre topological vector space and F a family of continous linear functionals, if for each x the set $\{||f(x)||:f\in F\}$ is bounded, then $\{||f(x)||:f\in F\}$ is bounded on a nhd of* 0.

Proof. Suppose otherwise. Then, for each n, there exist $x_n\in X$ and $f_n\in F$ such that $||f_n(x_n)||\geq n$. As f is continuous, $\{x:||f(x)||\leq n\}$ is closed, and so is

$$A_n:=\bigcap\nolimits_{f\in F}\{x:||f(x)||\leq n\},$$

so has the Baire property. By assumption

$$X=\bigcup\nolimits_n A_n.$$

So, since X is non-meagre, there is N such that A_N is non-meagre. So there are $t\in A_N$ and infinite $\mathbb{M}_t$ such that $x_m+t\in A_N$ for $m\in\mathbb{M}_t$. For $m\in\mathbb{M}_t$ one has

$$||f_m(x_m)||=||f_m(x_m+t)-f_m(t)||\leq||f_m(x_m+t)||+||f_m(t)||\leq 2N,$$

so $\{||f_m(x_m)||:m\in\mathbb{M}_t\}$ is bounded, a contradiction. ∎

Of course the above really proves a theorem about continous homomorphisms on a non-meagre normed topological group.

6.2. **Steinhaus-Piccard Theorem.** The following result refers to condition (wcc) studied in §9, which is satisfied for $\mathcal{E}$ the Euclidean topology and $\mathcal{D}$ the density topology on the line.

Theorem 6.2 (Fine Topology Interior Point Theorem)**.** *Let* $\mathbb{R}$ *be given a shift-invariant topology* $\mathcal{T}$ *under which it is a Baire space and suppose the homeomorphisms* $h_n(x) = x + z_n$ *satisfy (wcc), whenever* $\{z_n\} \to 0$ *is a null sequence (in the Euclidean topology). For* S *Baire and non-meagre in* $\mathcal{T}$, *the difference set* $S - S$ *contains an interval around the origin.*

Proof. Suppose otherwise. Then for each positive integer n we may select

$$z_n \in (-\,1/\,n, +\,1/\,n)\,\backslash(S - S).$$

Since $\{z_n\} \to 0$ (in the Euclidean topology), the Category Embedding Theorem applies, and gives an $s \in S$ and an infinite $\mathbb{M}_s$ such that

$$\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S.$$

Then for any $m \in \mathbb{M}_s$,

$$s + z_m \in S \text{ , i.e. } z_m \in S - S,$$

a contradiction. ∎

6.3. **The Effros Open Mapping Theorem.** We recall that a (classically) analytic set is the continuous image of a Polish space. Thus a continuous image of an analytic set is an analytic set.

Definition 6.3. A group $G \subset \mathcal{H}(X)$ acts *weakly* on a space X if $(g, x) \to g(x)$ is continuous *separately* in g and in x.
A group $G \subset \mathcal{H}(X)$ acts *transitively* on a space X if for each x, y in X there is g in X such that $g(x) = y$.
The group acts *micro-transitively* on X if for U open in G and $x \in X$ the set $\{h(x) : h \in U\}$ is a neighbourhood of x.

Theorem 6.4 (The Effros Open Mapping Principle, [Eff])**.** *Let G be a Polish topological group acting transitively on a separable metrizable space X. The following are equivalent.*
(i) G acts micro-transitively on X,
(ii) X is Polish,
(iii) X is non-meagre
More generally, for G an analytic ***normed group*** *acting transitively on a separable metrizable space X:*

$$\textit{(iii)} \implies \textit{(i)},$$

i.e., if X is non-meagre, then G acts micro-transitively on X.

Remark 6.5. Jan van Mill [vM1] gave the more general result here for G an analytic *topological* group, but actually his proof only assumes in effect a **normed** group structure. Of interest is the following result

Theorem 6.6 (Effros Theorem – Baire variant, [Ost-E])**.** *Let the normed group G have separately continuous and transitive action on X. If under either norm topology G is analytic and Baire and X is non-meagre, then the action of G is micro-transitive. That is, for U an open neighbourhood of e_G and for arbitrary $x \in X$ the set $Ux := \{u(x) : u \in U\}$ is a neighbourhood of x, so that the point-evaluation maps $g \to g(x)$ are open for each x.*

Remark 6.7. Analyticity here is needed principally to ensure that certain sets have the Baire property. Note that a group that is analytic and non-meagre is Baire (see the Characterization Theorem of Section 7.).

Lemma 6.8. *If G is a separable normed group, acting transitively on a non-meagre space X, then for each non-empty open U in G and each $x \in X$ the set Ux is non-meagre in X.*

Proof. We work in the right norm topology first. Suppose that $u \in U$ and so without loss of generality assume that $U = B_\varepsilon(u) = B_\varepsilon(e_G)u$ (for some $\varepsilon > 0$); put $y := ux$ and $W = B_\varepsilon(e_G)$. Then $Ux = Wy$. Next work, exceptionally, in the left norm topology (for which $W = B_\varepsilon(e_G)$ is a nhd of e_G); as each set hW for $h \in G$ is now open (since now the left shift $g \to hg$ is a homeomorphism), the open family $\{gW : g \in G\}$ covers G, and so has a countable sub-cover, $\{g_nW : n \in \omega\}$ say. As G is transitive, $X = Gy$ and so X is covered by $\{g_nWy : n \in \omega\}$. So for some n, the set g_nWy is non-meagre. As g_n^{-1} is a homeomorphism of X, the set $Wy = Ux$ is also non-meagre in X. ▮

The proof below refers to analyticity in order to use the fact that the continuous image of an analytic set is analytic and so has the Baire property (again Nikodym's Theorem cf. – [Jay-Rog], §2.9).

Proof of the Effros Theorem. Assume G acts transitively on X and that X is non-meagre. Let $B := B_\varepsilon(e_G)$ and suppose that for some x the set $T := Bx$ is not a nhd of x. Then there is $x_n \to x$ with $x_n \notin Bx$ for each n. Take $A := B_{\varepsilon/2}(e_G)$ and note that A is a symmetric open set ($A^{-1} = A$, by the inversion axiom). By the preceding lemma, Ax is non-meagre, since G is separable (being analytic). By Theorem S, since the evaluation map $g \to g(x)$ is continuous Ax is analytic and so Baire ([Jay-Rog]), so there are $a \in A$ (being open, has the Baire property) and a co-finite $\mathbb{M}_a$ such that $ax_m \in Ax$ for $m \in \mathbb{M}_a$. For any such m choose $b_m \in A$ with $ax_m = b_mx$. Then $x_m = a^{-1}b_mx \in A^2x \subseteq Bx$, a contradiction (note that $a^{-1} \in A$, by symmetry).

7. ANALYTICITY

The theme of this section is to be more persuasive about analytic spaces. The point is that analytic sets have the Baire property, so are almost absolute-$\mathcal{G}_\delta$, so almost complete and their structure often **directly** provides completeness arguments without the need to re-metrize their absolute-$\mathcal{G}_\delta$ 'cores' (see the Convergence Criterion below). It may be that p-spaces have overtaken analyticity – see the papers by Bouziad ([Bou1], [Bou2]) in which p-spaces ride harmoniously with α-favourability (refining a key feature of Baire spaces), and the concluding discussion in [Ost-AH]. Here we argue that they must now again play their role in mainstream topology. So we say: now they are back!

Analytic sets, after their classical phase in the hands of the founding fathers of topology, have been a central topic for Logicians whose 'neoclassical' contributions have been spectacular (especially Silver's Theorem, cf. [Rog]). Early interest was shown by F. B. Jones, who showed that a Hamel basis for the reals as a vector space over the rationals cannot be analytic (as did also W. Sierpiński), and applied this to an analysis of the Hamel pathology of additive functions in [Jones] (cf. §11.7).

Recall that Cantor's Theorem on the intersection of a nested sequence of closed (or compact, as appropriate) sets has two formulations: (i) referring to vanishing diameters (in a complete-space setting), and (ii) to (countable) compactness. Our first aim in this section is to give a topological version that is in this same spirit but appropriate to an analytic, rather than complete or compact, context.

Recall again that, in a metric space, a set is *analytic* if it is the continuous image of a Polish space P, i.e. of the form $f(P)$ for f continuous and P Polish – see [Jay-Rog] for details. Although our concern here is with metric spaces, there are several advantages in discussing analytic sets in the broader context of Hausdorff topological spaces, arising from explicitly exposing their underlying topological nature. The brief account below will suffice here – see [Ost-AH] for a wider discussion.

For X a Hausdorff space write $\mathcal{K} = \mathcal{K}(X)$ for the family of compact subsets of a space X, and $\wp(X)$ for the power set. Following the the notation of [Jay-Rog], write I for $\mathbb{N}^{\mathbb{N}}$ endowed with the product topology (treating $\mathbb{N}$ as discrete) and $i|n := (i_1, ..., i_n)$ for $i \in I$. For X a Hausdorff space a map $K : I \to \wp(X)$ is called *compact-valued* if $K(i)$ is compact for each $i \in I$, and *singleton-valued* if each $K(i)$ is a singleton. K is *upper semicontinuous* if, for each $i \in I$ and each open U in X with $K(i) \subseteq U$, there is a neighbourhood $N = \{j \in I : j|n = i|n\}$ of i such that $K(j) \subseteq U$ for each j in N. A subset of X is $\mathcal{K}$-*analytic* if it is the image $K(I)$ under an upper semicontinuous compact-valued map. We put

$$K(i_1, ..., i_n) = K(I(i_1, ..., i_n)) = \bigcup\nolimits_{i \in I(i_1,...,i_n)} K(i).$$

By a theorem of Jayne this is equivalent to other definitions studied by: Choquet 1951, Sion 1960. Fremlin, 1980, defines a more general notion of *Čech-analyticity* (for which see §11.6)

The following result is implicit in a number of situations, and goes back to Frolík's characterization of completely regular Čech-complete spaces as $\mathcal{G}_\delta$ in some compactification ([Frol-60]; see [Eng] §3.9). See [Ost-AH] for this and other versions. It is the 'completeness-compactness' property below that motivates the Choquet-style α-favourability perspective on classical analytic sets, and exposes their inherently topological nature.

Theorem 7.1 (Analytic Cantor Theorem)**.** *Let X be a Hausdorff space and $A = K(I)$ be $\mathcal{K}$-analytic in X, with K compact-valued and upper-semicontinuous.*
If F_n is a decreasing sequence of (non-empty) closed sets in X such that $F_n \cap K(I(i_1, ..., i_n)) \neq \emptyset$, for some $i = (i_1, ...) \in I$ and each n, then $K(i) \cap \bigcap_n F_n \neq \emptyset$.
Equivalently, if there are open sets V_n in I with $clV_{n+1} \subseteq V_n$ and $diam_I V_n \downarrow 0$ such that $F_n \cap K(V_n) \neq \emptyset$, for each n, then
(i) $\bigcap_n clV_n$ is a singleton, $\{i\}$ say;
(ii) $K(i) \cap \bigcap_n F_n \neq \emptyset$.

Proof. If not, then $\bigcap_n K(i) \cap F_n = \emptyset$ and so, by compactness, $K(i) \cap F_p = \emptyset$ for some p, i.e. $K(i) \subseteq X \backslash F_p$. So by semicontinuity $F_p \cap K(I(i_1, ..., i_n)) = \emptyset$ for some $n \geq p$, yielding the contradiction $F_n \cap K(I(i_1, ..., i_n)) = \emptyset$. ∎

The theorem that follows the Convergence criterion is crucial; this may be deduced from results due to van Mill [vM1] and Levi [Lev] – see also [Ost-AH]. A proof may be given based on the Convergence criterion, so is specific to the normed-group context (compare also Oxtoby's Theorem, e.g. in [Kech, Th. 8.17(i)]); it is thus based on Th. AC via this immediate corollary:

Proposition 7.2 (Convergence criterion)**.** *In an analytic normed group $X = K(I)$, for $\alpha_n = a_n \cdot \ldots \cdot a_1$ with $\mathrm{cl}B_{r_{n+1}}(a_{n+1}) \subseteq B_{r_n}(e)a_n$ and $K(i_1, ..., i_n) \cap B_{r_n}(\alpha_n) \neq \emptyset$, the sequence $\{\alpha_n\}$ is convergent.*

Proof. Indeed, $\alpha_n \to \alpha$, where

$$\{\alpha\} = K(i) \cap \bigcap\nolimits_n F_n \text{ for } F_n = \mathrm{cl}(B_{r_n}(\alpha_n)).$$

∎

Theorem 7.3 (Analytic Baire Theorem)**.** *In a normed group X under d_R^X, if X contains a non-meagre classically analytic set, then X is Baire. In fact, up to a meagre set, X is analytic (and separable).*

As a corollary we obtain the following.

Theorem 7.4 (Characterization Theorem for Almost completeness)**.** *In a separable normed group X under d_R^X, the following are equivalent:*
(i) X is a non-meagre absolute $\mathcal{G}_\delta$ modulo a meagre set (i.e. is almost complete);
(ii) X contains a non-meagre analytic subset;
(iii) X is non-meagre analytic modulo a meagre set.

Armed with these facts we may state a result that relies on analyticity.

Theorem 7.5 (Analytic Shift Theorem)**.** *In a normed group under the topology d_R^X, for $z_n \to e_X$ null, A classically-analytic and non-meagre:*
for a non-meagre set of $t \in A$ (in fact with co-meagre Baire envelope), there is an infinite set $\mathbb{M}_t$ and points $a_n \in A$ converging to t such that

$$\{ta_m^{-1}z_m a_m : m \in \mathbb{M}_t\} \subseteq A.$$

In particular, if the normed group is topological, for quasi all $t \in A$, there is an infinite set $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq A.$$

7.1. **Fine Analytic Baire Theorem.** The Gandy-Harrington Theorem asserts that Baire's Theorem holds for $\mathcal{GH}$, the fine topology of the reals generated by declaring the effective analytic sets to be open. We generalize this theorem to embrace a wide family of fine topologies. The key idea is of course analyticity. Refinement topologies are on the whole regarded by topologists as 'strange beasts', unless one is a functional analyst where the interplay of weak and strong topologies is common. But refinement is an important tool; our own contribution aside, let us briefly give chapter and verse.

One says $\mathcal{D}$ is a *fine* topology on $\mathbb{R}$. Fine topologies on $\mathbb{R}$ capture different notions of 'typicality' or 'randomness'. The alternative intuitive view is that they allow in more 'exceptions' – as e.g. in *Denjoy continuity*.

Mathematical Logic has been particularly successful in exhibiting various natural forms of 'typicality' in the form of 'genericity' constructed through 'forcing' methods. Refinements of the usual (Euclidean) topology of the reals play a special role in 'neo-classical' descriptive set theory.

Less spectacular examples come from Analysis, with $\mathcal{D}$ (cf. §4 above) the most widely known, the fine topology' a standard tool in potential theory in dimension ≥ 2. Lesser known topologies used to advantage in analysis include O'Malley's [O] *r-topology* $\mathcal{R}$ on $\mathbb{R}$ (or *resolvable*-topology – for this term see [Kur-1] §12. III, V), which he used to study approximate differentiability of real-valued functions; $\mathcal{R} \subseteq \mathcal{D}$ and is generated by taking as base $\mathcal{B}:= \mathcal{D} \cap \mathcal{G}_\delta \cap \mathcal{F}_\sigma$ the sets of $\mathcal{D}$ that are ambiguously both $\mathcal{G}_\delta$ and $\mathcal{F}_\sigma$ in the real line. (Here we see it is a generalized Gandy-Harrington topology, avant la lettre.) Another is *Scheinberg's maximal topology* $\mathcal{U} \supseteq \mathcal{D}$ (see [Sch]), which has the following important lifting property: any bounded measurable real-valued function f is equal a.e. to a unique function $\tilde{f}$ continuous relative to $\mathcal{U}$. (His modification refers to an ultrafilter of measurable sets extending the filter $\mathcal{D}_0 := \{D \in \mathcal{D} : 0 \in \mathcal{D}\}$.) Frolík and Holický use [FH] fine uniformities to study non-separable analyticity.

Actually topologists might profitably review even the best-known Logic examples: the Cohen reals (generic in the Euclidean topology) and the Solovay reals (generic in the density topology). Both are generic in avoiding 'ground-model sets' in the respective σ-ideals

$\mathcal{M}$ (the meagre sets) and $\mathcal{N}$ (the null sets). Here one might consider the infinitary combinatorics associated with these reals in reference to the established *ideal-neglecting topologies.* In a group context, if a σ-ideal is translation-invariant and $\mathcal{I}$ satisfies the localization property (see below), then a topology that neglects members of $\mathcal{I}$ may be defined so that G is open in the *ideal-neglecting topology*, in brief *i*-open, iff G takes the form $U \backslash Z$ with U arbitrary open and any $Z \in \mathcal{I}$; see [LMZ] p. 25. The case $\mathcal{I} = \mathcal{N}$, studied in [Sch], gives a topology $\mathcal{T}$ with $\mathcal{T}_d \subseteq \mathcal{T} \subseteq \mathcal{D}$.

Set-theoretic topologists are familar with the Ellentuck topology $\mathcal{E}l$ used to study the Ramsey property which corresponds to Mathias forcing (and Mathias reals). Less well-known, alas, are the Gandy "reals" (Gandy-Sacks degrees), which pre-date the Gandy-Harrington topology $\mathcal{GH}$ used to prove Silver's Theorem ([Jay-Rog] p. 466), a remarkable 'neo-classical' result of descriptive set-theory.

Our view-point here is that the introduction of $\mathcal{GH}$ by Harrington and of $\mathcal{E}l$ by Ellentuck (and Louveau) should both be viewed as two examples of a canonical construction. We need some preparatory tools and an old result of Kuratowski.

Definition 7.6. For $\mathcal{I}$ a σ-ideal, say that S is **$\mathcal{I}$-heavy**, resp. *locally $\mathcal{I}$-heavy on G*, in X if $S \cap U \notin \mathcal{I}$ for every open set U in X meeting S, resp. for every open U meeting $S \cap G$.

(The term 'heavy' is established, going back to [BrGo]; van Mill [vM1] calls a dense, heavy set 'fat'. See also [St] for a general 'kernel' approach.)

Definition 7.7. The *$\mathcal{I}$-light part* of A is defined to be the set $L_{\mathcal{I}}(A) := \bigcup\{V \cap A : V$ open and $A \cap V \in \mathcal{I}\}$. The *heavy part* of A is the complementary set $H_{\mathcal{I}}(A) = A \backslash L_{\mathcal{I}}(A)$. So A is *heavy* (*heavy* on G) iff $L_{\mathcal{I}}(A)$ is empty ($L_{\mathcal{I}}(A) \cap G$ is empty).

Definition 7.8. Say that $\mathcal{I}$ has the *localization property* if $L_{\mathcal{I}}(A) \in \mathcal{I}$ for each A.

Remark 7.9. 1. In a second-countable space any σ-ideal has the localization property. In a metric space the σ-ideal $\mathcal{M}$ of meagre sets has the localization property: this is Banach's Category Theorem ([Oxt] Ch. 16, [Jay-Rog] p. 42, [Kel] Th. 6.35, or [Kur-1] §10.III under the name Union theorem). In a locally compact metric group equipped with a Haar measure the σ-ideal $\mathcal{N}$ of null sets has the localization property (this follows from the appropriate Lebesgue Density Theorem, or its generalization – see [BOst-N]).
2. Under such circumstances, $H = H_{\mathcal{I}}(A)$ is $\mathcal{I}$-heavy, since $H \subseteq A$. (Otherwise there is open U with $\emptyset \neq U \cap H \in \mathcal{I}$, so $U \cap A \subseteq (U \cap H) \cup L_{\mathcal{I}}(A)) \in \mathcal{I}$; then $U \cap A \subseteq L_{\mathcal{I}}(A)$ so $(U \cap A) \cap H = \emptyset$, and so $\emptyset \neq U \cap H = U \cap H \cap A = \emptyset$, a contradiction.)

When the heavy part is non-empty it is dense-in-itself, and so some work of Kuratowski [Kur-34], concerning closed sets that are dense-in themselves relative some open set, can be easily amended to give the following results. (Compare also Jay-Rog §2.4.)

Lemma 7.10. *In a metric space, let $\mathcal{I}$ be a σ-ideal of subsets with the localization property. For G open and F closed with $F \cap G$ non-empty and $\mathcal{I}$-heavy, there exist for each $\delta > 0$ a sequence of non-empty, closed $\mathcal{I}$-heavy subsets H_i of G with $d(H_i) < \delta$ such that*

$$F \cap G = \bigcup\nolimits_i H_i \text{ with each } H_i \backslash \bigcup\nolimits_{j<i} H_j \text{ non-empty and } \mathcal{I}\text{-heavy.}$$

Theorem 7.11 (Generalized Kuratowski Representation)**.** *For a metric space X and any σ-ideal of subsets $\mathcal{I}$ with the localization property, any non-empty absolute-$\mathcal{G}_\delta$ subset A that is $\mathcal{I}$-heavy, there exists an upper semicontinuous representation $K : I \to X$ with each $K(I(i|n))$ an intersection of an open set $G(i|n)$ and a closed $\mathcal{I}$-heavy subset $F(i|n)$ of A of diameter at most 2^{-n}.*

Since all points of a topologically complete metric space dense-in-itself are condensation points, Kuratowski's original theorem refers de facto to properties of $\mathcal{I}_\omega$-heavy sets, for $\mathcal{I}_\omega$ the σ-ideal of countable subsets rather than the de iure more suggestive $\mathcal{I}_{\text{Fin}}$-heavy sets, for $\mathcal{I}_{\text{Fin}}$ the ideal of finite subsets.

Take $\mathcal{I} = \mathcal{N}$ and note that for the density topology $\mathcal{D}$ the family $\mathcal{H} = \mathcal{D} \cap \mathcal{F} \subseteq \mathcal{G}_\delta$ is a weak base comprising $\mathcal{N}$-heavy sets.

So these have a Kuratowski representation with each $K(I(i|n)) = G(i|n) \cap F(i|n)$, a $\mathcal{D}$-open set (since each $G(i|n)$ is also $\mathcal{D}$-open). This motivates a definition.

Definition 7.12. ($\mathcal{K}$-analytically heavy topologies). 1. For $(X, \mathcal{T})$ a topological space denote by $\mathcal{A}(\mathcal{T})$ the family of $\mathcal{K}$-analytic subsets of $(X, \mathcal{T})$.
2. $\mathcal{H}$ is a *topological base* for X if ([Eng] §1.1) $\mathcal{H}$ covers X, and for $H_1, H_2 \in \mathcal{H}$, whenever $x \in H_1 \cap H_2$, there is $H_3 \in \mathcal{H}$ with $x \in H_3 \subseteq H_1 \cap H_2$. We write $\mathcal{G}_{\mathcal{H}}$ for the topology generated by $\mathcal{H}$.
3. $\mathcal{B}$ is a *weak base* for a topology $\mathcal{T}$ if for each non-empty $V \in \mathcal{T}$ there is $B \in \mathcal{B}$ with $\emptyset \neq B \subseteq V$. In fact, sometimes we need only a *very weak base:* for each non-empty $V \in \mathcal{T}$ there is $B \in \mathcal{B}$ with $\emptyset \neq B \cap V$.
4. Let $(X, \mathcal{T})$ be a regular Hausdorff space and $\mathcal{T}' \supseteq \mathcal{T}$ a refinement topology. We say $\mathcal{T}'$ is *analytically heavy*, or *weakly $\mathcal{K}$-analytically generated* in $\mathcal{T}$, if $\mathcal{T}'$ possesses a weak base $\mathcal{H} \subseteq \mathcal{A}(\mathcal{T})$ comprising sets that are $\mathcal{K}$-analytic sets in $\mathcal{T}$ with a *$\mathcal{T}'$-open representation*, i.e. an upper semicontinuous representation $K : I \to \mathcal{K}(X)$ with $K(U) \in \mathcal{T}'$ for U open in I.

Remark 7.13. The term *analytically heavy* is only suggestive since one has for any non-empty open $G \supseteq A \neq \emptyset$ for some A analytic.

We are now able to give a very general Baire Theorem.

Theorem 7.14 (Fine Analytic Baire Theorem, Generalized Gandy-Harrington Theorem, cf. [Jay-Rog] p. 466)**.** *In a regular Hausdorff space $\mathcal{T}$, if $\mathcal{T}'$ is a refinement topology of $\mathcal{T}$, possessing a weak base (pseudo-base) $\mathcal{H} \subseteq \mathcal{A}(\mathcal{T}) \cap \mathcal{T}'$ whose members have an analytic representation that is $\mathcal{T}'$-open, then $\mathcal{T}'$ is Baire.*
In particular, this applies to a Polish space, the Gandy-Harrington $\mathcal{GH}$, the density $\mathcal{D}$, the Ellentuck $\mathcal{E}l$ tand O'Malley $\mathcal{R}$ topologies.

8. Semi-Polish Theorem

We sketch below the underlying theme of the proof of the

Theorem 8.1 (Semi-Polish Theorem)**.** *For a normed group X under d_R^X, if the space X is non-meagre and semi-analytic (i.e. is analytic under d_S, e.g. Polish under d_S), then it is a Polish topological group.*

Remark 8.2. This asserts that the topology determined by d_R^X is Polish and is *admissible*, i.e. under this topology X is a topological group.

The proof applies the Baire Homomorphism Theorem of Section 5, the following Open Mapping Theorem and shift-compactness.

Lemma 8.3 (Levi's Open Mapping Theorem, [Lev])**.** *Let X be a regular (classically) analytic space. Then X is Baire iff $X = f(P)$ for some f continuous and defined on some Polish space P with the property that there exists a set X' which is a dense metrizable $\mathcal{G}_\delta$ in X such that the* ***restriction map*** *$f|P' : P' \to X'$ where $P' = f^{-1}(X')$* ***is open****.*

Lemma 8.4. *For X a normed group, if (X, d_S) is Polish and (X, d_R) non-meagre, then there is a subset Y of X which is a dense absolute-$\mathcal{G}_\delta$ in (X, d_R), and on which the d_S and d_R topologies agree.*

Proof. The embedding $j : (X, d_S) \to (X, d_R)$ with $j(x) = x$ makes (X, d_R) is analytic, and being non-meagre is Baire.
Apply Levi's Theorem to $f = j$ to obtain a set Y that is a *dense* absolute $\mathcal{G}_\delta$ in (X, d_R), s.t. every open set in (Y, d_S) is open in (Y, d_R). Every open set in (Y, d_R) is open in (Y, d_S), since d_S is a refinement of d_R.
Thus the two topologies agree on the $\mathcal{G}_\delta$ subset Y.
As Y is a $\mathcal{G}_\delta$ subset of (X, d_R), it is also a $\mathcal{G}_\delta$ subset in the complete space (X, d_S), and so (Y, d_S) is topologically complete. So (Y, d_R) is an absolute $\mathcal{G}_\delta$, being homeomorphic to (Y, d_S). Working in Y, we have $y_n \to_R y$ iff $y_n \to_S y$ iff $y_n \to_L y$. ∎

Lemma 8.5. *If in the setting of the preceding Lemma the three topologies d_R, d_L, d_S agree on a dense absolutely-$\mathcal{G}_\delta$ set Y of (X, d_R), then for any $\tau \in Y$ the conjugacy $\gamma_\tau(x) := \tau x \tau^{-1}$ is continuous.*

The proof uses the analytic shift theorem to show first that $\gamma(x) := \tau^{-1} x \tau$ is continuous fo $\tau \in Y$. One then sees that γ^{-1} is a Baire homomorphism, so in the current setting is continuous. We can now prove the main result.

Proof of the Semi-Polish Theorem. Under d_R, the set $Z_\Gamma := \{x : \gamma_x \text{ is continuous}\}$ is a *close*d (subsemigroup) of X .

So $X = \mathrm{cl}_R Y \subseteq Z_\Gamma$, i.e. γ_x is continuous for all x, and so X is a topological group under the topology of d_R^X.

So $x_n \to_R x$ iff $x_n^{-1} \to_R x^{-1}$ iff $x_n \to_L x$ iff $x_n \to_S x$. So the topology of d_R^X is Polish as the topology of d_S^X is a Polish. □

9. Life without translations

First recall in telegraphic style some background on the density topology $\mathcal{D}$ on $\mathbb{R}$. Members of $\mathcal{D}$ are measurable sets such that all points of D are points of density 1. An arbitrary union of these is also measurable. So $\mathcal{D}$ is a refinement topology of the Euclidean one, which we denote by $\mathcal{E}$, i.e. $\mathcal{E} \subseteq \mathcal{D}$. One says it is a *fine* topology on $\mathbb{R}$. A set A is Baire in $\mathcal{D}$ iff A is measurable. A is meagre in $\mathcal{D}$ iff A is null. So $\mathcal{D}$ is a Baire space. Translations are homeomorphisms: so $\mathcal{D}$ gives a semitopological group that is not paratopological ([Sch] Prop. 1.9). Note that $\mathcal{D}$ has a *weak base* consisting of the $\mathcal{G}_\delta$ subsets which are in $\mathcal{D}$.

Now comes the key idea of this section.

Definition 9.1. (**weak category convergence**). A sequence of homeomorphisms h_n of a topological space $\mathfrak{X} = (X, \mathcal{T}_X)$ satisfies the *weak category convergence* condition (wcc) if for any non-meagre open set U there is an non-meagre open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap\nolimits_{n \geq k} V \backslash h_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ h_n(t) \in V.$$

The condition (wcc) may variously be interpreted as a topological convergence condition: see below and the next subsection.

Theorem 9.2 (Bitopological Shift-Theorem, aka Category Embedding Theorem)**.** *Let* $\mathcal{T}_X$ *be a submetrizable topology on* X*, i.e. a refinement topology of some metric topology* $(X,\mathcal{T}_d)$*.*
For a subgroup $G\subseteq\mathcal{H}(X,\mathcal{T}_d)\cap Auth(\mathcal{T}_X)$ *under the right norm topology, put* $H(g,x)=g(x)$ *for* $g\in G$ *and* $x\in X$*.*
Then the mapping $H_g:x\rightarrow g(x)$ *is continuous.*
Suppose further that for any $h_n\rightarrow e_G$ *in norm,* h_n *satisfies the (wcc).*
Let $T\subseteq X$ *be non-meagre and Baire in* $\mathcal{T}_X$*.*
Then there exists $t\in T$ *such that* $h_n(t)\in T$ *infinitely often.*

Remark 9.3. 1. Generalizations exist with consecutive embeddings of fixed length 'in van der Waerden style' (e.g. both h_{2m} and h_{2m+1} in T) and with multiple embeddings (into a sequence of sets T_n – this is Kingman's theorem) For a note on consequences see §11.8, or for details [BOst-KCC].
2. If $\mathcal{T}$ is the density topology, the set T above may w.l.o.g. be a density-open set W. One may show that under certain circumstances, which include the case of the real line under the density topology, the (wcc) condition is a continuity condition (see [MilOst]).
3. On the real line one proves that, for $z_n\rightarrow 0$, the shifts $h_n(t)=t+z_n$ satisfy wcc both
(i) in the usual (Euclidean norm) topology and
(ii) in the density topology.
Hence both versions of KBD follow from the last theorem.
4. Recall the **Birkhoff-Kakutani Theorem** (cited in Section 2) that a metrizable *topological* group has a right-invariant metric. In this case:
i) if X is a Baire space under the norm topology, then the wcc holds under the norm topology,
ii) if, additionally, X is locally compact, X has a Haar measure and the wcc can be verified for the *Haar-density topology.*

Definition 9.4. Say that $\{h_n\}$ *$\mathcal{I}$-converges to the identity* and write $\{h_n\} \rightrightarrows_{\mathcal{I}} e_G$ if for any open U on X there is a non-empty open $W \subseteq U$ such that for every increasing sequence $\{m(n)\}$ of natural numbers,

$$\bigcap\nolimits_n V \backslash h_{m(n)}^{-1}(V) \in \mathcal{I}.$$

Taking in particular for $m(n) = n + k$, one retrieves the old wcc condition for $k = 1, 2, ...$ as part of the new more demanding condition. For the group of translates on $\mathbb{R}^d$ this condition holds equally well, since $z_{m(n)}$ is a null sequence whenever z_n is a null sequence.

In [MilOst] studies when the convergence structure $\{h_n\} \rightrightarrows_{\mathcal{I}} e_G$ is topological, i.e. generates a topology $\mathcal{T}_{\mathcal{I}}$. One has the following

Theorem 9.5 ([MilOst])**.** *For the group of translations of the real line under the supremum norm defined in (sup) and with $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$, the topology $\mathcal{T}_{\mathcal{I}}$ on G is well-defined and is coarser than the right-norm topology.*

9.1. **From weak category to coarse convergence.** This section is devoted to interpreting (wcc). Refining an argument developed in [BOst-LBII], one obtains the following improvement. (For background on submetrizability see [Grue-84]).

Theorem 9.6 (Convergence to the identity)**.** *Assume that the homeomorphisms $h_n : X \to X$ satisfy the weak category convergence condition (wcc) and that X is a sub-metrizable Baire space. Then, for quasi-all t, there is an infinite $\mathbb{N}_t$ such that in the coarser associated metrizable topology*

$$\lim\nolimits_{m\in\mathbb{N}_t} h_m(t) = t.$$

Proof. Working first in the coarser metrizable topology, let $\mathcal{B} = \bigcup_{m\in\omega} \mathcal{B}_m$ be a basis with each $\mathcal{B}_n$ discrete. Now work in the finer (submetrizable) topology until further notice. In this finer topology each $\mathcal{B}_n$ is still open and discrete, and so the members of each $\mathcal{B}_n$ may be assumed non-empty, so non-meagre (as the finer topology is Baire). By (wcc), select for each $U \in \mathcal{B}_m$ and $k \in \omega$ a non-empty open $V_k(U) \subseteq U$ such that $M_k(U) := \bigcap_{n\geq k} V_k(U) \backslash h_n^{-1}(V_k(U))$ is a

meagre subset of $V_k(U)$.
For $k \in \omega$ and $W \in \mathcal{B}$ choose a maximal family $\mathcal{V}_k^W := \{V_k(U_i^W) : i \in I^W\}$ of disjoint non-empty open subsets of the form $V_k(U)$ for $U \subseteq W$. Let V_k^W denote its union and $F_k(W)$ the closure of V_k^W. Then $F_k(W) \subseteq \bar{W}$. Furthermore $W \subseteq F_k(W)$, otherwise $U := W \backslash F_k(W)$ is non-empty, open and disjoint from V_k^W, so that $V_k(U)$ is non-empty and disjoint from the members of $\mathcal{V}_k^W$, contradicting maximality.
We now construct two meagre sets N and M as follows.
Observe that $N_k(W) := F_k(W) \backslash V_k^W$ is closed and nowhere dense as V_k^W is open; one has $N_k(W) \subseteq \bar{W}$. For $m \in \omega$, note that $B_m := \bigcup\{\bar{W} \backslash W : W \in \mathcal{B}_m\}$ is nowhere dense. (Any point x of X has a nhd G meeting at most one set $W \in \mathcal{B}_m$, say W_G, but otherwise for $W \in \mathcal{B}_m$ the set G misses W and so also $\bar{W}$. As $\bar{W}_G \backslash W_G$ is nowhere dense, there is a non-empty subset G' of G avoiding $\bar{W}_G \backslash W_G$.) By the Banach Category Theorem (for which see [Oxt] Ch. 16, or as the Localization Theorem [Kel], Th. 6.35), since $\mathcal{B}_m$ is a discrete open family, $\bigcup\{N_k(W) \cap W : W \in \mathcal{B}_m\}$ is meagre. So $N_{k,m} := \bigcup\{N_k(W) : W \in \mathcal{B}_m\}$ is meagre, being a subset of $B_m \cup \bigcup\{N_k(W) \cap W : W \in \mathcal{B}_m\}$. So

$$N := \bigcup\nolimits_{k,m \in \omega} N_{km} = \bigcup\{N_k(W) : W \in \mathcal{B}_m \text{ and } k, m \in \omega\}$$

is meagre. Note that $W \backslash N_k(W) \subseteq F_k(W) \backslash N_k(W) \subseteq V_k^W$.
Next for $W \in \mathcal{B}$ and $k \in \omega$, put

$$M_k^W := \bigcup\{M_k(U_i^W) : i \in I^W\} \subseteq W,$$

which is meagre – again by the Banach Category Theorem, as $M_k(U_i^W) \subseteq V_k(U_i^W)$ and $\mathcal{V}_k^W$ is disjoint. Put

$$M := \bigcup\nolimits_{k,m \in \omega} \bigcup\{M_k^W : W \in \mathcal{B}_m\},$$

which is likewise meagre (by the discreteness of $\mathcal{B}_m$).
Consider $t \notin N \cup M$. For $W \in \mathcal{B}$ with $t \in W$ and $k \in \omega$, one has $t \in V_k(U_i^W)$ for some $U_i^W \subseteq W$ with $i \in I^W$, as $t \in W \backslash N_k(W) \subseteq V_k^W$. Also, since $t \notin M$, one has $t \in V_k(U_i^W) \backslash M_k(U_i^W)$, so $t \in h_m^{-1}(V_k(U_i^W))$ for some $m = m(t, k, W, i) \geq k$. So $h_m(t) \in V_k(U_i^W) \subseteq U_i^W \subseteq W$. Passing now to the coarser metrizable topology in which $\mathcal{B}_t := \{W \in \mathcal{B} : t \in W\}$ is a basis for the neighbourhoods of t,

it follows that there is an infinite set $\mathbb{N}_t$ of integers m for which $h_m(t) \to t$ in the coarser topology. ∎

9.2. **A generalized Pettis theorem.** As a final example of the way in which homeomorphisms take over the role of translations we mention that in topological regular variation theory, whenever $x_n \to x_0$ one needs the following property: the existence of a sequence of bounded self-homeomorphisms, i.e. in $\mathcal{H}(X)$, converging to the identity (i.e. $\psi_n \to id_X$ in $\hat{d}^X$, the supremum metric defined in (sup) above) with

$$\psi_n(x_0) = x_n.$$

We call this a 'crimping sequence' and think of this as a sequence of diminishing topological shifts (in the absence of algebraic shifts in X). This property is in fact equivalent to the microtransitivity of the Effros Theorem ([Eff], [vM1]). If $\psi_n \to id_X$ in $\hat{d}^X$, then ψ_n satisfies wcc, so this remains in our ambit.

Here is a continuous analogue which leads to a generalized Pettis theorem in the current setting.

Definition 9.7. Let $\{\psi_u : u \in U\}$ for U an open set in X be a family of homeomorphisms in $\mathcal{H}(X)$. Let $u_0 \in U$. Say that ψ_u *converges to the identity as* $u \to u_0$ if

$$\lim_{u \to u_0} \|\psi_u\|_{\mathcal{H}} = 0.$$

The setting of the next theorem is quite general: homogeneity (relative to $\mathcal{H}(X)$), i.e. all we require is that $\mathcal{H}(X)$ acts transitively on X.

Theorem 9.8 (Generalized Piccard-Pettis Theorem). *Let (X, d^X) be homogenous under $\mathcal{H}(X)$. Suppose that the homeomorphisms ψ_u converge to the identity as $u \to u_0$, and that A is Baire and non-meagre in X. Then, for some $\delta > 0$, we have* $\Psi\Psi$

$$A \cap \psi_u(A) \neq \emptyset, \text{ for all } u \text{ with } d^X(u, u_0) < \delta,$$

or equivalently, for some $\delta > 0$

$$A \cap \psi_u^{-1}(A) \neq \emptyset, \text{ for all } u \text{ with } d^X(u, u_0) < \delta.$$

Example 1. Take $X = \mathbb{R}$, $u_0 = 0$ and $\psi_u(x) = x + u$, then $\lim_{u \to u_0} \|\psi_u\|_{\mathcal{H}} = 0$. For A Baire non-meagre, there is some $\delta > 0$ such that for each u with $|u| < \delta$ there is a_1, a_2 with $a_1 = a_2 + u \in A \cap (A + u)$. So $a_1, a_2 \in A$ and $u = a_1 - a_2$, i.e. $(-\delta, \delta) \subseteq A - A$.

Remark 9.9. Writing $\Psi(u)$ for the map $x \to \psi_u(x)$, the property here for A an open set U may be rephrased as $B_\delta(u_0) \subseteq \Psi^{-1}\{h \in H(X) : U \cap h(U) \neq \emptyset\}$ indicating that Ψ is continuous relative to the lower Fox-Mosco topology on $\mathcal{H}(X)$, as defined in [MilOst].

10. Oscillation and Cauchy dichotomy

In this section we return to normed groups with the intention of showing how close they are to topological groups. We need

Definition 10.1. Recall from Section 2.1, $\gamma_t(x) := txt^{-1}$ denotes conjugacy. Write

$$\omega(t) := \lim_{\delta \searrow 0} \omega_\delta(t), \text{ where } \omega_\delta(t) := \sup_{||z|| \leq \delta} ||\gamma_t(z)||,$$

and call the function $\omega(\cdot)$ the *oscillation of the norm.*

The conjugacy γ_t is continuous (equivalently continuous at e) for those t for which $\omega(t) = 0$. Write

$$\Omega(\varepsilon) := \{t : \omega(t) < \varepsilon\}.$$

As γ_t is a homomorphism, normed groups inherit the Cauchy dichotomy: just like homomorphisms, normed groups are pathological or, given a modicum of regularity, topological.

Note that, working in the right norm topology (X, d_R), one may call t an *ε-shifting point* (on the left) if there is $\delta > 0$ such that for $\|z\| \leq \delta$

$$d_R(t, tz) < \varepsilon,$$

and this is equivalent, in oscillation function terms, to $\omega_\delta(t) \leq \varepsilon$ (since $\|tzt^{-1}\| \leq \varepsilon$ for $\|z\| \leq \delta$).

Theorem 10.2 (Uniform continuity of oscillation)**.** *For X a normed group*

$$\omega(t) - 2||s|| \leq \omega(st) \leq \omega(t) + 2||s||, \textit{ for all } s, t \in X.$$

Hence

$$0 \leq \omega(s) \leq 2||s||, \text{ for all } s \in X,$$

implying uniform continuity in the d_R^X *topology and norm-boundedness of the oscillation.*

Theorem 10.3 (Dense Oscillation Theorem)**.** *In a normed group* X

$$\bigcap_{n\in\mathbb{N}} cl[\Omega(1/n)] = \bigcap_{n\in\mathbb{N}} \Omega(1/n) = Z_\Gamma.$$

Hence, if for each $\varepsilon > 0$ *the* ε*-shifting points are dense, equivalently* $\Omega(\varepsilon) = \{t : \omega(t) < \varepsilon\}$ *is dense for each* $\varepsilon > 0$, *then the normed group is topological.*

More generally, if for some open W *and all* $\varepsilon > 0$ *the set* $\Omega(\varepsilon) \cap W$ *is dense in* W, *then* $\omega = 0$ *on* W; *n particular,*

(i) if $e_X \in W$ *and* X *is connected and Baire under its norm topology, then* X *is a topological group,*

(ii) if X *is separable, connected and topologically complete in its norm topology, then* X *is a topological group.*

Among its corollaries is the following result.

Theorem 10.4. *In a topologically complete, separable, connected normed group* X, *if the topological centre is non-meagre, then* X *is a topological group.*

11. Concluding remarks

1. *Van Mill's analytic version of the Baire Theorem.*

An observation of van Mill in the metric setting of [vM2] (Prop. 2.2) prompted an investigation of analyticity in the normed group setting. He proves what is an equivalent analytic-sets restatement of the standard Baire Theorem (i.e., *in a Polish space, the intersection of dense open sets is dense),* with the advantages of symmetry: analytic sets replace both the space of the premise (that the space is Polish), and the intersection set of the conclusion, and with a weaker hypothesis of largeness, since 'dense *open*' is replaced throughout by 'dense and everywhere *non-meagre and analytic*'.

Passage to other σ-ideals immediately suggests itself and is studied in [Ost-AH], which notes that this analytic Baire Theorem is a direct deduction from Nikodym's Theorem (the Baire property of analytic sets) by an appeal to the fact that dense and heavy analytic sets are co-meagre. An alternative proof can be built on the Convergence criterion.

2. *Normed groups are topological or pathological.*

A slight amount of regularity in the relationship between the left and right norm topologies often in the presence of some topological completeness such as the analyticity of X under d_R^X draws the two into coincidence. Straightforward instances (for which see [Ost-LBIII]) are:
(i) if the graph of the self-homeomorphism $x \to x^{-1}$ is analytic;
(ii) if all the conjugacies $\gamma_t(x) = txt^{-1}$ are Baire under d_R^X;
(iii) if X is locally compact and all the conjugacies $\gamma_t(x) = txt^{-1}$ are Haar-measurable;
(iv) if the norm has the property that there exists a sequence of constants $\kappa_n \to \infty$ such that $\kappa_n||x|| \leq ||x^n||$ for each $n \in \mathbb{N}$ and $x \in X$ ([BOst-N], Th. 3.39, where the normed group is said to be *Darboux-normed*).
As was pointed out earlier, the result (ii) is connected with the Cauchy dichotomy governing automatic continuuity of homomorphisms. More subtle connections, based on conjugacy, can be formulated in terms of the behaviour of the group's oscillation function on a dense subspace, mentioned in Section 10 (for details see [BOst-N]).

3. *Baire versus Baire.*

In this paper the term 'Baire' is used in relation to Baire category ideas, which occur in several related aspects: the Baire categories (meagre, non-meagre sets); the Baire Category Theorem, that in a complete metric space the intersection of dense open sets is *dense* (in fact is an absolute $\mathcal{G}_\delta$ set, i.e. is $\mathcal{G}_\delta$ in any completion and is competely metrizable); Baire spaces (spaces where this theorem holds); sets with the Baire property (i.e. sets open modulo a meagre set); and Baire measurable functions (for which the inverse image of an open sets is a set with the Baire property). For general information on Baire spaces, we refer to [Eng] §3.9 (especially for

topological completeness and the absolute $\mathcal{G}_\delta$ property), and also to [Kech] §8. Note that:
(i) a space that is the union of an absolute $\mathcal{G}_\delta$ set and a meagre set (if the $\mathcal{G}_\delta$ set is dense, then this is 'an almost complete space') is a Baire space [AL];
(ii) in a complete metric space, a set with the Baire property is the union of an absolute $\mathcal{G}_\delta$ set and a meagre set [Kech], 8.23;
(iii) a topologically well-defined subset, such as an analytic set, has the Baire property, so its heavy part, if non-empty, is almost complete by (i);
(iv) a Baire measurable function between separable metric spaces is continuous on a co-meagre absolute $\mathcal{G}_\delta$ set, (see [BOst-N], §11);
(v) a metric space X is Baire iff X has the Blumberg property: for any (arbitrary) function $f : X \to \mathbb{R}$ there is a dense subspace $D = D_f$ on which the restriction function $f|D$ is continuous. (See [BrGo]; [HMc] §3.1 has a wide ranging discussion of Blumberg-type characterizations of Baire spaces. Compare [Mich] Prop. 7.8, for a more recent application.)
So, when working with 'decently defined' spaces and sets, all these notions merge into one notion, that of almost completeness.

For metric spaces our category definition in (i) above is equivalent (and more directly connects with completeness). Indeed, on the one hand a completely regular space is almost Čech-complete iff it contains a dense Čech-complete (or topologically complete) subspace, i.e. one that is absolutely $\mathcal{G}_\delta$ (is $\mathcal{G}_\delta$ in some/any compactification). On the other hand a metrizable Baire space X contains a dense completely metrizable $\mathcal{G}_\delta$-subset iff X is a completely metrizable $\mathcal{G}_\delta$-set up to a meagre set.

We summarize these observations in one formal assertion (see [Ost-AH] and [Ost-AB], for additional material):

Proposition 11.1. *In an almost-complete space, a subset B has the Baire property iff the subspace B is almost-analytic.*

4. *Strong shift-compactness.*

a) The importance of null-sequences may be seen by observing that in the additive group $\mathbb{R}$, the subgroup $\mathbb{Z}$ is closed and locally compact, so shift-compact. Of course, $\mathbb{Z}$ is too small to contain shifts of arbitrary null sequences.

b) The argument in Section 3 shows the following. *In a normed topological group, if a subgroup S is locally right-shift compact, then S is closed and locally compact. Conversely, a closed, locally compact subgroup is locally right-shift compact.*

c) Note that $A \subseteq \mathbb{R}$ is density-open (open in the density topology) iff each point of A is a density point of A. Suppose a_0 is a limit point (in the usual topology) of such a set A; then, for any $\varepsilon > 0$, we may find a point $\alpha \in A$ within $\varepsilon/2$ of a_0 and hence some $t \in A$ within $\varepsilon/2$ of the point α such that some subsequence $t + a_m$ is included in A, with limit $t + a_0$ and with $|t| < \varepsilon$. That is, a density-open set is strongly shift-compact.

d) Suppose that $a_n = (a_n^i) \in A = \prod A_i$. Pick t^i and inductively infinite $\mathbb{M}_i \subseteq \mathbb{M}_{i-1}$ so that $a_n^i t^i \to a_0^i t^i$ along $n \in \mathbb{M}_i$ with $a_n^i t^i \in A_i$ for $n \in \omega$. Diagonalize $\mathbb{M}_i$ by setting $\mathbb{M} := \{m_i\}$, where $m_{n+1} = \min\{m \in \mathbb{M}_{n+1} : m > m_n\}$. Then the subsequence $\{a_m : m \in \mathbb{M}\}$ satisfies, for each J finite,

$$\mathrm{pr}_J a_m t \subseteq \prod\nolimits_{j \in J} A_j \text{ for eventually all } m \in \mathbb{M},$$

and so in the product topology $a_m t \to a_0 t$ through $\mathbb{M}$, where $(a^i)(t^i)$ is defined to be $(a^i t^i)$.

5. *Subgroup dichotomies.*

a) The Subgroup Dichotomy Theorem – in the normed topological groups context, known as the Banach-Kuratowski Theorem (for which see [Ban-G, Satz 1], [Kur-1, Ch. VI. 13. XII]; cf. [Kel, Ch. 6 Pblm P] ; cf. [BGT, Cor. 1.1.4] and also [BCS] and [Be] for the measure variant) asserts the following. *Let X be a topologically complete normed group which is non-meagre and A any Baire subgroup. Then A is either meagre or clopen in X.*

We quote the proof from [BOst-N] as it again illustrates the use of shift-compactness.

Proof. Suppose that A is non-meagre. We show that e is an interior point of A, from which it follows that A is open. Suppose otherwise. Then there is a sequence $z_n \to e$ with $z_n \in B_{1/n}(e)\backslash A$. Now for some $a \in A$ and infinite M we have $z_n a \in A$ for all $n \in M$. But A is a subgroup, hence $z_n = z_n a a^{-1} \in A$ for $n \in M$, a contradiction.

Now suppose that A is not closed. Let a_n be a sequence in A with limit x. Then $a_n x^{-1} \to e$. Now for some $a \in A$ and infinite

M we have $a_n x^{-1} a \in A$ for all $n \in M$. But A is a subgroup, so a_n^{-1} and a^{-1} are in A and hence, for all $n \in M$, we have $x^{-1} = a_n^{-1} a_n x^{-1} a a^{-1} \in A$. Hence $x \in A$, as A is a subgroup.

Banach's proof in [Ban-G] is purely topological, so applies to all topological groups (even though originally stated for metric groups), and relies on the mapping $x \to ax$ being a homeomorphism, likewise Kuratowski's proof, which proceeds via another dichotomy as detailed below.

b) [Kuratowski-McShane Dichotomy – [Kur-33], [Kur-1] Ch 1 §13.XI, [McSh, Cor. 1]] *Suppose* $H \subseteq Auth(X)$ *acts transitively on the topological space* X, *and* $Z \subseteq X$ *is Baire and has the property that for each* $h \in H$

$$Z = h(Z) \text{ or } Z \cap h(Z) = \emptyset,$$

i.e. under each $h \in H$, *either* Z *is invariant or* Z *and its image are disjoint. Then, either* Z *is meagre or it is clopen.*

6. *Čech-analyticty.*

Say that A in X is obtained from a family $\mathcal{H}$ of subsets of X by the Souslin operation, i.e. is *Souslin-*$\mathcal{H}$, if for each $i|n$ there are sets $H(i|n) \in \mathcal{H}$ such that

$$A = \bigcup\nolimits_{i \in I} \bigcap\nolimits_{n \in \omega} H(i|n).$$

Any $\mathcal{K}$*-analytic set is Souslin-*$\mathcal{F}$ (take $H(i|n) = \text{cl}K(I(i|n))$. In similar spirit to Frolík's characterization [Frol-70] of $\mathcal{K}$-analyticity via a Souslin-$\mathcal{F}$ representation in some compactification, Fremlin defines A, for A completely regular, to be *Čech-analytic* if A is Souslin-$(\mathcal{F} \cup \mathcal{G})$ in some compactification. (See Hansell [Han], Th. 5.3.; Fremlin's website.)

7. *From trace to global behaviour ('analytic thinning')*

For T analytic and spanning $\mathbb{R}$ over $\mathbb{Q}$ (eg containing a Hamel basis), the following hold via shift-compactness arguments (see [BOst-GenSub], [BOst-Aeq], [BOst-Thin]):

i) subadditive functions locally bounded on T are locally bounded above.

ii) convex functions locally bounded on T are continuous.

iii) (Kominek's Theorem) additive functions bounded on T are continuous on $\mathbb{R}$.

iv) (Jones' Theorem) additive functions continuous on T are continuous on $\mathbb{R}$.

Example: $T =$ Cantor set, so is 'thin' in contrast to the 'thick' sets (such as non-null measureable sets, or non-meagre Baire sets) needed by the corresponding Darboux Theorem.

8. *Applications to additive combinatorics: van der Waerden theorems.*

Recall van der Waerden's theorem of 1927, that in any finite colouring of the natural numbers, one colour contains arbitrarily long arithmetic progressions (one of Khintchine's three pearls of number theory [Kh], Ch. 1), with its enormous impact on Ramsey theory ([GRS], [HS], Ch. 18) and additive combinatorics [TV]; [HS], Ch. 14).

An earlier theorem of the same type, but for finite partitions of the *reals* into measurable cells, is immediately implied by the theorem of Ruziewicz [Ruz] in 1925, quoted here. Its category and measure forms follow from the consecutive form of the KBD Theorem (specifically of the Bitopological Shift Theorem of §9). For bibliography and details see [BOst-KCC]. (The Baire case is new).

Theorem 11.2 (Ruziewicz's Theorem [Ruz]; cf. [Kem] after Lemma 2.1 for the measure case)**.** *Given p positive real numbers $k_1, ..., k_p$ and any Baire non-meagre/measurable non-null set T, there exist d and points $x_0 < x_1 < ... < x_p$ in T such that*

$$x_i - x_{i-1} = k_i d, \qquad i = 1, ..., p.$$

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