UNIFORMITY AND SELF-NEGLECTING FUNCTIONS

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Abstract. We relax the continuity assumption in Bloom's uniform convergence theorem for Beurling slowly varying functions φ. We assume that φ has the Darboux property, and obtain results for φ measurable or having the Baire property.
Keywords: Karamata slow variation, Beurling slow variation, Wiener's Tauberian theorem, Beurling's Tauberian theorem, uniform convergence theorem, Darboux property, Kestelman-Borwein-Ditor theorem, Baire's category theorem, measurability, Baire property, affine group action.
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1. Definitions and motivation

The motivation for this paper may be traced back to two classic papers. First, in 1930, Karamata [Kar] introduced his theory of *regular variation* (in particular, of *slow variation*). Also in 1930 Karamata simplified the Hardy-Littlewood approach to *Tauberian theorems*; in 1931 he applied his theory of regular variation very successfully to Tauberian theory. For textbook accounts, see e.g. [BinGT] (BGT below), Ch. 4, Korevaar [Kor], IV. Secondly, in 1932 Wiener [Wie] transformed Tauberian theory by working with general kernels (rather than special kernels as Hardy and Littlewood had done); his method was based on Fourier transforms. There are two common forms for Wiener's Tauberian theorem, one for the additive group of reals, one for the multiplicative group of positive reals. Both concern convolutions: additive convolutions for the first, with Fourier transforms and Haar measure being Lebesgue measure dx; multiplicative convolutions for the second, with Mellin transforms and Haar measure dx/x.

Theorem W (Wiener's Tauberian theorem). *For $K \in L_1(\mathbb{R})$ with the Fourier transform $\hat{K}$ of K non-vanishing on the real line, and $H \in L_\infty(\mathbb{R})$: if*

$$\int K(x-y)H(y)dy \to c\int K(y)dy \qquad (x \to \infty),$$

then for all $G \in L_1(\mathbb{R})$*,*

$$\int G(x-y)H(y)dy \to c \int G(y)dy \qquad (x \to \infty).$$

The corresponding multiplicative form, with $\int_0^\infty K(x/y)H(y)dy/y$, is left to the reader. For textbook accounts, see e.g. Hardy [Har], XII, Widder [Wid], V, BGT Ch. 4, [Kor], II. Usually, the multiplicative form is preferred for applications, the additive form for proofs.

The classic summability methods of Cesàro and Abel fall easily into this framework. The next most important family is that of the Euler and Borel methods; these are tractable by Wiener methods, but are much less amenable to them; see e.g. [Har] VIII, IX, §12.15. Indeed, Tenenbaum [Ten], motivated by analytic number theory, gives an approach to Tauberian theory for the Borel method by Hardy-Littlewood rather than Wiener methods.

The Borel method is at the root of our motivation here. It is of great importance, in several areas: analytic continuation by power series ([Har] VIII, IX, Boas [Boa1] §5.5); analytic number theory [Ten]; probability theory ([Bin1] – [Bin4]).

In unpublished lectures, Beurling undertook the task of bringing the Borel method (and its numerous relatives) within the range of easy applicability of Wiener methods. His work was later published by Peterson [Pet] and Moh [Moh]. We state his result as *Beurling's Tauberian theorem* below, but we must first turn to the two themes of our title.

A function $f : \mathbb{R} \to \mathbb{R}_+$ is *regularly varying* in Karamata's sense if for some function g,

$$f(ux)/f(x) \to g(u) \qquad (x \to \infty) \qquad \forall\ u > 0. \qquad (RV)$$

It turns out that to get a fruitful theory, one needs *some* regularity condition on f. Karamata himself used continuity. This was weakened to (Lebesgue) measurability by Korevaar et al. [KvAEdB] in 1949. Matuszewska [Mat] in 1962 showed that one could also use functions with the Baire property (briefly: Baire functions). Note that neither of the measurable and Baire cases contains the other. There are extensive and useful parallels between the measure and Baire (or category) cases – see e.g. Oxtoby [Oxt], [BinO3], [BinO4], [Ost3]; furthermore these have wide-ranging applications, see e.g. [BinO9], [Ost1], [Ost2] – particularly to the Effros Theorem, cf. [Ost4] and §5.5.

Subject to a regularity condition (f measurable or Baire, say), one has:
(i) the *uniform convergence theorem* (UCT): (RV) holds *uniformly* on compact u-sets;
(ii) the *characterization theorem*: $g(u) = u^\rho$ for some ρ, called the *index* of regular variation.
The class of such f, those regularly varying with index ρ, is written R_ρ. One can reduce to the case $\rho = 0$, of the (Karamata) *slowly varying* functions, R_0. Then

$$f(xu)/f(x) \to 1 \qquad (x \to \infty) \qquad \forall\, u \tag{SV}$$

working multiplicatively, or

$$f(x+u) - f(x) \to 0 \qquad (x \to \infty) \qquad \forall\, u \tag{SV_+}$$

working additively; either way the convergence is uniform on compact u-sets (in the line for (SV), the half-line for (SV_+)).

Beurling observed in his lectures that the function $\sqrt{x}$ – known to be crucial to the Tauberian theory of the Borel method – has a property akin to Karamata's slow variation. We say that $\varphi > 0$ is *Beurling slowly varying*, $\varphi \in BSV$, if $\varphi(x) = o(x)$ as $x \to \infty$ and

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1 \qquad (x \to \infty) \qquad \forall\, t. \tag{BSV}$$

If (as in the UCT for Karamata slow variation) the convergence here is locally uniform in t, we say that φ is *self-neglecting*, $\varphi \in SN$; we write (SN) for the corresponding strengthening of (BSV).

We may now state Beurling's extension to Wiener's Tauberian theorem (for an extension see [FeiS]).

Theorem (Beurling's Tauberian theorem). *If $\varphi \in BSV$, $K \in L_1(\mathbb{R})$ with $\hat{K}$ non-zero on the real line, H is bounded, and*

$$\int K\Big(\frac{x-y}{\varphi(x)}\Big) H(y)dy/\varphi(x) \to c \int K(y)dy \qquad (x \to \infty),$$

then for all $G \in L_1(\mathbb{R})$,

$$\int G\Big(\frac{x-y}{\varphi(x)}\Big) H(y)dy/\varphi(x) \to c \int G(y)dy \qquad (x \to \infty).$$

Notice that the arguments of K and G here involve *both* the additive group operation on the line *and* the multiplicative group operation on the half-line. Thus Beurling's Tauberian theorem, although closely related to Wiener's (which it contains, as the case $\varphi \equiv 1$), is structurally different from it. One may also see here the relevance of the *affine group*, already well used for regular variation (see e.g. BGT §8.5.1, and §3 below).

Analogously to Karamata's UCT, the following result was proved by Bloom in 1976 [Blo]. A slightly extended and simplified version is in BGT, Th. 2.11.1.

Theorem (Bloom's theorem). If $\varphi \in BSN$ with φ continuous, then $\varphi \in SN$: (BSN) holds locally uniformly.

The question as to whether one can extend this to φ measurable and/or Baire has been open ever since; see BGT §2.11, [Kor] IV.11 for textbook accounts. Our purpose here is to give some results in this direction. This paper is part of a series (by both authors, and by the second author, alone and in [MilO] with Harry I. Miller) on our new theory of *topological regular variation*; see e.g. [BinO1-9], [Ost1-4] and the references cited there. One of the objects achieved was to find the common generalization of the measurable and Baire cases. This involves infinite combinatorics, in particular such results as the Kestelman-Borwein-Ditor theorem (KBD – see e.g. [MilO]), the category embedding theorem [BinO4] (quoted in §3 below) and shift-compactness [Ost3]. A by-product was the realization that, although the Baire case came much later than the measurable case, it is in fact the more important. One can often handle both cases together bitopologically, using the Euclidean topology for the Baire case and the density topology for the measurable case; see §3 below and also [BinO4]. Such measure-category duality only applies to *qualitative* measure theory (where all that counts is whether the measure of a set is zero or positive, not its numerical value). We thus seek to avoid *quantitative* measure-theoretic arguments; see §5.4. Recall ([Oxt, Ch. 8]) that Lusin's theorem is qualitative, while Egorov's theorem is quantitative, so does not dualize to the category case. Thus the previous proof of the UCT for Karamata slow variation nearest in spirit to our methods, by Trautner [Tra], is not available, as it uses Egorov's theorem.

Our methods of proof (as with our previous studies in this area) involve tools from infinite combinatorics, and replacement of quantitative measure theory by qualitative measure theory.

2. Extensions of Bloom's Theorem: Monotone functions

We suggest that the reader cast his eye over the proof of Bloom's theorem, in either [Blo] or BGT §2.11 – it is quite short. Like most proofs of the UCT for Karamata slow variation, it proceeds by contradiction, assuming that the desired uniformity fails, and working with two sequences, $t_n \in [-T, T]$ and $x_n \to \infty$, witnessing to its failure.

The next result, in which we assume φ monotone (φ increasing to infinity is the only case that requires proof) is quite simple. But it is worth stating explicitly, for three reasons:

1. It is a complement to Bloom's theorem, and to the best of our knowledge the first new result in the area since 1976.
2. The case φ increasing is by far the most important one for applications. For, taking G the indicator function of an interval in Beurling's Tauberian theorem, the conclusion there has the form of a *moving average*:

$$\frac{1}{a\varphi(x)} \int_x^{x+a\varphi(x)} H(y)dy \to c \qquad (x \to \infty) \qquad \forall a \ > 0.$$

Such moving averages are *Riesz (typical) means* and here φ increasing to ∞ is natural in context. For a textbook account, see [ChaM]; for applications, in analysis and probability theory, see [Bin5], [BinG1], [BinG2], [BinT]. The prototypical case is $\varphi(x) = x^\alpha$ $(0 < \alpha < 1)$; this corresponds to $X \in L_{1/\alpha}$ for the probability law of X.

3. Theorem 1 below is closely akin to results of de Haan on the Gumbel law Λ in extreme-value theory; see §5.7 below.

We offer three proofs (two here and a third after Theorem 2M) of the result, as each is short and illuminating in its own way.

For the first, recall that if a sequence of monotone functions converges pointwise to a continuous limit, the convergence is uniform on compact sets. See e.g. Pólya and Szegő [PolS], Vol. 1, p.63, 225, Problems II 126, 127, Boas [Boa2], §17, p.104-5. (The proof is a simple compactness argument. The result is a complement to the better-known result of Dini, in which it is the convergence, rather than the functions, that is monotone; see e.g. [Rud], 7.14.)

Theorem 1 (Monotone Beurling UCT). *If $\varphi \in BSV$ is monotone, $\varphi \in SN$: the convergence in (BSV) is locally uniform.*

First proof. As in [Blo] or BGT §2.11, we proceed by contradiction. Pick $T > 0$, and assume the convergence is not uniform on $[-T, 0]$ (the case $[0, T]$ is similar). Then there exists $\varepsilon_0 \in (0, 1)$, $t_n \in [-T, 0]$ and $x_n \to \infty$ such that

$$|\varphi(x_n + t_n\varphi(x_n))/\varphi(x_n) - 1| \geq \varepsilon_0 \qquad \forall\ n.$$

Write

$$f_n(t) := \varphi(x_n + t\varphi(x_n))/\varphi(x_n) - 1.$$

Then f_n is monotone, and tends pointwise to 0 by (BSV). So by the Pólya-Szegő result above, the convergence is uniform on compact sets. This contradicts $|f_n(t_n)| \geq \varepsilon_0$ for all n. □

The second proof is based on the following result, thematic for the approach followed in §4. We need some notation that will also be of use later. Below, $x > 0$ will be a continuous variable, or a sequence $x := \{x_n\}$ diverging to $+\infty$ (briefly, *divergent sequence*), according to context. We put

$$V_n^x(\varepsilon) := \{t \geq 0 : |\varphi(x_n + t\varphi(x_n))/\varphi(x_n) - 1| \leq \varepsilon\},\ H_k(\varepsilon) := \bigcap\nolimits_{n \geq k} V_n^x(\varepsilon).$$

Lemma 1. *For $\varphi > 0$ monotonic increasing and $\{x_n\}$ a divergent sequence, each set $V_n^x(\varepsilon)$, and so also each set $H_k^x(\varepsilon)$, is an interval containing* 0.

Proof of Lemma. Since $x + s\varphi(x) > x$ for $s > 0$, one has $1 \leq \varphi(x + t\varphi(x))/\varphi(x)$. Also if $0 < s < t$, then, as $\varphi(x) > 0$, one has

$$x + s\varphi(x) < x + t\varphi(x).$$

So if $t \in V_n^x(\varepsilon)$, then

$$1 \leq \varphi(x_n + s\varphi(x_n))/\varphi(x_n) \leq \varphi(x_n + t\varphi(x_n))/\varphi(x_n) \leq 1 + \varepsilon,$$

and so $s \in V_n^x(\varepsilon)$. The remaining assertions now follow, because an intersection of intervals containing 0 is an interval containing 0. □

Second proof of Theorem 1. Suppose otherwise; then there are $\varepsilon_0 > 0$ and sequences $x_n := x(n) \to \infty$ and $u_n \to u_0$ such that

$$|\varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) - 1| \geq \varepsilon_0,\ \text{for all } n. \qquad \text{(all)}$$

Since φ is Beurling slowly varying the increasing sets $H_k^x(\varepsilon_0)$ cover $\mathbb{R}_+$ and so being increasing intervals (by Lemma 1) their interiors cover the compact set $K := \{u_n : n = 0, 1, 2, ...\}$. So for some integer k the set $H_k^x(\varepsilon_0)$ already covers K, and then so does $V_k^x(\varepsilon_0)$. But this implies that

$$|\varphi(x_k + u_k\varphi(x_k))/\varphi(x_k) - 1| < \varepsilon_0,$$

contradicting (all) at $n = k$. □

Remark. Of course the uniformity property of φ is equivalent to the sets $H_k^x(\varepsilon)$ containing arbitrarily large intervals $[0, t]$ for large enough k (for all divergent $\{x_n\}$).

3. Combinatorial preliminaries

We work in the affine group $\mathcal{A}ff$ acting on $(\mathbb{R}, +)$ using the notation

$$g_n(t) = c_n t + z_n,$$

where $c_n \to c_0 = c > 0$ and $z_n \to 0$ as $n \to \infty$. These are to be viewed as (self-) homeomorphisms of $\mathbb{R}$ under either $\mathcal{E}$, the Euclidean topology, or $\mathcal{D}$, the Density topology. Recall that the open sets of $\mathcal{D}$ are measurable subsets, all points of which are (Lebesgue) density points, and that (i) Baire sets under $\mathcal{D}$ are precisely the Lebesgue measurable sets, (ii) the nowhere dense sets of $\mathcal{D}$ are precisely the null sets, and (iii) Baire's Theorem holds for $\mathcal{D}$. (See Kechris [Kec] 17.47.) Below we call a set *negligible* if, according to the topological context of $\mathcal{E}$ or $\mathcal{D}$, it is meagre/null. A property holds for 'quasi all' elements of a set if it holds for all but a negligible subset. We recall the following definition and Theorem from [BinO4], which we apply taking the space X to be $\mathbb{R}$ with one of $\mathcal{E}$ or $\mathcal{D}$.

Definition. A sequence of homeomorphisms $h_n : X \to X$ satisfies the *weak category convergence* condition (wcc) if:

For any non-meagre open set $U \subseteq X$, there is a non-meagre open set $V \subseteq U$ such that for each $k \in N$,

$$\bigcap_{n \geq k} V \backslash h_n^{-1}(V) \text{ is meagre.} \qquad \text{(wcc)}$$

Theorem CET (Category Embedding Theorem). *Let X be a topological space and $h_n : X \to X$ be homeomorphisms satisfying (wcc). Then for* *any Baire set T, for quasi-all $t \in T$ there is an infinite set $\mathbb{M}_t \subseteq \mathbb{N}$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

From here we deduce:

Lemma 2 (Affine Two-sets Lemma). *For $c_n \to c > 0$ and $z_n \to 0$, if $cB \subseteq A$ for A, B non-negligible (measurable/Baire), then for quasi all $b \in B$ there exists an infinite set $\mathbb{M} = \mathbb{M}_b \subseteq \mathbb{N}$ such that*

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A.$$

Proof. It is enough to prove the existence of one such point b, as the Generic Dichotomy Principle (for which see [BinO7, Th. 3.3]) applies here, because we may prove existence of such a b in any non-negligible $\mathcal{G}_\delta$-subset B' of B, by replacing B below with B'. (One checks that the set of bs with the desired property is Baire, and so its complement in B cannot contain a non-negligible $\mathcal{G}_\delta$.)

Writing $T := cB$ and $w_n = c_n c^{-1}$, so that $c_n = w_n c$ and $w_n \to 1$, put

$$h_n(t) := w_n t + z_n.$$

Then h_n converges to the identity in the supremum metric, so (wcc) holds by Th. 6.2 of [BinO6] (First Verification Theorem) and so Theorem CET above applies for the Euclidean case; applicability in the measure case is established as Cor. 4.1 of [BinO2]. (This is the basis on which the affine group preserves negligibility.) So there are $t \in T$ and an infinite set of integers $\mathbb{M}$ with

$$\{w_m t + z_m : m \in \mathbb{M}\} \subseteq T.$$

But $t = cb$ for some $b \in B$ and so, as $w_m c = c_m$, one has

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq cB \subseteq A,$$

as required. □

4. Extensions to Bloom's theorem: Darboux property

Bloom uses continuity only through the intermediate value property – that if a (real-valued) function attains two values, it must attain all intermediate values. This is the *Darboux property.* It is much weaker than continuity

– it does not imply measurability, nor the Baire property. For measurability, see the papers of Halperin [Halp1,2]; for the Baire property, see e.g. [Por-WBW] and also §5.3 below.

We use Lemma 2 above to prove Theorem 3B below, which implies Bloom's Theorem, as continuous functions are Baire and have the Darboux property. We note a result of Kuratowski and Sierpiński [KurS] that for a function of Baire class 1 (for which see §5.2) the Darboux property is equivalent to its graph being connected; so Theorem 2 goes beyond the class of functions considered by Bloom.

We begin with some infinite combinatorics associated with a positive function $\varphi \in BSV$.

Definitions. Say that $\{u_n\}$ with limit u is a *witness sequence at* u (for non-uniformity in φ) if there are $\varepsilon_0 > 0$ and a divergent sequence x_n such that

$$|\varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) - 1| > \varepsilon_0. \qquad \text{(eps-0)}$$

Say that $\{u_n\}$ with limit u is a *divergent witness sequence* if there is a divergent sequence x_n such that

$$\text{either } \varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) \to +\infty, \text{ or } \varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) \to 0.$$

Thus a divergent witness sequence is a special type of witness sequence, but, as we now show, they characterize absence of uniformity in the class BSV. We begin with a lemma that yields simplifications later.

Lemma 3. *For any* u, *convergence in* (BSV) *is uniform near* $t = 0$ *iff it is uniform near* $t = u$.

Proof. For $z_n \to 0$ one has

$$\begin{aligned} & \varphi(x_n + (u + z_n)\varphi(x_n))/\varphi(x_n) \\ = \; & (\varphi([x_n + u\varphi(x_n)] + z_n\varphi(x_n))/\varphi(x_n + u\varphi(x_n))) \cdot (\varphi(x_n + u\varphi(x_n)/\varphi(x_n)) \\ = \; & (\varphi([x_n + u\varphi(x_n)] + z_n\gamma_n\varphi(x_n + u\varphi(x_n))/\varphi(x_n + u\varphi(x_n))) \\ & \cdot (\varphi(x_n + u\varphi(x_n))/\varphi(x_n)), \end{aligned}$$

with $\gamma_n := \varphi(x_n)/\varphi(x_n + u\varphi(x_n))$. Likewise

$$\begin{aligned}
& \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \\
= \; & (\varphi([x_n - u\varphi(x_n)] + (u + z_n)\varphi(x_n))/\varphi(x_n - u\varphi(x_n))) \cdot (\varphi(x_n - u\varphi(x_n)/\varphi(x_n)) \\
= \; & (\varphi([x_n - u\varphi(x_n)] + z_n\gamma_n\varphi(x_n - u\varphi(x_n)))/\varphi(x_n - u\varphi(x_n))) \\
& \cdot (\varphi(x_n - u\varphi(x_n))/\varphi(x_n))
\end{aligned}$$

with $\gamma_n := \varphi(x_n)/\varphi(x_n - u\varphi(x_n)$. So, since $\varphi(x_n)/\varphi(x_n \pm u\varphi(x_n))) \to 1$ and $\varphi(x_n \pm u\varphi(x_n)/\varphi(x) \to 1$, the conclusion is clear. □

Theorem 2B (Divergence Theorem – Baire version). *If $\varphi \in BSV$ has the Baire property and u_n with limit u is a witness sequence, then u_n is a divergent witness sequence.*

Proof. As u_n is a witness sequence, for some $x_n \to \infty$ and $\varepsilon_0 > 0$ one has (eps-0). By Lemma 3 we may assume that $u = 0$. So (as in the Proof of Lemma 3) we will write z_n for u_n. If z_n is not a divergent witness sequence, then $\{\varphi(x_n{+}z_n\varphi(x_n))/\varphi(x_n)\}$ contains a bounded subsequence and so a convergent sequence. W.l.o.g. we thus also have

$$c_n := \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \longrightarrow c \in (0, \infty). \qquad \text{(lim)}$$

So we may now also assume that

$$\varepsilon_n := |\varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) - 1| \downarrow \varepsilon_0$$

(pass to a subsequence, and increase ε_0 if necessary). Thus $\varepsilon_n \geq \varepsilon_0 > 0$, and w.l.o.g. for all n we have $\varepsilon_n \leq 3\varepsilon_0/2$. Noting that for some choice of sign $\alpha = \pm 1$ one has for infinitely many n that

$$\varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) = 1 + \alpha\varepsilon_n, \qquad \text{(signum)}$$

and passing to a subsequence, we may assume w.l.o.g. that (signum) holds *for all* n. Now choose η with $0 < \eta < 1$ so small that both

$$\frac{1+\eta}{1-\eta} < 1 + \varepsilon_0, \text{ and } 1 - \varepsilon_0 < \frac{1-\eta}{1+\eta}.$$

Write $y_n := x_n + z_n\varphi(x_n)$. Then $y_n = x_n(1 + z_n\varphi(x_n)/x_n) \to \infty$.

With the notation of §2, put $V_n(\eta) = V_n^x(\eta)$, $W_n(\eta) = V_n^y(\eta)$ and $\gamma_n(t) := c_n t + z_n$ and let

$$W_n'(\eta) := \gamma_n\left[W_n(\eta)\right] = \{t : t = z_n + sc_n \text{ for some } s \in W_n(\eta)\}.$$

These are Baire sets, and

$$\mathbb{R} = \bigcup\nolimits_k H_k^x(\eta) = \bigcup\nolimits_k H_k^y(\eta), \tag{cov}$$

as $\varphi \in BSV$. The increasing sequence of sets $\{H_k^x(\eta)\}$ covers $\mathbb{R}$. So for some k the set $H_k^x(\eta)$ is non-negligible. Furthermore, as $c > 0$, the set $c^{-1}H_k^x(\eta)$ is non-negligible and so, by (cov), for some l the set

$$B := (c^{-1}H_k^x(\eta)) \cap H_l^y(\eta)$$

is also non-negligible. Taking $A := H_k^x(\eta)$, one has $B \subseteq H_l^y(\eta)$ and

$$cB \subseteq A$$

with A, B non-negligible. Applying Lemma 2 to the maps $\gamma_n(s) = c_n s + z_n$, there exist $b \in B$ and an infinite set $\mathbb{M}$ such that

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A = H_k^x(\eta).$$

That is, as $B \subseteq H_l^y(\eta)$, there exists $\mu \in H_l^y(\eta)$ such that,

$$\{c_m \mu + z_m : m \in \mathbb{M}\} \subseteq H_k^x(\eta).$$

There is now $\mu \in H_k^y(\eta)$ such that for infinitely many m one has

$$z_m + c_m \mu \in H_l^x(\eta).$$

In particular, for such m with $m > k, l$ one has

$$\mu \in W_m(\eta) \text{ and } z_m + c_m \mu \in W_m'(\eta) \cap V_m(\eta).$$

Fix such an m and write

$$w := z_m + \mu\varphi(y_m)/\varphi(x_m),$$

which is a point common to $V_m(\eta)$ and $W_m'(\eta)$ for some $\mu \in W_m(\eta)$. So

$$|\varphi(y_m + \mu\varphi(y_m))/\varphi(y_m) - 1| \leq \eta.$$

But $w \in V_m(\eta)$, so, since $x_m + w\varphi(x_m) = x_m + z_m\varphi(x_m) + \mu\varphi(y_m) = y_m + \mu\varphi(y_m)$, we have

$$|\varphi(y_m + \mu\varphi(y_m))/\varphi(x_m) - 1| \leq \eta.$$

The proof now depends on the sign of α.
Case (a). $\varphi(x_n+z_n\varphi(x_n))/\varphi(x_n) = 1 + \varepsilon_n$ for all n. Then

$$\begin{aligned} 1+\eta &\geq \frac{\varphi(y_m + \mu\varphi(y_m))}{\varphi(x_m)} = \frac{\varphi(y_m + \mu\varphi(y_m))}{\varphi(y_m)} \cdot \frac{\varphi(y_m)}{\varphi(x_m)} \\ &\geq (1-\eta)\cdot(1+\varepsilon_m), \end{aligned}$$

or

$$(1+\eta)/\,(1-\eta) \geq 1 + \varepsilon_m \geq 1 + \varepsilon_0,$$

since $\varepsilon_0 \leq \varepsilon_m$. But this is a contradiction as we selected η so small that

$$(1+\eta)/\,(1-\eta) < 1 + \varepsilon_0.$$

Case (b). $\varphi(x_n+z_n\varphi(x_n))/\varphi(x_n) = 1 - \varepsilon_n$ for all n. This is entirely similar with inequalities reversed. ■

Theorem 2M (Divergence Theorem – Measure version). *If $\varphi \in BSV$ is measurable and u_n with limit u is a witness sequence, then u_n is a divergent witness sequence.*

Proof. The argument above applies, with the density topology $\mathcal{D}$ in place of the Euclidean topology $\mathcal{E}$ (the real line is still a Baire space, as remarked earlier). ■

As an immediate corollary we have:

Third Proof of Theorem 1. If not, then there exists a witness sequence u_n with limit u. By Lemma 3, w.l.o.g. $u > 0$. Let $v > u > w > 0$. Since $\varphi \in BSV$,

$$\varphi(x_n + v\varphi(x_n))/\varphi(x_n) \to 1 \text{ and } \varphi(x_n + w\varphi(x_n))/\varphi(x_n) \to 1,$$

so there is N such that both $(1/2)\varphi(x_n) < \varphi(x_n + w\varphi(x_n))$ and $\varphi(x_n + v\varphi(x_n)) < 2\varphi(x_n)$ for all $n > N$. By increasing N if necessary we may assume that $w < u_n < v$ for $n > N$. But then

$$(1/2)\varphi(x_n) < \varphi(x_n+w\varphi(x_n)) < \varphi(x_n+u_n\varphi(x_n)) < \varphi(x_n+v\varphi(x_n)) < 2\varphi(x_n)$$

implies that $1/2 < \varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) < 2$, contradicting Theorem 2B/2M, as φ is Baire/measurable. ■

We now deduce

Theorem 3B (Beurling-Darboux UCT: Baire version). *If $\varphi \in BSV$ has the Baire and Darboux properties, then $\varphi \in SN$: (BSV) holds locally uniformly.*

Proof. Suppose the conclusion of the theorem is false. Then for there exists a witness sequence v_n with limit v and in particular for some $x_n \to \infty$ and $\varepsilon_0 > 0$ one has

$$|\varphi(x_n + v_n\varphi(x_n))/\varphi(x_n) - 1| \geq \varepsilon_0.$$

We construct below a convergent sequence u_n, with limit u say, such that

$$c_n := \varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) \longrightarrow c \in (0, \infty), \qquad \text{(lim)}$$

and also (eps-0) holds. This will contradict Theorem 2B.

The proof here splits according as the ratios $\varphi(x_n + v_n\varphi(x_n))/\varphi(x_n)$ are bounded away from 0 and ∞ with n.

Case (i) The ratios $\varphi(x_n + v_n\varphi(x_n))/\varphi(x_n)$ converge to 0 or diverge to ∞.

Here we appeal to the Darboux property to replace the sequence $\{v_n\}$ with another sequence $\{u_n\}$ for which the corresponding ratios are convergent.

Now $f_n(t) = \varphi(x_n + t\varphi(x_n))/\varphi(x_n) - 1$ has the Darboux property and $f_n(0) = 0$. Either $f_n(v_n) \geq \varepsilon_0$ and so there exists u_n between 0 and v_n with $f_n(u_n) = \varepsilon_0$, or $-f_n(v_n) \geq \varepsilon_0$, and so there exists u_n with $-f_n(u_n) = \varepsilon_0$. Either way $|f_n(u_n)| = \varepsilon_0$. W.l.o.g. $\{u_n\}$ is convergent with limit u say, since $\{v_n\}$ is so, and now (lim) and (eps-0) hold, the latter as in fact

$$|\varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) - 1| = \varepsilon_0.$$

Case (ii) The ratios $\varphi(x_n + v_n\varphi(x_n))/\varphi(x_n)$ are bounded away from 0 and ∞ with n. In this case we can get (lim) by passing to a subsequence.

In in either case we contradict Theorem 2B. ■

Theorem 3M (Beurling-Darboux UCT - Measure version). *If $\varphi \in BSV$ is measurable and has the Darboux property, then $\varphi \in SN$: (BSV) holds locally uniformly.*

Proof. The argument above applies, appealing this time to Theorem 2M. ■

Remarks. 1. The Darboux property in Theorems 3 above may be replaced with a weaker local property. It is enough to require that φ be *locally range-dense* – i.e. that at each point t there is an open neigbourhood I_t such that the range $\varphi[I_t]$ is dense in the interval $(\inf \varphi[I_t], \sup \varphi[I_t])$ – so be in the class $\mathfrak{A}_0$ of [BruC, §2], cf. also [BruCW].

2. The proofs of Theorems 3B and 3M begin as Bloom's does, but only in the case (i) of the first step, and even then we appeal to the Darboux property rather than the much stronger assumption of continuity. Thereafter, we are able to use Theorem CET to base the rest of the proof on Baire's category theorem. This enables us to handle Theorems 3B and 3M together, by qualitative measure theory; see the end of §1 and §5.4 below. By contrast, the proofs of Bloom's theorem in [Blo] and BGT §2.11 use quantitative measure theory; see §5.4.

5. Complements

0. *Beurling's Tauberian theorem: approximation form.* Recall (see e.g. [Kor] II.8) that Wiener's Tauberian theorem is a consequence of Wiener's approximation theorem: that for $f \in L_1(\mathbb{R})$ the following are equivalent:
(i) linear combinations of translates of f are dense in $L_1(\mathbb{R})$,
(ii) the Fourier transform $\hat{f}$ of f has no real zeros.
The result is the key to Beurling's Tauberian theorem ([Kor] IV Th. 11.1). Rate of convergence results (Tauberian remainder theorems) are also possible; see e.g. [FeiS], [Kor] VII.13).
The theory extends to Banach algebras (indeed, played a major role in their development). In this connection ([Kor],V.4) we mention weighted versions of L_1 : for *Beurling weights* ω – positive measurable functions on $\mathbb{R}$ with

subadditive logarithms

$$\omega(t+u) \le \omega(t)\omega(u) \qquad (\forall t, u \in \mathbb{R}),$$

define $L_\omega = L_{1,\omega}$ the set of f with

$$||f|| = ||f||_{1,\omega} := \int_{\mathbb{R}} |f(t)|\omega(t)dt < \infty.$$

Theorem (Beurling's approximation theorem) Wiener's approximation theorem extends to the weighted case when ω satisfies the quasi-analyticity condition

$$\int_{\mathbb{R}} \frac{|\log \omega(t)|}{1+t^2} dt < \infty.$$

This condition has been extensively studied (see e.g. [Koo]) and is important in probability theory (work by Szegő – see e.g. [Bin6]).

1. *Representation.* As with Karamata slow variation, Beurling slow variation has a representation theorem: $\varphi \in SN$ iff $\varphi > 0$ and

$$\varphi(x) = c(x) \int_0^x e(u)du,$$

with $e(.) \to 0, c(.) \to c \in (0, \infty)$; as in BGT §2.11, [BinO5, Part II] we may take $e(.) \in \mathcal{C}^\infty$ (so the integral is smooth), and then $c(.)$ has the same degree of regularity (Baire/measurable, descriptive character, etc.) as $\varphi(.)$. The treatment of BGT §2.11 goes over to the setting here without change. So too does the drawback that the representation on the right above does not necessarily imply that φ is positive – this has to be assumed, or to be given from context.

2. *Functions of Baire class 1.* Recall that Baire class 1 functions – briefly, Baire-1 functions – are limits of sequences of continuous functions, and then the Baire hierarchy is defined by successive passages to the limit. See e.g. Bruckner and Leonard [BruL] §2, and the extensive bibliography given there, [Sol]. Compare [Kech] §24.B. The union of the classes in the Baire hierarchy gives the Borel functions; see e.g. [Nat] Ch. XV. Since Borel functions are (Lebesgue) measurable and Baire (have the Baire property)[1], the Baire 1

[1] In general one needs to distinguish between Borel and Baire measurability (cf. [Halm, §51] and [BinO5, §11]), but the two coincide for real analysis, our context here – see [Kech, 24.3].

functions are both measurable and Baire (see e.g. [Kur] §11).
Lee, Tang and Zhao [LeeTZ] define a concept of *weak separation.* They also show thatfor real-valued functions on a Polish space this is equivalent to being of Baire class 1. Their result is greatly generalized by Bouziad [Bou].
3. *Darboux functions of Baire class 1.* We recall that a Darboux function need be neither measurable nor (with the property of) Baire – hence the need to impose Darboux-Lebesgue or Darboux-Baire as double conditions in our results.

While Darboux functions in general may be badly behaved, Darboux functions of Baire class 1 are more tractable; recall the Kuratowski-Sierpiński theorem of §4. See e.g. [BruL, §5], [BruC, §6], [CedP], [EvH] for Darboux functions of Baire class 1, and Marcus [Mar], [GibN1], [GinN2] for literature and illuminating examples in the study of the Darboux property.
4. *Qualitative versus quantitative measure theory.* Bloom's proof of his theorem used quantitative measure theory. Our proof replaces this by qualitative measure theory, thus allowing use of measure-category duality.

The application of Theorem CET above requires the verification of (wcc), and in the measure case this calls for just enough of the quantitative aspects to suffice – see [BinO6, §6]. One brings the Baire and measure cases together here via the coincidence between measure and metric for real intervals.
5. *Beyond the reals.* Theorem CET above was conceived to capture topologically the embedding properties enjoyed by non-negligible sets under translation as typified by the Steinhaus Theorem, or Sum-Set Theorem (that $A - A$ has 0 in its interior). Thus CET refers to the underlying group of homeomorphisms of a space. A more general setting involves the apparatus of group action on a topological space (see [MilO]). Here the central result is the Effros Theorem, which may be deduced from CET-like theorems (see [Ost3]).

The context in our results here is real analysis, as in BGT and [Blo]. But the natural setting is much more general. One such setting is the normed groups of [BinO6] (where one has the dichotomy: normed groups are either topological, or pathological); see [BinO5, Part I] for a development of slow variation in that context. Other possible settings include semitopological groups, paratopological groups, etc.; see e.g. [ElfN].
6. *Monotone rearrangements.* The theory of monotone rearrangements is considered in the last chapter of Hardy, Littlewood and Pólya [HLP] Ch. X. For a function f its distribution function $|\{u : f(u) \leq x\}|$ is non-decreasing and so has a non-decreasing inverse function, the non-decreasing (briefly, *increasing*) *rearrangement* $f_\uparrow$ (thus f and $f_\uparrow$ have the same distribution func-

tion). Such rearrangements are of great interest, and use, in a variety of contexts, including
(i) *probability* (Barlow [Bar], Marcus and Rosen [MarR] §6.4);
(ii) *statistics*: estimation under monotonicity constraints [JanW];
(iii) *optimal transport*: in transport problems with f a strategy $f_\uparrow$ gives the optimal strategy [Vil];
(iv) *analysis*: [HorW], [BerLR].
As we have seen, the uniformity result for the *monotone* case is quite simple – much simpler than for the general case – and also, φ monotone will typically be clear from context. In the general case we may aim to replace φ by $\varphi_\uparrow$; for specific φ, replacing e by e_+ may well suffice.
7. *The class* Γ (BGT §3.10) consist of functions $f : \mathbb{R} \to \mathbb{R}$, non-decreasing right-continuous which, for some measurable $g : \mathbb{R} \to (0, \infty)$, the *auxiliary function* of f,

$$f(x + ug(x))/f(x) \to e^u \qquad x \to \infty \qquad (\forall u \in \mathbb{R}).$$

It turns out that the convergence here is uniform on compact u-sets (from f being monotone – as in Th. 1), and hence that g is self-neglecting.

The class Γ originates in extreme-value theory (EVT) in probability theory, in connection with de Haan's work on the domain-of-attraction problem for the Gumbel (double-exponential) extremal law Λ ($\Lambda(x) := \exp\{-e^{-x}\}$). See BGT §8.13, [BalE].
8. *Beurling regular variation.* In a sequel [BinO10] we explore the consequences of the *Beurling regular variation* property

$$f(x + t\varphi(x))/f(x) \to g(t) \qquad x \to \infty \qquad (\forall t \in \mathbb{R}). \qquad (BRV)$$

We obtain, in particular, a characterization theorem

$$g(t) = e^{\rho t}$$

for some ρ. We also relax the condition above that f be monotone.
9. *Continuous and sequential aspects.* The reader will have noticed that Beurling slow variation is (like Karamata slow variation) a continuous-variable property, while the proofs here are by contradiction, and use sequences (bearing witness to the contradiction). This is a recurrent theme; see e.g. BGT §1.9, [BinO6].
10. *Questions.* We close with two questions.

1. Does Bloom's theorem extend to measurable/Baire functions – that is, can one omit the Darboux requirement? Does it even extend to Baire-1 functions?

2. Are the classes BSV, SN closed under monotone rearrangement?

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