Uniformity and self-neglecting functions:
II. Beurling Regular Variation and the class Γ
by
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To Paul Embrechts on his 60$^{\text{th}}$ *birthday.*

Abstract. Beurling slow variation is generalized to Beurling regular variation. A Uniform Convergence Theorem, not previously known, is proved for those functions of this class that are measurable or have the Baire property. This permits their characterization and representation. This extends the gamma class of de Haan theory studied earlier.
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1. Introduction

This paper is a sequel, both to our recent joint paper [BinO11] (Part I below), and to the second author's earlier paper [Ost1], in which regular variation was studied from the viewpoint of topological dynamics in general and cocycles in particular. It is inspired also by [BinO1]. Our reference for regular variation is [BinGT] (BGT below).

We begin by setting the context of what we call *Beurling regular variation* as an extension of the established notion of Beurling slow variation, and then in Section 2 recall from Part I combinatorial preliminaries, used there to expand Bloom's analysis of self-neglecting functions; they are key in enabling us to establish (in Section 4) a Beurling analogue of the *Uniform Convergence Theorem* (UCT) of Karamata theory. In Section 3 we return to the flow issues raised in this introduction below; there flow rates, time measures and cocycles are introduced. Here we discuss the connection between the orbits of the relevant flows and the Darboux property that plays such a prominent role in Part I. Incidentally, this explains why the Darboux property is quite natural in the context of Part I. These ideas prepare the ground for a Beurling version of the UCT. We deduce a *Characterization Theorem* in Section 5.

Armed with these two theorems, we are able in Section 6 to establish various *Representation Theorems* for Beurling regularly-varying functions, but only after a review of Bloom's work on the representation of self-neglecting functions, from which we glean *Smooth Variation Theorems.* We close in Section 7 by commenting on the place of Karamata theory, and of de Haan's theory of the gamma class, relative to the new Beurling Theory.

As in Part I the reader should have clearly in mind two isometric topological groups: the real line under addition with (the Euclidean topology and) Haar measure Lebesgue measure dx, and the positive half-line under multiplication, with Haar measure dx/x, and metric $d_W(x,y) = |\log y - \log x|$. ("W for Weil", as this generates the underlying Weil topology of the Haar measure – for which see [Halm, §62], [Wei]; cf. [BinO7, Th. 6.10], and Part I, § 5.11.) As usual (see again e.g. Part I, § 5.11) we will move back and forth between these two as may be convenient, by using their natural isomorphism exp/log. Again as usual, we work additively in proofs, and multiplicatively in applications; we use the convention (as in Part I)

$$h := \log f, \qquad k := \log g.$$

The new feature Beurling regular variation presents, beyond Karamata regular variation, is the need to use both addition and multiplication simultaneously. It is this that makes the affine group $\mathcal{A}ff$ a natural ingredient here. Recall that on the line the *affine group* $x \to ux + v$ with $u > 0$ and v real has (right) Haar measure $u^{-1}du\, dv$ (or $(du/u)dv$, as above) – see [HewR, IV, (15.29)]. This explains the presence of the two measure components in the representation of a Beurling regularly varying function with index ρ:

$$f(x) = d(x) \exp\left(\rho \int_1^x \frac{u}{\varphi(u)} \frac{du}{u}\right) \left(\int_0^x \frac{e(v)}{\varphi(v)} dv\right), \qquad (\Gamma_\rho)$$

with d converging to a constant, e smooth and vanishing at infinity, and integrals initialized at the appropriate group identity (0 for dv and 1 for du/u). Here $u/\varphi(u)$ should be viewed as a *density function* (for the Haar measure du/u – unbounded, as $\varphi(x) = o(x)$ in this 'Beurling case', but with $\varphi(x) = x$ giving the 'Karamata case' in the limit) – see Section 3.

Generalizations of Karamata's theory of regular variation (BGT; cf. [Kor]), rely on a group G acting on a space X in circumstances where one can interpret 'limits to infinity' $x \to \infty$ in the following expression:

$$g(t) := \lim_{x\to\infty} f(tx)/f(x), \text{ for } t \in G \text{ and } x \in X.$$

Here an early treatment is [BajK] followed by [Bal], but a full topological development dates from the more recent papers [BinO5], [BinO6] and [Ost1] – see also [BinO10] for an overview. Recall that a group action $A : G \times X \to X$ requires two properties:
(i) *identity*: $A(1_G, x) = x$ for all x, i.e. $1_G = \mathrm{id}$, and
(ii) *associativity*: $A(gh, x) = A(g, A(h, x))$,
with the maps $x \to g(x) := A(g, x)$, also written gx, often being homeomorphisms. An action A defines an A-*flow* (also referred to as a G-flow), whose orbits are the sets $Ax = \{gx : g \in G\}$.

In fact (i) follows from (ii) for *surjective* A (as $A(g, y) = A(1_G, A(g, y))$), so we will say that A is a *pre-action* if just (i) holds, and then continue to use the notation $g(x) := A(g, x)$; it is helpful here to think of the corresponding sets Ax as orbits of an A-*preflow,* using the language of flows and topological dynamics [Bec].

Beurling's theory of slow variation, introduced in order to generalize the Wiener Tauberian Theorem ([Kor], Part I), is concerned with consequences of the equation

$$f(x + t\varphi(x))/f(x) \to 1 \text{ as } x \to \infty \ \forall t, \qquad (BSV)$$

for $f : \mathbb{R} \to \mathbb{R}_+$, equivalently,

$$h(x + t\varphi(x)) - h(x) \to 0 \text{ as } x \to \infty \ \forall t, \qquad (BSV_+)$$

for $h : \mathbb{R} \to \mathbb{R}$, where $\varphi > 0$ itself satisfies the stronger property:

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1 \text{ as } x \to \infty \text{ locally uniformly in } t. \qquad (SN)$$

Such a function φ is said to be *self-neglecting*, briefly $\varphi \in SN$. A function φ is said to be *Beurling slowly varying* if (BSV) holds for $f = \varphi$ together with the side condition $\varphi(x) = o(x)$. A measurable self-neglecting function is necessarily Beurling slowly varying in this sense (as $\varphi(x) = o(x)$; see Theorem 4). There are issues surrounding the converse direction, for which see Part I (as side-conditions are needed for uniformity).

Below we relax the definition of regular variation so that it relies not so much on group-action but on asymptotic "cocycle action" associated with a group G. This will allow us to develop a theory of Beurling regular variation analogous to the Karamata theory, in which the regularly varying functions are those functions f with the Baire property (briefly, *Baire*) or measurable that for some fixed self-neglecting φ possess a *non-zero* limit function g (not identically zero modulo null/meagre sets) satisfying

$$f(x+t\varphi(x))/f(x) \to g(t), \text{ as } x \to \infty, \ \forall t, \qquad (BRV)$$

(so that $g(0)=1$). Equivalently,

$$h(x+t\varphi(x)) - h(x) \to k(t), \text{ as } x \to \infty, \ \forall t, \qquad (BRV_+)$$

(so that $k(0)=0$). This latter equivalence is non-trivial: it follows from Theorem 3 below that if g is a non-zero function, then it is in fact positive. Specializing (BRV) to the sequential format

$$g(t) = \lim_{n\in\mathbb{N}} f(n+t\varphi(n))/f(n),$$

one sees that the limit function g is Baire/measurable if f is so. We refer to functions f satisfying (BRV) as (Beurling) *φ-regularly varying*.

This takes us beyond the classical development of such a theory restricted to the class Γ of monotonic functions f satisfying the equation (BGT §3.10; de Haan [deH]), and comes on the heels of a recent breakthrough concerning local uniformity of Beurling slow variation in Part I. We prove in Theorem 2 below a Uniform Convergence Theorem for Baire/measurable functions f with non-zero limit g, not previously known, and in Theorem 3 the Characterization Theorem that a Baire/measurable function f is Beurling φ-regularly varying with non-zero limit iff for some ρ one has

$$f(x+t\varphi(x))/f(x) \to e^{\rho t} \ \forall t,$$

where ρ is the *Beurling φ-index* of regular variation. Baire and measurable (positive) functions of this type form the class $\Gamma_\rho(\varphi)$ (cf. Mas [Mas, §3.2], Omey [Om], for f measurable; see also [Dom] for the analogous power-wise approach to Karamata regular variation).

2. Combinatorial preliminaries

As usual with proofs involving regular variation the nub lies in infinite combinatorics, to which we now turn. We recall that one can handle Baire and measurable cases together by working bi-topologically, using the Euclidean topology in the Baire case (the primary case) and the density topology in the measure case; see [BinO4], [BinO8], [BinO7]. The *negligible* sets are the meagre sets in the Baire case and the null sets in the measure case; we say that a property holds *quasi everywhere* if it hold off a negligible set.

We work in the affine group $\mathcal{A}ff$ acting on $(\mathbb{R},+)$ using the notation

$$\gamma_n(t) = c_n t + z_n,$$

where $c_n \to c_0 = c > 0$ and $z_n \to 0$ as $n \to \infty$, as in Theorem 0 below. These are to be viewed as (self-) homeomorphisms of $\mathbb{R}$ under either the Euclidean topology, or the Density topology. We recall the following definition from [BinO4] and a result from [BinO10].

Definition. A sequence of homeomorphisms $h_n : X \to X$ satisfies the *weak category convergence* condition (wcc) if:

For any non-meagre open set $U \subseteq X$, there is a non-meagre open set $V \subseteq U$ such that for each $k \in N$,

$$\bigcap\nolimits_{n\geq k} V \backslash h_n^{-1}(V) \text{ is meagre.}$$

Theorem 0 (Affine Two-sets Theorem). *For $c_n \to c > 0$ and $z_n \to 0$, if $cB \subseteq A$ for A, B non-negligible (measurable/Baire), then for quasi all $b \in B$ there exists an infinite set $\mathbb{M} = \mathbb{M}_b \subseteq \mathbb{N}$ such that*

$$\{\gamma_m(b) = c_m b + z_m : m \in \mathbb{M}\} \subseteq A.$$

Below we use only the case $c = 1$, for which γ_m converge to the identity (in supremum norm) as a sequence with (wcc), a matter verified in the Baire case in [BinO6, Th. 6.2] and in the measure case in [BinO3, Cor. 4.1].

3. Flows, orbits, cocycles and the Darboux property.

Our approach is to view Beurling regular variation as a generalization of Karamata regular variation obtained by replacing the associativity of group action by a form of asymptotic associativity. To motivate our definition below, take X and G both to be $(\mathbb{R},+)$, $\varphi \in SN$ and consider the map

$$T^{\varphi} : (t, x) \mapsto x + t\varphi(x).$$

One wants to think of t as representing translation. For fixed t put

$$t(x), \text{ or just } tx := T_t^{\varphi}(x) = T^{\varphi}(t, x) = x + t\varphi(x),$$

so that $0(x) = x$, and so T^φ is a pre-action. Here we have $T^\varphi_{s+t}(x) = x + (s + t)\varphi(x)$, so that

$$T^\varphi_s(T^\varphi_t(x)) = x + t\varphi(x) + s\varphi(x + t\varphi(x)) \neq T^\varphi_{s+t}(x).$$

So $T^\varphi : G \times X \to X$ is not a group action, as associativity fails. However, just as in a proper flow context, here too one has a well-defined *flow rate*, or infinitesimal generator, at x, for which see [Bec], [Rud, Ch. 13], [BinO1], cf. [Bal],

$$\dot{T}^\varphi_0 x = \frac{d}{dt} T^\varphi_t x \Big|_{t=0} = \lim\nolimits_{t\to 0} \frac{T^\varphi_t x - x}{t} = \varphi(x).$$

There is of course an underlying true flow here – in the measure case, generated[1] by $\varphi > 0$ (with $1/\varphi$ locally integrable) and described by the system of differential equations (writing $u_x(t)$ for $u(t, x)$)

$$\dot{u}_x(t) = du_x(t)/dt = \varphi(u_x(t)) \text{ with } u_x(0) = x, \text{ so that } \quad (t, x) \mapsto u_x(t). \quad (1)$$

(The inverse problem, for $t(u)$ with $t(0) = 1$, has an explicit increasing integral representation, yielding $u_x(t) := u(t + t(x))$, where $u(t(x)) = x$, as u and t are inverse.) The 'differential flow' $\Phi : (t, x) \mapsto u_x(t)$ is continuous in t for each x. As such, Φ is termed by Beck a *quasi-flow.*[2] By contrast 'translation flow', i.e. $(t, x) \mapsto x + t$, being jointly continuous, is a 'continuous flow', briefly a *flow*. It is interesting to note that, by a general result of Beck (see [Bec] Ch. 4 – Reparametrization, Th. 4.4.), if the *orbits* of Φ (i.e. the sets $\mathcal{O}(x) := \{\Phi(t, x) : t \in \mathbb{R}\}$) are continua then, even though φ need not be continuous, there still exists a unique *continuous* 'local time-change' system of maps $t \mapsto f_x(t)$ embedding the quasi-flow in the translation flow, i.e. $\Phi(t, x) = x + f_x(t)$; here f_x has the *cocycle property* (cf. Theorem 1 below),

$$f_x(s + t) = f_x(s) + f_y(t) \text{ for } y = x + f_x(s),$$

and $f_x(0) = 0$ for all x. This will be the case when φ has the intermediate value property, so here the Darboux property says simply that *orbits embed.* Cocycles are thus central to the flow-analysis of regular variation, central to our earlier index theory of regular variation [BinO1].

[1] Positivity is key here; $x = 0$ is a fixed-point of the flow $\dot{u} = \varphi(u)$ when $\varphi(x) = \sqrt{|x|}$.

[2] Beck denotes flows by $\varphi(t, x)$ and uses f where we use φ. As we follow the traditional notation of φ for self-neglecting functions, the flow here is denoted Φ.

It is this differential flow that the, algebraically much simpler, Beurling preflow circumvents, working not with the continuous translation function $f_x(t)$ but $t\varphi(x)$, now only measurable, but with the variables separated. Nevertheless, the differential equation above is the source of an immediate interpretation of the integral

$$\tau_x := \int_1^x \frac{du}{\varphi(u)},$$

arising in the representation formula (Γ_ρ) for a regularly varying function f, as the metric of *time measure* (in the sense of Beck – [Bec, p. 153]). The metric is the *occupation-time measure* (cf. BGT §8.11) of the interval $[1, x]$ under the φ-generated flow started at the natural origin of the mutiplicative group $\mathbb{R}_+$. For $\varphi(x) = x$, the φ-time measure is Haar measure, and the associated metric is the Weil (multiplicatively invariant) metric with $d_W(1, x) = |\log x|$, as in Section 1. In general, however, the φ-time measure μ_φ is obtained from Haar measure via the density $x/\varphi(x)$, interpretable as a *time-change* 'multiplier' $w(x) := \varphi(x)/x$ (cf. [Bec, 5.41]).

Granted its interpretation, it is only to be expected in (Γ_ρ) that τ_x multiplies the index ρ describing the asymptotic behaviour of the function f. The time integral τ_x is in fact asymptotically equal to the time taken to reach x from the origin under the Beurling pre-action T^φ, when $\varphi \in SN$, namely $x/\varphi(x)$. We hope to return to this matter elsewhere.

Actually, T^φ is even closer to being an action: it is an *asymptotic action* (i.e. asymptotically an action), in view of two properties critical to the development of regular variation. The first refers to the dual view of the map $(t, x) \mapsto x(t) = x + t\varphi(x)$ with x fixed (rather than t, as at the beginning of the section). Here we see the affine transformation $\alpha_x(t) = \varphi(x)t + x$. This *auxiliary group* plays its part through allowing the absorption of a small "time" variation $t + s$ of t into a small "space" variation in x involving a concatenation formula, earlier identified in [BinO1] as a component in the abstract theory of the index of regular variation.

Lemma 1 (Near-associativity, almost absorption). *For* $\varphi \in SN$,

$$T^\varphi(t+s, x) = T^\varphi(\gamma t, T^\varphi(s, x)), \text{ where } \gamma = \gamma_x^\varphi(y_s) = \varphi(x)/\varphi(y) \text{ and } y_s := T^\varphi(s, x).$$

and

$$\gamma_x^\varphi(y_s) \to 1 \text{ as } s \to 0.$$

Here γ satisfies the concatenation formula

$$\gamma_x^\varphi(z) = \gamma_x^\varphi(y)\gamma_y^\varphi(z), \qquad \forall x, y, z.$$

Alternatively,

$$T_{t+s}^\varphi x = T_{\gamma t}^\varphi y_s, \text{ where } y_s = \alpha_x(s) = s\varphi(x) + x,$$

equivalently

$$T_{t+s}^\varphi x = T_{\beta(t+s)}^\varphi \alpha_x(s), \text{ for } \beta(t) = \gamma_x(y_s)(t-s).$$

Proof.

$$\begin{aligned} T_{t+s}^\varphi x &= x + (t+s)\varphi(x) \\ &= (x + s\varphi(x)) + \frac{\varphi(x)}{\varphi(x+s\varphi(x))} t\varphi(x + s\varphi(x)). \end{aligned}$$

As for the concatenation formula, one has

$$\gamma_x^\varphi(z) = \frac{\varphi(x)}{\varphi(z)} = \frac{\varphi(x)}{\varphi(y)} \cdot \frac{\varphi(y)}{\varphi(z)} = \gamma_x^\varphi(y)\gamma_y^\varphi(z). \qquad \square$$

As for the second property of T^φ, recall that for G a group acting on a second group X, a G-cocycle on X is a function $\sigma : G \times X \to X$ defined by the condition

$$\sigma(gh, x) = \sigma(g, hx)\sigma(h, x).$$

This definition is already meaningful if a pre-action rather than an action is defined from $G \times X$ to X; so for the purposes of asymptotic analysis one may capture a weak form of associativity as follows using Lemma 1. For a Banach algebra X, we let X_* denote its invertible elements.

Definition. For X a Banach algebra, given a pre-action $T : G \times X \to X$ (i.e. with $1_G x = x$ for all x, where, as above, $gx := T(g, x)$), an *asymptotic G-cocycle* on X is a map $\sigma : G \times X \to X_*$ with the property that for all $g, h \in G$ and $\varepsilon > 0$ there is $r = r(\varepsilon, g, h)$ such that for all x with $||x|| > r$

$$||\sigma(gh, x) - \sigma(g, hx)\sigma(h, x)||_X < \varepsilon.$$

Say that the cocycle is *locally uniform*, if the inequality holds uniformly on compact (g,h)-sets.

Remark. Baire and measurable cocycles are studied in [Ost1] for their uniform boundedness properties. One sees that the Second and Third Boundedness Theorems proved there hold in the current setting with asymptotic cocycles replacing cocycles. We now verify that taking $X_* = \mathbb{R}_+$, $T = T^\varphi$, and the natural cocycle of regular variation $\sigma^f(t,x) := f(tx)/f(x)$ the above property holds. (More general contexts are considered in [BinO12] – Part III.) The case $f = \varphi$ comes first; the general case of φ-regularly varying f must wait till after Theorem 3.

Theorem 1. *For (positive) $\varphi \in SN$ the map $\sigma^\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$*

$$\sigma^\varphi(t,x) := \varphi(tx)/\varphi(x), \text{ where } tx := T_t^\varphi(x),$$

regarded as a map into the Banach algebra $\mathbb{R}$ is a locally uniform asymptotic $(\mathbb{R},+)$-cocycle, i.e. for every $\varepsilon > 0$ and compact set K there is r such that for all $s,t \in K$ and all x with $||x|| > r$

$$|\sigma^\varphi(s+t,x) - \sigma^\varphi(s,tx)\sigma^\varphi(t,x)| < \varepsilon,$$

i.e.

$$\left|\frac{\varphi(T_{s+t}^\varphi(x))}{\varphi(x)} - \frac{\varphi(T_s^\varphi(T_t^\varphi(x)))}{\varphi(T_t^\varphi(x))} \cdot \frac{\varphi(T_t^\varphi(x))}{\varphi(x)}\right| < \varepsilon,$$

or

$$|\varphi(T_{s+t}^\varphi(x))/\varphi(x) - \varphi(T_s^\varphi(T_t^\varphi(x)))/\varphi(x)| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Given s,t let I be any open interval with $s+t \in I$.

Pick $\delta > 0$ so that the interval $J = (1-\delta, 1+\delta)$ satisfies $t + sJ \subseteq I$. Next pick r such that for $x > r$ both

$$|\sigma^\varphi(t,x) - 1| = |\varphi(x + t\varphi(x))/\varphi(x) - 1| < \delta$$

and

$$|\sigma^\varphi(u,x) - 1| = |\varphi(x + u\varphi(x))/\varphi(x) - 1| < \varepsilon/2, \text{ for all } u \in I.$$

In particular

$$|\sigma^\varphi(s+t,x) - 1| = |\varphi(x + (s+t)\varphi(x))/\varphi(x) - 1| < \varepsilon/2.$$

Noting, as in Lemma 1, that

$$T_s^{\varphi}(T_t^{\varphi}x) = (x + t\varphi(x)) + s\varphi(x + t\varphi(x)) = x + \varphi(x)\left(t + s\frac{\varphi(x + t\varphi(x))}{\varphi(x)}\right),$$

so that

$$w := t + s\sigma^{\varphi}(t, x) = t + s\frac{\varphi(x + t\varphi(x))}{\varphi(x)} \in t + sJ \subseteq I,$$

one has

$$|\sigma^{\varphi}(w, x) - 1| < \varepsilon/2,$$

i.e.

$$|\varphi(T_s^{\varphi}(T_t^{\varphi}x))/\varphi(x) - 1| < \varepsilon/2.$$

But

$$\begin{aligned}\sigma^{\varphi}(w, x) &= \frac{\varphi(T_s^{\varphi}(T_t^{\varphi}x))}{\varphi(x)} = \frac{\varphi(T_s^{\varphi}(T_t^{\varphi}(x)))}{\varphi(T_t^{\varphi}(x))}\frac{\varphi(T_t^{\varphi}(x))}{\varphi(x)} \\ &= \sigma^{\varphi}(s, T_t^{\varphi}(x))\sigma^{\varphi}(t, x),\end{aligned}$$

so for $x > r$ one has

$$\begin{aligned}&|\sigma^{\varphi}(w, x) - \sigma^{\varphi}(s + t, x)| \\ \leq\ &|[\sigma^{\varphi}(s, T_t^{\varphi}(x))\sigma^{\varphi}(t, x)] - 1| + |\sigma^{\varphi}(s + t, x)] - 1| \\ \leq\ &\varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \square\end{aligned}$$

4. Uniform Convergence Theorem

We begin with a lemma that yields simplifications later; it implies a Beurling analogue of the Bounded Equivalence Principle in the Karamata theory, first noted in [BinO2, Th. 4]. As it shifts attention to the origin, we call it the Shift Lemma. It has substantially the same statement and proof as the corresponding Shift Lemma of Part I except that here $h = \log f$ whilst there one has $h = \log \varphi$, so that here the difference $h(x_n + u\varphi(x_n)) - h(x_n)$ tends to $k(u)$ rather than to zero. So we omit the proof. Below *uniform near* a point u means 'uniformly on sequences converging to u' and is equivalent to local uniformity at u (i.e. on compact neighbourhoods of u).

Lemma 2 (Shift Lemma: uniformity preservation under shift). *For any u, convergence in (BRV_+) is uniform near $t = 0$ iff it is uniform near $t = u$.*

Definition. Say that $\{u_n\}$ with limit u is a *witness sequence at* u (for non-uniformity in h) if there are $\varepsilon_0 > 0$ and a divergent sequence x_n such that for $h = \log f$

$$|h(x_n + u_n\varphi(x_n)) - h(x_n)| > \varepsilon_0 \qquad \forall\, n \in \mathbb{N}. \tag{2}$$

Theorem 2 (UCT for φ-regular variation). *For $\varphi \in SN$, if f has the Baire property (or is measurable) and satisfies (BRV) with limit g strictly positive on a non-negligible set, then f is locally uniformly φ-RV.*

Proof. Suppose otherwise. We modify a related proof from Part I (concerned there with the special case of φ itself) in two significant details. In the first place, we will need to work relative to the set $S := \{s > 0 : g(s) > 0\}$, ("S for support"), so that $k(s) = \log g(s)$ is well-defined on S. Now S is Baire/measurable; as S is assumed non-negligible, by passing to a Baire/measurable subset of S if necessary, we may assume w.l.o.g. that the restriction $k|S$ is continuous on S, by [Kur, §28] in the Baire case (cf. [BinO6] Th. 11.8) and Luzin's Theorem in the measure case ([Oxt], Ch. 8, cf. [BinO7]).

Let u_n be a witness sequence for the non-uniformity of h so, for some $x_n \to \infty$ and $\varepsilon_0 > 0$ one has (2). By the Shift Lemma (Lemma 2) we may assume that $u = 0$. So we will write z_n for u_n. As φ is self-neglecting we have

$$c_n := \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \longrightarrow 1. \tag{3}$$

Write $\gamma_n(s) := c_n s + z_n$ and $y_n := x_n + z_n\varphi(x_n)$. Then $y_n = x_n(1 + z_n\varphi(x_n)/x_n) \to \infty$, and as $k(0) = 0$,

$$|h(y_n) - h(x_n)| \geq \varepsilon_0. \tag{4}$$

Now take $\eta = \varepsilon_0/4$ and for $x = \{x_n\}$, working in S, put

$$V_n^x(\eta) := \{s \in S : |h(x_n + s\varphi(x_n)) - h(x_n) - k(s)| \leq \eta\}, \;\; H_k^x(\eta) := \bigcap\nolimits_{n \geq k} V_n^x(\eta),$$

and likewise for $y = \{y_n\}$. These are Baire sets, and

$$S = \bigcup\nolimits_k H_k^x(\eta) = \bigcup\nolimits_k H_k^y(\eta), \tag{5}$$

as $h \in BRV_+$. The increasing sequence of sets $\{H_k^x(\eta)\}$ covers S. So for some k the set $H_k^x(\eta)$ is non-negligible. As $H_k^x(\eta)$ is non-negligible, by (5), for some l the set

$$B := H_k^x(\eta)) \cap H_l^y(\eta)$$

is also non-negligible. Taking $A := H_k^x(\eta)$, one has $B \subseteq H_l^y(\eta)$ and $B \subseteq A$ with A, B non-negligible. Applying Theorem 0 to the maps $\gamma_n(s) = c_n s + z_n$ with $c = \lim_n c_n = 1$, there exist $b \in B$ and an infinite set $\mathbb{M}$ such that

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A = H_k^x(\eta).$$

That is, as $B \subseteq H_l^y(\eta)$, there exist $t \in H_l^y(\eta)$ and an infinite $\mathbb{M}_t$ such that

$$\{\gamma_m(t) = c_m t + z_m : m \in \mathbb{M}_t\} \subseteq H_k^x(\eta).$$

In particular, for this t and $m \in \mathbb{M}_t$ with $m > k, l$ one has

$$t \in V_m^y(\eta) \text{ and } \gamma_m(t) \in V_m^x(\eta).$$

As $t \in S$ and $\gamma_m(t) \in S$ (a second, critical, detail), we have by continuity of $k|S$ at t, since $\gamma_m(t) \to t$, that for all m large enough

$$|k(t) - k(\gamma_m(t))| \leq \eta. \tag{6}$$

Fix such an m. As $\gamma_m(t) \in V_m^x(\eta)$,

$$|h(x_m + \gamma_m(t)\varphi(x_m)) - h(x_m) - k(\gamma_m(t))| \leq \eta. \tag{7}$$

But $\gamma_m(t) = c_m t + z_m = z_m + t\varphi(y_m)/\varphi(x_m)$, so

$$x_m + \gamma_m(t)\varphi(x_m) = x_m + z_m\varphi(x_m) + t\varphi(y_m) = y_m + t\varphi(y_m),$$

'absorbing' the affine shift $\gamma_m(t)$ into y. So, by (7),

$$|h(y_m + t\varphi(y_m)) - h(x_m) - k(\gamma_m(t))| \leq \eta.$$

But $t \in V_m^y(\eta)$, so

$$|h(y_m + t\varphi(y_m)) - h(y_m) - k(t))| \leq \eta.$$

By the triangle inequality, combining the last two inequalities with (6),

$$\begin{aligned} & |h(y_m) - h(x_m)| \\ \leq\ & |h(y_m + t\varphi(y_m)) - h(y_m) - k(t)| + |k(t) - k(\gamma_m(t))| + |h(y_m + t\varphi(y_m)) - h(x_m) - k(\gamma_m(t))| \\ \leq\ & 3\eta < \varepsilon_0, \end{aligned}$$

contradicting (4). ■

5. Characterization Theorem

We may now deduce the characterization theorem which implies in particular that the support set S of the last proof is in fact all of $\mathbb{R}$.

Theorem 3 (Characterization Theorem). *For $\varphi \in SN$, if $f > 0$ is φ-regularly varying and Baire/measurable and satisfies*

$$f(x + t\varphi(x))/f(x) \to g(t), \ \forall t,$$

with non-zero limit, i.e. $g > 0$ on a non-negligible set, then for some ρ (the index of φ-regular variation) one has

$$g(t) = e^{\rho t}.$$

Proof. Proceed as in Theorem 1: writing $y := x + s\varphi(x)$, and recalling from Th. 1 the notation $\gamma = \varphi(x)/\varphi(y)$, one has

$$h(x + (s + t)\varphi(x)) - h(x) = [h(y + t\gamma\varphi(y)) - h(y)] + [h(y) - h(x)]. \quad (8)$$

Fix s and $t \in \mathbb{R}$; passing to limits and using uniformity (by Theorem 2), we have

$$k(s + t) = k(t) + k(s), \quad (CFE)$$

since $\gamma = \varphi(x)/\varphi(y) \to 1$. This is the Cauchy functional equation; as is well-known, for k Baire/measurable (see Banach [Ban, Ch. I, §3, Th. 4] and Mehdi [Meh] for the Baire case, [Kucz, 9.4.2] for the measure case, and [BinO8] for an up-to-date discussion) this implies $k(x) = \rho x$ for some $\rho \in \mathbb{R}$, and so $g(x) = e^{\rho x}$. □

Remark. The conclusion that $k(x) = \rho x$ ($\forall x$) for some ρ tells us that in fact $g > 0$ everywhere, which in turn implies the cocycle property below. (If we assumed that $g > 0$ everywhere, we could argue more naturally, and more nearly as in the Karamata theory, by establishing the cocycle property first and from it deducing the Characterization Theorem.)

As an immediate corollary, we now have an extension to Theorem 1:

Corollary 1 (Cocycle property). *For $\varphi \in SN$, if $f > 0$ is φ-regularly varying and Baire/measurable and satisfies*

$$f(x + t\varphi(x))/f(x) \to g(t),\ \forall t,$$

with non-zero limit, i.e. $g > 0$ on a non-negligible set, then

$$\sigma^f(t, x) := f(x + t\varphi(x))/f(x)$$

is a locally uniform asymptotic cocycle.

Proof. With the notation of Theorem 3, rewrite (8) as

$$\frac{f(x + (s+t)\varphi(x))}{f(x)} = \frac{f(y + t\gamma\varphi(y))}{f(y)} \cdot \frac{f(x + s\varphi(x))}{f(x)}, \tag{9}$$

where by Theorem 3 both ratios on the right-hand side have non-zero limits $g(t)$ and $g(s)$, as x (and so y) tend to infinity. Given $\varepsilon > 0$ it now follows from (9) using (CFE), and uniformity in compact neighbourhoods of s, t and $s + t$ (by Theorem 2), that for all large enough x

$$\left| \frac{f(x + (s+t)\varphi(x))}{f(x)} - \frac{f(x + t\varphi(x))}{f(x)} \cdot \frac{f(x + s\varphi(x))}{f(x)} \right| < \varepsilon,$$

so that σ^f is a locally uniform asymptotic cocycle. □

6. Smooth Variation and Representation Theorems

Before we derive a representation theorem for Baire/measurable Beurling regularly varying functions, we need to link the Baire case to the measure case. Recall the *Beck iteration* of $\gamma(x) := T_1^\varphi(x) = x + \varphi(x)$ (so that $\gamma_{n+1}(x) = \gamma(\gamma_n(x))$ with $\gamma_1 = \gamma$, for which see [Bec, 1.64] in the context of bounding a flow) and Bloom's result for $\varphi \in SN$ concerning the sequence $x_{n+1} = \gamma_n(x_1)$, i.e. $x_{n+1} := x_n + \varphi(x_n)$, that for all x_1 large enough one has $x_n \to \infty$, i.e. the sequence gives a *Bloom partition* of $\mathbb{R}_+$ (see [Blo], or BGT §2.11). We next need to recall a construction due to Bloom in detail as we need a slight amendment.

Lemma 4 (Interpolation Lemma). *For $\varphi \in SN$, set $x_{n+1} := x_n + \varphi(x_n)$, with x_1 large enough so that $x_n \to \infty$. Put $x_0 = 0$.*

For $\psi > 0$ a φ-slowly varying function, there exists a continuously differentiable function $\phi > 0$ such that
i) $\phi(x_n) = \psi(x_n)$ for $n = 0, 1, 2, ...,$
ii) $\phi(x)$ *lies between* $\psi(x_n)$ *and* $\psi(x_{n-1})$ *for* x *between* x_{n-1} *and* x_n *for* $n = 1, 2, ...,$
iii) $|\phi'(x)| \leq 2|\psi(x_n) - \psi(x_{n-1})|/\varphi(x_{n-1})$, *for* x *between* x_{n-1} *and* x_n *for* $n = 1, 2, ...$

Proof. Proceed as in [Blo] or BGT §2.11; we omit the details. □

Definition. Call any function ϕ with the properties (i)-(iii) a (φ-) *interpolating function* for ψ.

We now deduce an extension of the Bloom-Shea Representation Theorem in the form of a Smooth Beurling Variation Theorem (for smooth variation, see BGT §2.1.9, following Balkema et al. [BalGdH]). Indeed, the special case $\psi = \varphi$ is included here. Our proof is a variant on Bloom's. It will be convenient to introduce:

Definition (Asymptotic equivalence). For $\varphi, \phi > 0$ write $\varphi \sim \phi$ if $\varphi(x)/\phi(x) \to 1$ as $x \to \infty$. If $\phi \in \mathcal{C}^1$, say that ϕ is a *smooth representation* of φ.

Theorem 4 (Smooth Beurling Variation). *For $\varphi \in SN$ and ψ a φ-slowly varying function, if ϕ is any continuously differentiable function interpolating ψ w.r.t. φ as in Lemma 4, then*

$$\psi(x) = c(x)\phi(x), \text{ for some positive } c(.) \to 1, \text{ i.e. } \psi \sim \phi \in \mathcal{C}^1,$$

so that ϕ is φ-slowly varying, and also

$$\left|\varphi(x)\frac{\phi'(x)}{\phi(x)}\right| \to 0.$$

This yields the representation

$$\psi(x) = c(x)\phi(x) = c(x)\exp\left(\int_0^x \frac{e(u)}{\varphi(u)}du\right), \text{ for } e \in \mathcal{C}^1 \text{ with } e \to 0.$$

Moreover, if ψ is Baire/measurable, then so is $c(x)$.

Furthermore, if $\psi \in SN$, in particular when $\psi = \varphi$, then:

(i) *$\phi(x)$ is self-neglecting and $\psi \sim \phi \sim \int_0^x e(u)du$ for some continuous e with $e \to 0$;*

(ii) *both $\phi(x)/x \to 0$ and $\psi(x)/x \to 0$, as $x \to \infty$;*

(iii) *if f is ψ-regularly varying with index ρ, then f is ϕ-regularly varying with index ρ with $\psi \sim \phi \in \mathcal{C}^1 \cap SN$.*

Proof. Note that for any y_n between x_n and x_{n+1} one has $\varphi(y_n)/\varphi(x_n) \to 1$, and so also $\psi(y_n)/\psi(x_n) = [\psi(y_n)/\varphi(x_n)][\varphi(x_n)/\varphi(y_n)][\varphi(y_n)/\psi(x_n)] \to 1$; indeed $y_n = x_n + t_n\varphi(x_n)$ for some $t_n \in [0,1]$, and so the result follows from local uniformity in ψ and φ and because ψ is φ-slowly varying. This implies first that, if (say) $\psi(x_n) \leq \psi(x_{n+1})$, then for ϕ as in the statement of the theorem,

$$\frac{\psi(x_n)}{\psi(x_{n+1})} \leq \frac{\phi(y_n)}{\psi(y_n)} \leq \frac{\psi(x_{n+1})}{\psi(y_{n+1})},$$

and so $\phi(y_n)/\psi(y_n) \to 1$; similarly for $\psi(x_{n+1}) \leq \psi(x_n)$. So $\phi(x)/\psi(x) \to 1$, as $x \to \infty$. So, by Lemma 4 (i) and as ψ is φ-slowly varying,

$$\frac{\phi(y_n)}{\varphi(x_n)} = \frac{\phi(y_n)}{\psi(y_n)} \cdot \frac{\psi(y_n)}{\varphi(x_n)} = \frac{\psi(y_n)}{\psi(y_n)} \cdot \frac{\psi(y_n)}{\varphi(x_n)} \to 1,$$

i.e. ϕ is φ-slowly varying. Furthermore, by Lemma 4 (iii) and since ϕ and ψ are φ-slowly varying,

$$\left|\frac{\varphi(x)}{\phi(x)}\phi'(x)\right| \leq 2\left(\frac{\psi(x_{n+1})}{\psi(x_n)} - 1\right)(\psi(x_n)/\varphi(x_n)) \cdot (\varphi(x)/\phi(x)) \to 0.$$

Take $c(x) := \psi(x)/\phi(x)$; then $\lim_{x\to\infty} c(x) = 1$, and re-arranging one has

$$\psi(x) = c(x)\phi(x),$$

as asserted.

From here we have, setting $e(u) := \varphi(u)\phi'(u)/\phi(u) \to 0$ and noting that $e(x)/\varphi(x) \geq 0$ is the derivative of $\log\phi(x)$,

$$\psi(x) = c(x)\phi(x) = c(x)\exp\int_0^x \frac{e(u)}{\varphi(u)}du.$$

Conversely, such a representation yields slow φ-variation: by the Mean Value Theorem, for any t there is $s = s(x) \in [0,t]$ such that

$$\int_x^{x+t\varphi(x)} \frac{e(u)}{\varphi(u)}du = \frac{e(x+s\varphi(x))}{\varphi(x+s\varphi(x))}\varphi(x+s\varphi(x)),$$

which tends to 0 uniformly in t as $x \to \infty$, since $e(.) \to 0$ and $\varphi \in SN$.

Now suppose additionally that $\psi \in SN$ (e.g. if $\psi = \varphi$). We check that then $\phi \in SN$. Indeed, suppose that $u_n \to u$; then as $\phi(x_n) = \psi(x_n)$, and since $\psi \in SN$, writing $y_n = x_n + u_n\phi(x_n)$ one has

$$\frac{\phi(y_n)}{\phi(x_n)} = \frac{\psi(x_n + u_n\phi(x_n))/c(y_n)}{\psi(x_n)/c(x_n)} = \frac{\psi(x_n + u_n\psi(x_n))/c(y_n)}{\psi(x_n)/c(x_n)} \to 1,$$

as asserted in (i). Given this ϕ apply Lemma 4 with ϕ for φ and $\psi = \varphi = \phi$ to yield a further smooth representing function $\bar{\phi} \sim \phi$. Then by Lemma 4 (iii) we have for a corresponding sequence $\bar{x}_n$

$$|\bar{\phi}'(x)| \leq 2|\phi(\bar{x}_n) - \phi(\bar{x}_{n-1})|/\phi(\bar{x}_{n-1}) = 2|\phi(\bar{x}_n)/\phi(\bar{x}_{n-1}) - 1| \to 0.$$

So taking $e(x) = \bar{\phi}'(x)$ one has $\lim_{x\to\infty} e(x) = 0$ and for $\bar{c}(x) := \phi(x)/\bar{\phi}(x)$, one has again $\lim_{x\to\infty} \bar{c}(x) = 1$. So integrating $\bar{\phi}'$, one has

$$\phi(x) = \bar{c}(x)\bar{\phi} = \bar{c}(x)\int_0^x e(u)du \text{ with } e(u) \to 0.$$

From this integral representation, one can check that ϕ is self-neglecting (as in [Blo], BGT §2.11).

As to (ii), we first prove this for φ itself. So specializing (i) to $\psi = \varphi$, write

$$\varphi(x) \sim \int_0^x e(u)du \text{ with } e(u) \to 0,$$

we deduce immediately that $\varphi(x)/x \sim \int_0^x e(u)du/x \to 0$.

We now use the fact that $\varphi(x)/x \to 0$ to consider a general $\psi \in SN$ that is φ-slowly varying. Take $\psi \sim \phi \in \mathcal{C}^1$.Take $a_n = \psi(x_n)/\varphi(x_n)$, $b_n = \varphi(x_n)/\varphi(x_{n-1}) > 0$, so that $a_n \to 1$ and $b_n \to 1$. Put $z_n := \varphi(x_n)/x_n > 0$, so that $z_n \to 0$, as just shown. Now one has by Lemma 4 (i) that

$$\begin{aligned}\frac{\phi(x_n)}{x_n} &= \frac{\psi(x_n)}{x_{n-1} + \varphi(x_n)} = \frac{a_n}{1 + x_{n-1}/\varphi(x_n)} \\ &= \frac{a_n}{1 + (1/z_{n-1})\varphi(x_{n-1})/\varphi(x_n)} = \frac{a_n}{1 + 1/(z_{n-1}b_n)} \to 0,\end{aligned}$$

as required.

As to (iii) for $\psi \in SN$, if $\psi \sim \phi \in \mathcal{C}^1$, then $\phi \in SN$ (by (ii)). So as $\psi(x)/\phi(x) \to 1$, by Theorem 2

$$\lim_{x\to\infty} \frac{f(x+t\psi(x))}{f(x)} = \lim_{x\to\infty} \frac{f(x+t[\psi(x)/\phi(x)]\phi(x))}{f(x)}.$$

That is, f is ϕ-regularly varying. □

We have just seen that self-neglecting functions are necessarily $o(x)$. We now see that, for $\varphi \in SN$, a φ-slowly-varying function is also SN if it is $o(x)$.

Theorem 5. *For $\varphi \in SN$, if $\psi > 0$ is φ-slowly varying and $\psi(x) = o(x)$, then ψ is SN, and so has a representation*

$$\psi \sim \int_0^x e(u)du \text{ with continuous } e(.) \to 0.$$

Proof. Since self-neglect is preserved under asymptotic equivalence, without loss of generality we may assume that ψ is smooth. Now $\psi(x)/\varphi(x) \to 1$ (by definition), so for fixed u, $u[\psi(x)/\varphi(x)] \to u$. For ψ a φ-slowly varying function, by the UCT for φ-regular variation

$$\psi(x+t\varphi(x))/\varphi(x) \to 1, \text{ loc. unif. in } u,$$

as ψ is measurable. So in particular,

$$\frac{\psi(x+t\psi(x))}{\psi(x)} = \frac{\psi(x+t[\psi(x)/\varphi(x)]\varphi(x))}{\varphi(x)} \frac{\varphi(x)}{\psi(x)} \to 1.$$

That is, ψ is BSV, since $\psi(x) = o(x)$. But ψ is continuous, so by Bloom's theorem ([Blo]) $\psi \in SN$. □

Lemma 5. *For measurable $\varphi \in SN$, the function*

$$f_\rho(x) := \exp\left(\rho \int_1^x \frac{du}{\varphi(u)}\right)$$

is φ-regularly varying with index ρ.

Proof. With $h_\rho = \log f_\rho$ one has that

$$\begin{aligned} & h_\rho(x+t\varphi(x)) - h_\rho(x) - \rho t \\ = \ & \rho \int_x^{x+t\varphi(x)} \frac{du}{\varphi(u)} - \rho t = \rho \int_x^{x+t\varphi(x)} \left(\frac{\varphi(x)}{\varphi(u)} - 1 \right) \frac{du}{\varphi(x)} \\ = \ & \rho \int_0^t \left(\frac{\varphi(x)}{\varphi(x+v\varphi(x))} - 1 \right) dv = o(1). \qquad \square \end{aligned}$$

We may now establish our main result with f_ρ as above.

Theorem 6 (Beurling Representation Theorem). *For $\varphi \in SN$ with φ Baire/measurable eventually bounded away from 0, and f measurable and φ-regularly varying: for some $\rho \in \mathbb{R}$ and φ-slowly varying function $\tilde{f}$, one has*

$$f(x) = f_\rho(x)\tilde{f}(x) = \exp\left(\rho \int_1^x \frac{du}{\varphi(u)} \right) \tilde{f}(x).$$

Any function of this form is φ-regularly varying with index ρ.

So $f \sim f_\rho \phi$ for some smooth representation ϕ of $\tilde{f}$.

Proof. By Theorem 4(iii), we may assume that φ is smooth. Choose ρ as in Theorem 3 and, referring to the flow rate $\varphi(x) > 0$ at x, put

$$\tilde{h}(x) := h(x) - \rho \int_1^x \frac{du}{\varphi(u)},$$

where $h = \log f$. So $\tilde{h}(x)$ is Baire/measurable as h is.

By Theorem 2 (UCT), locally uniformly in t one has a 'reduction' formula for $\tilde{h}$:

$$\tilde{h}(x+t\varphi(x)) - \tilde{h}(x) - \rho t + \rho \int_x^{x+t\varphi(x)} \frac{du}{\varphi(u)} = h(x+t\varphi(x)) - h(x) - \rho t = o(1).$$

So substituting $u = x + v\varphi(x)$ in the last step,

$$\begin{aligned} \tilde{h}(x+t\varphi(x)) - \tilde{h}(x) &= \rho t - \rho \int_x^{x+t\varphi(x)} \frac{du}{\varphi(u)} + o(1) \\ &= \rho \int_x^{x+t\varphi(x)} \left(1 - \frac{\varphi(x)}{\varphi(u)} \right) \frac{du}{\varphi(x)} + o(1) \\ &= \rho \int_0^t \left(1 - \frac{\varphi(x)}{\varphi(x+v\varphi(x))} \right) dv + o(1) \\ &= o(1), \end{aligned}$$

and the convergence under the integral here is locally uniform in t since $\varphi \in SN$. So $\exp(\tilde{h})$ is Beurling φ-slowly varying. The converse was established in Lemma 5. The remaining assertion follows from Theorem 4. □

As a second corollary of Theorems 2 and 3 and of the de Bruijn-Karamata Representation Theorem (see BGT, Ths. 1.3.1 and 1.3.3 and the recent generalization [BinO10]), we deduce a Representation Theorem for Beurling regular variation which extends previous results concerned with the class Γ – see BGT, Th. 3.10.6. We need the following result, which is similar to Bloom's Th. 4 except that we use regularity of φ rather than assume conditions on convergence rates.

Lemma 6 (Karamata slow variation). *If $\varphi \in SN$ with φ Baire/measurable eventually bounded away from 0, then*

$$\varphi(x+v)\,/\varphi(x) \to 1 \text{ as } x \to \infty, \text{ locally uniformly in } v.$$

Proof. W.l.o.g suppose that $0 < K < \varphi(x)$ for all x. Fix v; then $0 \leq |v|/\varphi(x) \leq |v|K^{-1}$ for all x. Let $\varepsilon > 0$. Since $\varphi \in SN$, there is $X = X(\varepsilon, v)$ such that

$$|\varphi(x + t\varphi(x))\,/\varphi(x) - 1| < \varepsilon, \tag{10}$$

for all $|t| \leq |v|K^{-1}$ and all $x \geq X$. So in particular, for $x \geq X$ and $t := v/\varphi(x)$, since $|t| \leq vK^{-1}$, substitution in (10) yields

$$|\varphi(x+v)\,/\varphi(x) - 1| < \varepsilon,$$

for $x \geq X$. This shows that for each $v \in \mathbb{R}$

$$\varphi(x+v)\,/\varphi(x) \to 1.$$

So $\log \varphi$ is Karamata slowly varying in the additive sense; being Baire/measurable, by the UCT of additive Karamata theory, convergence to the limit for $\log \varphi$, and so convergence for φ as above, is locally uniform in v. □

An alternative 'representation' follows from Lemma 6.

Theorem 6′ (Beurling Representation Theorem). *For $\varphi \in SN$ with φ Baire/measurable eventually bounded away from 0, and f measurable and*

φ-regularly varying: there are $\rho \in \mathbb{R}$, measurable $d(.) \to d \in (0,\infty)$ and continuous $e(.) \to 0$ such that

$$f(x) = d(x)\exp\left(\rho\int_1^x \frac{du}{\varphi(u)} + \int_0^x e(v)dv\right) = d(x)\exp\left(\rho\int_1^x \frac{u}{\varphi(u)}\frac{du}{u} + \int_0^x e(v)dv\right),$$

where for each t

$$\int_x^{x+t\varphi(x)} e(v)dv = o(1).$$

Proof. Choose ρ as in Theorem 3 and, referring to the flow rate $\varphi(x) > 0$ at x, put

$$\tilde{h}(x) := h(x) - \rho\int_1^x \frac{du}{\varphi(u)},$$

where $h = \log f$. Here, since $1/\varphi(x)$ is eventually bounded above as $x \to \infty$ and our analysis is asymptotic, w.l.o.g. we may assume again by Luzin's Theorem that φ here is continuous.

By Theorem 2 (UCT), locally uniformly in t one has, as in Theorem 6, a 'reduction' formula for $\tilde{h}$:

$$\begin{aligned} & \tilde{h}(x + t\varphi(x)) - \tilde{h}(x) - \rho t + \rho\int_x^{x+t\varphi(x)} \frac{du}{\varphi(u)} && (11)\\ = & \; h(x + t\varphi(x)) - h(x) - \rho t = o(1). && (12)\end{aligned}$$

Fix y and let $K > 0$ be a bound for $1/\varphi$, far enough to the right. We will use local uniformity in (12) on the interval $|t| \leq |y|K^{-1}$. First, take $t = y/\varphi(x)$, so $|t| \leq |y|/K$, so by (12),

$$\begin{aligned}\tilde{h}(x+y) - \tilde{h}(x) &= \frac{\rho y}{\varphi(x)} - \rho\int_x^{x+y}\frac{du}{\varphi(u)} + o(1)\\ &= \rho\int_x^{x+y}\left(\frac{1}{\varphi(x)} - \frac{1}{\varphi(u)}\right)du + o(1)\\ &= \frac{\rho}{\varphi(x)}\int_0^y\left(1 - \frac{\varphi(x)}{\varphi(x+w)}\right)d + o(1).\end{aligned}$$

By Lemma 6 applied to the set $\{v : |v| \leq |y|\}$, which corresponds to the w range in the integral above, we have

$$\tilde{h}(x+y) - \tilde{h}(x) = \frac{\rho}{\varphi(x)}\int_0^y\left(1 - \frac{\varphi(x)}{\varphi(x+v)}\right)dv + o(1) = o(1), \qquad (13)$$

since $1/\varphi(x)$ is bounded. That is, $\tilde{h}(x)$ is slowly varying in the additive Karamata sense (as with $\log\varphi$ in the lemma). So by the Karamata-de Bruijn representation (see BGT, 1.3.3),

$$\tilde{h}(x) = c(x) + \int_0^x e(v)dv, \tag{14}$$

for some measurable $c(.) \to c \in \mathbb{R}$ and continuous $e(.) \to 0$. Re-arranging yields

$$\log f(x) = h(x) = \tilde{h}(x) + \rho\int_1^x \frac{du}{\varphi(u)} = c(x) + \rho\int_1^x \frac{du}{\varphi(u)} + \int_0^x e(v)dv.$$

Taking $d(x) = e^{c(x)}$ we obtain the desired representation. To check this, w.l.o.g. we now take $d(x) = 1$, and continue by substituting $u = x + s\varphi(x)$ to obtain from (13) and (14)

$$\int_x^{x+t\varphi(x)} e(v)dv = \tilde{h}(x + t\varphi(x)) - \tilde{h}(x) = o(1),$$

since $\varphi \in SN$. $\square$

The proof above remains valid when $\rho = 0$ for arbitrary $\varphi \in SN$, irrespective of whether φ is bounded away from zero or not. Since φ is itself φ-regularly varying with corresponding index $\rho = 0$, we have an alternative to the Bloom-Shea representation of φ via the de Bruijn-Karamata representation. We record this as

Corollary 2. *For measurable $\varphi \in SN$ there are measurable $d(.) \to d \in (0,\infty)$ and continuous $e(.) \to 0$ such that*

$$\varphi(x) = d(x)\exp\left(\int_1^x e(v)dv\right),$$

where for each t

$$\int_x^{x+t\varphi(x)} e(v)dv = o(1).$$

We have in the course of the proof of Theorem 4 in fact also shown:

Corollary 3. *For $\varphi \in SN$ bounded below and f measurable and φ-regularly varying:*

$$f(x) = \tilde{f}(x) \exp\left(\rho \int_1^x \frac{du}{\varphi(u)}\right),$$

for some $\rho \in \mathbb{R}$ and some Karamata (multiplicatively) slowly varying $\tilde{f}$.

7. Complements

1. *Direct and indirect specialization to the Karamata framework.* That $\varphi(x) = o(x)$ is part of the definition of Beurling slow variation (see §1). We note that allowing $\varphi(x) = x$ formally puts us back in the framework of Karamata regular variation. The results above all extend in this way, and in particular one formally recovers the standard results of *multiplicative* Karamata theory:

$$\varphi(x) = x, \qquad T_t^{\varphi} x = (t+1)x, \qquad f(x) = d(x) \exp\left(\rho \int_1^x \frac{du}{u} + \int_0^x e(v)dv\right).$$

Of course $\varphi(x) = 1$ directly specializes Beurling to Karamata *additive* regular variation; the latter is equivalent under log/exp transformation to the Karamata multiplicative form, so in that sense the Beurling theory above incorporates the whole of Karamata theory, albeit indirectly.

2. *Karamata- versus Beurling-variation in the case $\varphi(x) = x$.* As the reader may verify, the proofs of Theorems 2-4 above apply also to the context of $\varphi(x) = x$ (amended around the origin to satisfy $\varphi > 0$ in Th. 4), where the restriction $\varphi(x) = o(x)$ fails (that is, $\varphi \notin SN$). However, for $\varphi(x) = x$ one has $\varphi(x + u\varphi(x))/\varphi(x) = 1 + u$ (locally uniformly), and indeed this latter fact suffices for the Shift Lemma and for Theorem 2 (UCT) as $y_n \to \infty$, and $c_n \to 1$ still hold. (This is a matter of separate interest, addressed in full generality via the λ-UCT of [Ost2]; $\lambda(t) = 1 + t$ is the case relevant here.) Thus Theorems 2 and 3 above actually contain their Karamata multiplicative counterparts ($\varphi(x) = x$) directly, rather than depending on them. This situation is akin to that of the relationship between the de Haan theory of BGT, Ch. 3 and the Karamata theory of BGT, Ch. 1, cf. the 'double-sweep' procedure of BGT of p. 128, and § 3.13.1 p. 188. Of course the exception is Theorem 6′ where, though the Beurling case specializes to the Karamata case, nevertheless it depends on the Karamata case, by design.

3. *Miller homotopy.* Given Theorems 4 and 5, w.l.o.g. in the asymptotic

analysis one may take $\varphi \in \mathcal{C}^1$ – continuously differentiable; then $H(x,t) = x + t\varphi(x)$ is a Miller homotopy in the sense of [BinO9], i.e. its three defining properties are satisfied: (i) $H(x,0) = x$, (ii) H_x, H_t exist, and (iii) $H_t(t,x) = \varphi(x) > 0$. In view of this, Miller's [Mil] generalization of a result due to Kestelman and also Borwein and Ditor (see [BinO9] for their context) asserts that *for any non-negligible (Baire/measurable) set* T *and any null sequence* z_n, *for quasi all* $t \in T$ *there is an infinite* $\mathbb{M}_t \subseteq \mathbb{N}$ *such that* $\{t + \varphi(t)z_m : m \in \mathbb{M}_t\} \subseteq T$; cf. Theorem 0 above.
4. *Self-neglect in auxiliary functions.* We have seen in the Beurling UCT how uniformity in the auxiliary function φ passes 'out' to φ-regularly varying f. For the converse (uniformity passing 'in' from f to φ), we note the following.

Proposition. *For Baire (measurable)* f, *if for some Baire (measurable) function* $\varphi > 0$ *and some real* $\rho \neq 0$

$$f(x + u\varphi(x))/f(x) \to e^{\rho u}, \text{ locally uniformly in } u,$$

then $\varphi \in SN$.

Proof. Replace u by ρu and $\varphi(x)$ by $\psi(x) = \varphi(x)/\rho$ to yield

$$f(x + u\psi(x))/f(x) \to e^u,$$

(allowing $u < 0$), then follow verbatim as in BGT, 3.10.6, which relies only on uniformity (and on the condition that $x + u\psi(x) \to \infty$ as $x \to \infty$, which is also deduced from uniformity in BGT 3.10.1). □

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