OPTIMAL EXECUTION IN A MULTIPLICATIVE LIMIT ORDER BOOK

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ABSTRACT. We introduce a limit order book model with a multiplicative relationship between the unaffected price process and the shadow limit order book. This model can be viewed as a multiplicative version of the limit order book model of Obizhaeva and Wang [11], Alfonsi, Fruth and Schied [4] and Predoiu, Shaikhet and Shreve [12], which have an additive relationship between the unaffected price process and the shadow limit order book. In the multiplicative limit order book, the shadow order book scales with the unaffected price process and bid prices remain positive for arbitrary sales orders, which is not the case for the additive model. In particular, we show that the optimal purchasing strategy and the optimal liquidation strategy are deterministic in the multiplicative limit order book model, and that if the additive and the multiplicative models have the same shadow limit order books at time 0, the optimal execution strategies for the two models coincide.

1. INTRODUCTION

Recent years have seen an increasing interest in limit order book models and problems related to optimal order execution. Obizhaeva and Wang [11] introduced a model for the limit order book and addressed the problem of optimal purchasing within a given time-horizon. In their model, they introduce the notion of the unaffected price process. The unaffected price process represents the ask price if the large investor does not submit any market orders. If the investor submits a market order, this has the effect of "eating into" the limit order book until the order is filled, thus affecting the ask price. The size of this effect is determined by the shape of the shadow limit order book, which is assumed deterministic and constant. In relation to our paper, we note that there is an additive relationship between the randomness, represented by the unaffected price process, and the effect the large investor's trading is having on the ask price process. Obizhaeva and Wang [11] assume that the ask price reverts to the unaffected price at an exponential rate, i.e. that the market has exponential resilience. The problem they study is that of minimising the expected cost of purchasing a given number of shares within a fixed time horizon. Because of the additive relationship between the unaffected price process and the effect on the ask price process due to the large investor's trading, it is quite straightforward to show that the optimal strategy is deterministic. Alfonsi, Fruth and Schied [4] generalised the limit order book model of Obizhaeva and Wang [11] to incorporate more general strictly positive

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shape functions. They also introduced two different notions of exponential resilience, one as in Obizhaeva and Wang [11], and one based on price. Predoiu, Shaikhet and Shreve [12] further generalised the limit order book model to allow for a general shadow limit order book shape (i.e. not necessarily strictly increasing and continuously differentiable as in [4]) and resilience based on a general function (i.e. not necessarily exponential resilience).

The aforementioned papers address the problem of minimising the expected cost of purchasing, and do not explicitly address the related problem of optimal liquidation (i.e. starting with a given position of shares, to maximise the expected cash position at a fixed future time). To address this problem one needs to specify the bid price limit order book. If these are specified according to the Obizhaeva and Wang [11] model, there is a positive probability of negative bid prices if the large investor trades according to any non-trivial strategy. Within the optimal execution literature, positivity of prices is not crucial. In fact the Bachelier model forms the basis of randomness in a number of the more popular models (see e.g. Almgren and Chriss [2], Almgren [3], Kissel and Malamut [10], Gatheral [7] and Schied and Schöneborn [14]). However, models which exhibits positive prices have a conceptual advantage. In this paper, we therefore propose a version of the Obizhaeva and Wang [11] model (with the general specification of Predoiu, Shaikhet and Shreve [12]) with a multiplicative relationship between the unaffected price process and the effect on the bid/ask price caused by the large investor's trading. The multiplicative relationship has the effect that the constant shadow limit order book scales with the unaffected price process and that prices therefore remain positive. A version of this multiplicative model with transaction costs, but without resilience, was introduced by Guo and Zervos [8], who use singular control theory to obtain the optimal execution strategy. In our paper, we show that the multiplicative model can be expressed in terms of the additive model of Obizhaeva and Wang [11], but with a time varying stochastic limit order book. Fruth, Schöneborn and Urusov [6] address the problem of optimal execution in an additive limit order book with stochastic, time-varying linear impact using dynamic programming. Bank and Fruth [5] address the problem of optimal execution in an additive limit order book with time-varying deterministic linear impact and exponential resilience using convex optimization.

We show that for the multiplicative limit order book that we introduce, regardless of the shape of the shadow order book, there exists an additive model with a shadow order book which coincides with that of the multiplicative model at time zero. In this case we say that the two models are equally calibrated at time zero, and provide an explicit formula for the relationship between the two order books when this is the case. We then prove that the optimal purchasing strategy for the multiplicative model involves no sales orders and is deterministic, and that the optimal liquidation strategy involves no purchasing orders and is deterministic. Due to the multiplicative relationship, this involves arguments which differ from the additive case. We then show that if the multiplicative model and the additive model are equally calibrated at time zero, the optimal purchasing strategy for the multiplicative model coincides with the optimal purchasing strategy for the additive model, and that the optimal liquidation strategy for the multiplicative model coincides with the optimal liquidation strategy for the additive model. In view of the results of Predoiu, Shaikhet and Shreve [12], which provide the solution to the optimal purchasing problem in the additive model, this provides the solution to the optimal execution problems for the multiplicative model and demonstrate a kind of robustness for the optimal strategy.

The paper is organized as follows. In Section 1 and 2 we specify the full limit order book corresponding to the model in Predoiu, Shaikhet and Shreve [12]. In Section 1 and 3, we specify the multiplicative version of the limit order book model, and in Section 4 we provide a basic connection between the additive and the multiplicative limit order book models. In Section 5, we review the solution to the optimal purchasing problem obtained by Predoiu, Shaikhet and Shreve [12], and provide a relationship between the optimal purchasing problem and the optimal liquidation problem for the additive limit order book model. In section 6 we provide the solution to the optimal purchasing problem and the optimal liquidation problem for the multiplicative model by showing that if the multiplicative limit order book has the same shadow limit order book as the additive model at time 0, then the optimal strategies for the two models coincide.

2. Mathematical preliminaries and basic concepts

Throughout the paper we will consider a fixed time-horizon $[0,T]$, which represents the time-horizon the large investor wants to either purchase $\overline{Y} > 0$ number of shares or liquidate a position of $\overline{Y}$ number of shares. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, supporting a one-dimensional, positive and continuous $(\mathcal{F}_t)$-martingale I satisfying

$$\mathbb{E}\left[\max_{0\le t\le T} I_t\right] < \infty. \tag{2.1}$$

From Karatzas and Shreve [9, Appendix B], it follows that there exists a continuous $(\mathcal{F}_t)$-martingale M such that I is the unique positive solution to

$$I_t = I_0 + \int_0^t I_u \, dM_u, \qquad I_0 > 0,$$

i.e. I is the Doleans-Dàdè exponential of M.

In the absence of trading by the large investor, we assume that the bid price B_t (i.e. the best offered bid price) at time t is given by $B_t = I_t$, and that the ask price A_t (i.e. the best offered ask price) at time t is given by $A_t = I_t$.

Definition 2.1. Let $\mathcal{A}_p$ denote the set of all pairs of $\mathcal{F}_t$-adapted processes (X, Y), where X and Y are càdlàg processes such that X is non-decreasing with $X_{0-} = 0$, Y is non-increasing with $Y_{0-} = 0$, and $X_T + Y_T = \overline{Y}$.

The set $\mathcal{A}_p$ represents the set of purchasing strategies which allow intermediate sales of shares, where X_t is the number of shares purchased up to time t, $-Y_t$ is the number of shares sold up to time t and $X_t + Y_t$ is the net position in shares at time t.

Definition 2.2. Let $\mathcal{A}_s$ denote the set of all pairs of $\mathcal{F}_t$-adapted processes (X, Y), where X and Y are càdlàg processes such that X is non-decreasing with $X_{0-} = 0$, Y is non-increasing with $Y_{0-} = \overline{Y}$, and $X_T + Y_T = 0$.

The set $\mathcal{A}_s$ corresponds to the set of liquidation strategies which allow intermediate purchases of shares, where X_t is the number of shares purchased up to time t, $\overline{Y} - Y_t$ is the number of shares sold up to time t, and $X_t + Y_t$ is the net position in shares at time t.

We will also consider the following smaller sets of strategies $\mathcal{P}$ and $\mathcal{S}$, which do not allow intermediate sales and purchases, respectively.

Definition 2.3. Let $\mathcal{P}$ denote the set of all $\mathcal{F}_t$-adapted, càdlàg, non-decreasing processes satisfying $X_{0-} = 0$ and $X_T = \overline{Y}$.

Definition 2.4. Let $\mathcal{S}$ denote the set of all $(\mathcal{F}_t)$-adapted, càdlàg, non-increasing processes satisfying $Y_{0-} = \overline{Y}$ and $Y_T = 0$.

Let h be a strictly increasing, locally lipschitz function defined on $\mathbb{R}$ satisfying

$$h(0) = 0, \qquad \lim_{x \to -\infty} h(x) < -\frac{\overline{Y}}{T}, \qquad \lim_{x \to \infty} h(x) > \frac{\overline{Y}}{T}. \tag{2.2}$$

The function h is called the resilience function and determines the speed at which the limit order book recovers in relation to the volume effect process E. We note that the properties of h listed in (2.2) correspond to the properties required for the resilience function in Predoiu, Shaikhet and Shreve [12].

For a càdlàg and $\mathcal{F}_t$-adapted process Z defined on $[0, T]$, which is either increasing or decreasing, denote by E^Z the $\mathcal{F}_t$-adapted process satisfying

$$E_t^Z = Z_t - Z_{0-} - \int_0^t h\big(E_{s-}^Z\big)\, ds. \tag{2.3}$$

According to Predoiu, Shaikhet and Shreve [12, Appendix A], equation (2.3) has a unique solution. In the next two sections, we will describe how the impact of the large investors trading can be expressed in terms of the volume effect process E in the additive and the multiplicative limit order book models.

3. The additive limit order book (ALOB) model

For some extended negative real number $\bar{x}^-$ and some extended positive real number $\bar{x}^+$, let μ_a be a measure on $(\bar{x}^-, \bar{x}^+)$ that is infinite on $(\bar{x}^-, 0]$ and $[0, \bar{x}^+)$, finite on each compact subset of $(\bar{x}^-, \bar{x}^+)$ and satisfies $\mu_a(\{0\}) = 0$. Define a function $F_a : (\bar{x}^-, \bar{x}^+) \to \mathbb{R}$ by

$$F_a(x) = \begin{cases} -\mu_a\big((x, 0]\big), & \bar{x}^- < x < 0, \\ 0, & x = 0, \\ \mu_a\big([0, x)\big), & 0 < x < \bar{x}^+. \end{cases}$$

We assume that $F_a(x) < 0$ for every $x < 0$ and that $F_a(x) > 0$ for every $x > 0$. With the exception of the condition $\mu_a(\{0\}) = 0$, this is the exactly the specification and assumptions corresponding to the full limit order book version of the model in Predoiu, Shaikhet and Shreve [12], which only considered the one sided case. The condition $\mu_a(\{0\}) = 0$ is natural if one want a symmetric specification of the ask limit order book and the bid limit order book. It follows from the definition of F_a and the assumptions on μ_a that $F_a(x)$ is continuous at $x = 0$, $F_a(x)$ is right-continuous for $x \leq 0$ and left-continuous for $x \geq 0$. Moreover, a function F_a satisfying these properties completely determines the corresponding measure μ_a.

If S is a measurable subset of $(\bar{x}^-, 0]$, then at time t, the number of bid limit orders in the shadow limit order book with prices in

$$B_t + S = \big\{B_t + x \mid x \in S\big\}$$

is $\mu_a(S)$. If S is a measurable subset of $[0, \bar{x}^+)$, then at time t, the number of ask limit orders in the shadow limit order book with prices in $A_t + S$ is $\mu_a(S)$.

Define a function $\psi_a : \mathbb{R} \to (\bar{x}^-, \bar{x}^+)$ by

$$\psi_a(y) = \begin{cases} \inf\big\{x \leq 0 \mid F_a(x) > y\big\}, & y < 0, \\ 0, & y = 0, \\ \sup\big\{x \geq 0 \mid F_a(x) < y\big\}, & y > 0. \end{cases} \tag{3.1}$$

It follows from the properties of F_a that $\psi_a(y)$ is continuous at $y = 0$, right-continuous for $y \leq 0$ and left-continuous for $x \geq 0$.

If the large investor trades according to a strategy $(X, Y) \in \mathcal{A}_p \cup \mathcal{A}_s$, then in the ALOB model, the ask price process is defined by

$$A_t + \psi_a\big(E_t^X\big), \qquad 0 \leq t \leq T,$$

which represents the best offered ask price observable in the market. The bid price process is defined by

$$B_t + \psi_a\big(E_t^Y\big), \qquad 0 \leq t \leq T,$$

which represents the best offered bid price observable in the market.

For a purchasing strategy $(X, Y) \in \mathcal{A}_p$, the total cost of purchasing $\overline{Y}$ number of shares is in the ALOB model given by

$$\begin{aligned} C_a^p(X, Y) = &\int_0^T \big\{A_t + \psi_a\big(E_{t-}^X\big)\big\}\, dX_t^c + \sum_{0 \leq t \leq T} \int_0^{\triangle X_t} \big\{A_t + \psi_a(E_{t-}^X + x)\big\}\, dx \\ &+ \int_0^T \big\{B_t + \psi_a\big(E_{t-}^Y\big)\big\}\, dY_t^c + \sum_{0 \leq t \leq T} \int_0^{\triangle Y_t} \big\{B_t + \psi_a(E_{t-}^Y + y)\big\}\, dy, \end{aligned}$$

which corresponds to the market sales orders being matched with the best bid prices available until the order is filled, and the market buy orders being matched with the best ask prices available until the order is filled. In particular, we can observe that the total cost of purchasing $\overline{Y}$ number of shares using a purchasing strategy $X \in \mathcal{P}$ is given by

$$C_a^p(X) = \int_0^T \big\{A_t + \psi_a\big(E_{t-}^X\big)\big\}\, dX_t^c + \sum_{0 \leq t \leq T} \int_0^{\triangle X_t} \big\{A_t + \psi_a(E_{t-}^X + x)\big\}\, dx,$$

which in view of [12, equation (3.7)], is exactly the formulation in Predoui, Shaikhet and Shreve [12]. If the large investor choose to liquidate his position using a strategy $(X,Y)\in\mathcal{A}_s$, then in the ALOB model, his cash position at time T is given by

$$\begin{aligned}C_a^s(X,Y) = &-\int_0^T \big\{B_t+\psi_a\big(E_{t-}^Y\big)\big\}\,dY_t^c - \sum_{0\le t\le T}\int_0^{\triangle Y_t}\big\{B_t+\psi_a(E_{t-}^Y+y)\big\}\,dy\\ &-\int_0^T \big\{A_t+\psi_a\big(E_{t-}^X\big)\big\}\,dX_t^c - \sum_{0\le t\le T}\int_0^{\triangle X_t}\big\{A_t+\psi_a(E_{t-}^X+x)\big\}\,dx.\end{aligned}$$

In particular, we can observe that if the large investor use a liquidation strategy $Y\in\mathcal{S}$, then the cash position at time T corresponding to the liquidation strategy Y is

$$C_a^s(Y) = -\int_0^T \big\{B_t+\psi_a\big(E_{t-}^Y\big)\big\}\,dY_t^c - \sum_{0\le t\le T}\int_0^{\triangle Y_t}\big\{B_t+\psi_a(E_{t-}^Y+y)\big\}\,dy.$$

Given a non-decreasing function $F=F(x)$ which is continuous at $x=0$, right-continuous for $x\le 0$ and left-continuous for $x\ge 0$, this uniquely defines a measure μ satisfying the properties listed in the beginning of this section. Therefore, the data for the ALOB model is a pair (h,F), where h is the resilience function and F describes the shape of the shadow limit order book as outlined in this section with F equal to F_a. We will refer to this model as the ALOB model with data (h,F).

Definition 3.1. The ALOB purchasing problem consists of minimising the expected cost $\mathbb{E}\big[C_a^p(X,Y)\big]$ of purchasing $\overline{Y}$ shares by time T. In particular, we are interested in finding a purchasing strategy $(X^*,Y^*)\in\mathcal{A}_p$ which attains the minimum, i.e. an admissible purchasing strategy (X^*,Y^*) such that

$$\mathbb{E}\big[C_a^p(X^*,Y^*)\big] = \min_{(X,Y)\in\mathcal{A}_p}\mathbb{E}\big[C_a^p(X,Y)\big]. \tag{3.2}$$

Without prior knowledge, it is not obvious that such a strategy $(X^*,Y^*)\in\mathcal{A}_p$ exists. However, the optimal purchasing problem formulated in Definition 3.1 coincides with the optimal execution problem in Predoiu, Shaikhet and Shreve [12], who completely solved the problem, and in particular proved the existence of an optimal strategy satisfying equation (3.2).

Definition 3.2. The ALOB liquidation problem consists of maximising the expected cash position $\mathbb{E}\big[C_a^s(X,Y)\big]$ at time T, given initial position of $\overline{Y}$ shares. In particular, we are interested in finding a liquidation strategy $(X^*,Y^*)\in\mathcal{A}_s$ which attains the maximum, i.e. an admissible liquidation strategy (X^*,Y^*) such that

$$\mathbb{E}\big[C_a^s(X^*,Y^*)\big] = \max_{(X,Y)\in\mathcal{A}_s}\mathbb{E}\big[C_a^s(X,Y)\big]. \tag{3.3}$$

In Section 5 we will establish a certain symmetry relation between the problem formulation in Definition 3.1 and the problem formulation in Definition 3.2, and the existence of an optimal strategy satisfying equation (3.3) will then follow from the existence of an optimal strategy satisfying equation (3.2).

4. The multiplicative limit order book (MLOB) model

For some extended negative real number $\bar{y}^-$ and some extended positive real number $\bar{y}^+$, let μ_m be a measure on $(\bar{y}^-, \bar{y}^+)$ that is infinite on $(\bar{y}^-, 0]$ and $[0, \bar{y}^+)$, finite on each compact subset of $(\bar{y}^-, \bar{y}^+)$ and satisfies $\mu_m(\{0\}) = 0$. Define a function $F_m : (\bar{y}^-, \bar{y}^+) \to \mathbb{R}$ by

$$F_m(x) = \begin{cases} -\mu_m\big((x, 0]\big), & \bar{y}^- < x < 0, \\ 0, & x = 0, \\ \mu_m\big([0, x)\big), & 0 < x < \bar{y}^+. \end{cases}$$

We assume that $F_m(x) < 0$ for every $x < 0$ and that $F_m(x) > 0$ for every $x > 0$. From the assumptions on μ_m, it follows that $F_m(x)$ is continuous at 0, right-continuous for $\bar{y}^- < x \leq 0$ and left-continuous for $0 \leq x < \bar{y}^+$.

If S is a measurable subset of $(\bar{y}^-, 0]$, then at time t, the number of bid limit orders in the shadow limit order book with prices in

$$B_t \mathrm{e}^S = \big\{ B_t \mathrm{e}^x \mid x \in S \big\}$$

is $\mu_m(S)$, and if S is a measurable subset of $[0, \bar{y}^+)$, then at time t, the number of ask limit orders in the shadow limit order book with prices in $A_t \mathrm{e}^S$ is $\mu_m(S)$.

Denote by $\psi_m : \mathbb{R} \to (\bar{y}^-, \bar{y}^+)$ the function given by

$$\psi_m(y) = \begin{cases} \inf\big\{ x \leq 0 \mid F_m(x) > y \big\}, & y < 0, \\ 0, & y = 0, \\ \sup\big\{ x \geq 0 \mid F_m(x) < y \big\}, & y > 0. \end{cases}$$

It follows from the properties of F_m that $\psi_m(y)$ is continuous at $y = 0$, right-continuous for $y \leq 0$ and left-continuous for $y \geq 0$.

If the large investor trades according to a strategy $(X, Y) \in \mathcal{A}_p \cup \mathcal{A}_s$, then in the MLOB model, the ask price process is given by

$$A_t \exp\big(\psi_m(E_t^X)\big), \qquad 0 \leq t \leq T,$$

which represents the best offered ask price observable in the market, and the bid price process is given by

$$B_t \exp\big(\psi_m(E_t^Y)\big), \qquad 0 \leq t \leq T,$$

which represents the best offered bid price observable in the market. Observe that the assumptions on μ_m ensure that for any admissible strategy, the bid price process remains strictly positive in the MLOB model. For the ALOB model however, there is typically a strictly positive probability that the bid price can be negative for any admissible strategy which involve sales orders.

For a purchasing strategy $(X,Y)\in\mathcal{A}_p$, the total cost of purchasing $\overline{Y}$ number of shares is in the MLOB model given by

$$\begin{aligned} C_m^p(X,Y) = &\int_0^T A_t \mathrm{e}^{\psi_m(E^X_{t-})}\,dX_t^c + \sum_{0\le t\le T}\int_0^{\triangle X_t} A_t\mathrm{e}^{\psi_m(E^X_{t-}+x)}\,dx \\ &+ \int_0^T B_t \mathrm{e}^{\psi_m(E^Y_{t-})}\,dY_t^c + \sum_{0\le t\le T}\int_0^{\triangle Y_t} B_t\mathrm{e}^{\psi_m(E^Y_{t-}+y)}\,dy, \end{aligned}$$

which corresponds to the best offered limit orders being executed first until the market orders are filled. For future reference, also observe that the total cost of purchasing $\overline{Y}$ number of shares using a strategy $X\in\mathcal{P}$ is given by

$$C_m^p(X) = \int_0^T A_t \mathrm{e}^{\psi_m(E^X_{t-})}\,dX_t^c + \sum_{0\le t\le T}\int_0^{\triangle X_t} A_t\mathrm{e}^{\psi_m(E^X_{t-}+x)}\,dx.$$

If the large investor choose to liquidate his position using a strategy $(X,Y)\in\mathcal{A}_s$, then in the MLOB model, his cash position at time T is given by

$$\begin{aligned} C_m^s(X,Y) = &-\int_0^T A_t \mathrm{e}^{\psi_m(E^X_{t-})}\,dX_t^c - \sum_{0\le t\le T}\int_0^{\triangle X_t} A_t\mathrm{e}^{\psi_m(E^X_{t-}+x)}\,dx \\ &- \int_0^T B_t \mathrm{e}^{\psi_m(E^Y_{t-})}\,dY_t^c - \sum_{0\le t\le T}\int_0^{\triangle Y_t} B_t\mathrm{e}^{\psi_m(E^Y_{t-}+y)}\,dy. \end{aligned}$$

In particular, the cash position at time T corresponding to a liquidation strategy $Y\in\mathcal{S}$ is given by

$$C_m^s(Y) = -\int_0^T B_t \mathrm{e}^{\psi_m(E^Y_{t-})}\,dY_t^c - \sum_{0\le t\le T}\int_0^{\triangle Y_t} B_t\mathrm{e}^{\psi_m\left(E^Y_{t-}+y\right)}\,dy.$$

Definition 4.1. The MLOB purchasing problem consists of minimising the expected cost $\mathbb{E}\big[C_m^p(X,Y)\big]$ of purchasing $\overline{Y}$ shares by time T. In particular, we are interested in finding a purchasing strategy $(X^*,Y^*)\in\mathcal{A}_p$ which attains the minimum, i.e. an admissible purchasing strategy (X^*,Y^*) such that

$$\mathbb{E}\big[C_m^p(X^*,Y^*)\big] = \min_{(X,Y)\in\mathcal{A}_p}\mathbb{E}\big[C_m^p(X,Y)\big]. \tag{4.1}$$

The existence of a strategy (X^*,Y^*) satisfying equation (4.1) will follow from the existence of an optimal purchasing strategy for the ALOB model and a correspondence between ALOB purchasing problem and the MLOB purchasing problem which we will establish in Section 7. The same applies to the existence of an optimal strategy for the optimal liquidation problem formulated in Definition 4.2.

Definition 4.2. The MLOB liquidation problem consists of maximising the expected cash position $\mathbb{E}\big[C_m^s(X,Y)\big]$ at time T, given an initial position of $\overline{Y}$ shares. In particular, we are interested in finding a liquidation strategy $(X^*, Y^* \in \mathcal{A}_s$ which attains the maximum, i.e. an admissible liquidation strategy (X^*, Y^*) such that

$$\mathbb{E}\big[C^s_m(X^*, Y^*)\big] = \max_{(X,Y)\in\mathcal{A}_s} \mathbb{E}\big[C^s_m(X, Y)\big]. \tag{4.2}$$

Remark 4.3. For a continuous stricly increasing function $\xi : \mathbb{R} \to \mathbb{R}$ satisfying $\xi(0) = 1$,

$$\lim_{z\to-\infty} \xi(z) \leq 0 \quad \text{and} \quad \lim_{z\to\infty} \xi(z) = \infty,$$

one can alternatively define the shadow limit order book for the MLOB model as follows. For a measurable set $S \subseteq \mathbb{R}^-$, let the number of bid limit orders at time t with prices in $B_t\xi(S) = \big\{B_t\xi(x) \mid x \in S\big\}$ be $\mu_m(S)$. For a measurable set $S \subseteq \mathbb{R}^+$, let the number of ask limit orders at time t with prices in the set $A_t\xi(S)$ be $\mu_m(S)$. One can then show that all the results and proofs in this paper remains true for ξ replaced by $\exp(\cdot)$, and where appropriate, $\ln(\cdot)$ replaced by ξ^{-1}. Thus, while the exponential function is a natural choice, it is the multiplicative relationship between the unaffected price process and the impact the large investor's trading has on bid and ask prices which is important.

5. Basic connections between the ALOB and the MLOB models

A natural question to ask is whether it is possible to specify the shape of the shadow limit order book for the ALOB model and the shape of the shadow limit order book for the MLOB model in such a way that they have the same shadow limit order book at a given time t. This involves specifying the measures μ_a and μ_m, or equivalently the functions F_a and F_m, in such a way that the ALOB model and the MLOB model have the same shadow limit order books at time t. This turns out to be possible, and in this section we explore the connection between the functions F_a and F_m when such a relationship to holds.

Recall that for $\bar{x}^- < y < 0$, $-F_a((y, 0]))$ is the number of bid limit orders in the ALOB model for which

$$B_t + x \in (B_t + y, B_t].$$

We can then observe that $B_t \mathrm{e}^x \in (B_t + y, B_t]$ if and only if

$$\ln\left(\frac{B_t + y}{B_t}\right) < x \leq 0, \quad \text{for } -B_t < y \leq 0.$$

It follows that the ALOB and the MLOB model have the same bid shadow limit order books at time t, if and only if

$$F_a(y) = F_m\left(\ln\left(\frac{B_t + y}{B_t}\right)\right), \quad \text{for } \bar{x}^- < y \leq 0 \quad \text{and} \quad \bar{x}^- = B_t \mathrm{e}^{\bar{y}^-} - B_t. \tag{5.1}$$

Further, recall that for $y > 0$, $F_a([0, y))$ is the number of ask limit orders in the ALOB model for which

$$B_t + x \in [B_t, B_t + y),$$

from which it follows that $B_t e^x \in [B_t, B_t + y)$ if and only if

$$0 \leq x < \ln\left(\frac{B_t + y}{B_t}\right), \qquad \text{for } y > 0.$$

Therefore the ALOB and the MLOB model have the same ask limit order books at time t if and only if

$$F_a(y) = F_m\left(\ln\left(\frac{B_t + y}{B_t}\right)\right), \qquad \text{for } 0 \leq y < \bar{x}+ \qquad \text{and} \qquad \bar{x}^+ = B_t e^{\bar{y}^+} - B_t. \tag{5.2}$$

Definition 5.1. We say that the ALOB and the MLOB models are equally calibrated at time 0 if equations (5.1) and (5.2) hold for $t = 0$.

From the considerations above, it follows that the MLOB model can be expressed in the ALOB form, but now with a time varying and stochastic limit order book shape given by

$$F_a(y;t) = F_m\left(\ln\left(\frac{B_t + y}{B_t}\right)\right), \qquad \text{for} \quad B_t e^{\bar{y}^-} - B_t < y < B_t e^{\bar{y}^+} - B_t. \tag{5.3}$$

For future reference, we note that if the ALOB and the MLOB models are equally calibrated at time 0 then it follows from the definition of ψ_m that

$$\psi_m(z) = \ln\left(\frac{B_0 + \psi_a(z)}{B_0}\right), \qquad \text{for } z \in \mathbb{R}.$$

Hence in particular,

$$\exp\bigl(\psi_m(z)\bigr) = \frac{B_0 + \psi_a(z)}{B_0}, \qquad \text{for } z \in \mathbb{R}, \tag{5.4}$$

as expected.

6. The solution to the optimal execution problems; the ALOB case

The optimal purchasing problem for the ALOB model given by Definition 3.1 was solved by Predoiu, Shaikhet and Shreve in [12]. They showed that the optimal purchasing strategy takes either one of two explicit forms, which they refer to as Type A and Type B strategies. Instead of going into more details regarding these forms and how the optimal strategy is obtained (the reader is referred to the paper by Predoiu, Shaikhet and Shreve [12] for the details), we will solve the optimal liquidation problem for the ALOB model given by Definition 3.2 by proving a symmetry relation between the optimal purchasing problem and the optimal liquidation problem, which will provide the solution to the optimal liquidation problem.

By the arguments given in Predoiu, Shaikhet and Shreve [12, Remark 3.1], it follows that the optimal purchasing strategy for the ALOB purchasing problem given in Definition 3.1 involves no sales orders, i.e.,

$$\min_{X \in \mathcal{P}} \mathbb{E}\bigl[C_a^p(X)\bigr] = \min_{(X,Y) \in \mathcal{A}_p} \mathbb{E}\bigl[C_a^p(X,Y)\bigr].$$

Observe that

$$(X,Y)\in\mathcal{A}_p \quad \text{if and only if} \quad (-Y,\overline{Y}-X)\in\mathcal{A}_s, \tag{6.1}$$

or alternatively

$$(X,Y)\in\mathcal{A}_s \quad \text{if and only if} \quad (\overline{Y}-Y,-X)\in\mathcal{A}_p. \tag{6.2}$$

For a function $f:(a,b)\to\mathbb{R}$, where $a<0<b$ are extended real numbers, define the symmetric mirror of f around the line $\{(x,y)\in\mathbb{R}^2 \mid y=-x\}$ by

$$\widetilde{f}(x)=-f(-x), \qquad -b<x<-a.$$

Let $(X,Y)\in\mathcal{A}_p$ and define $(X',Y')\in\mathcal{A}_s$ by $X'=-X$ and $Y'=\overline{Y}-X$. Let $C_a^s\big((X',Y');(h,F_a)\big)$ denote the cash position at time T in the ALOB model with data (h,F_a) corresponding to the liquidation strategy (X',Y') and let $C_a^p\big((X,Y);(\widetilde{h},\widetilde{F}_a)\big)$ denote the purchasing cost in the ALOB model with data $(\widetilde{h},\widetilde{F}_a)$ corresponding to the purchasing strategy $(X,Y)\in\mathcal{A}_p$. Now observe that

$$-E_t^X=-X_t+\int_0^t \widetilde{h}(E_{u-}^X)\,du=Y_t'-\overline{Y}-\int_0^t h(-E_{u-}^X)\,du.$$

On the other hand

$$E_t^{Y'}=Y_t'-\overline{Y}-\int_0^t h(E_{u-}^{Y'})\,du,$$

which verifies that $E^{Y'}=-E^X$, since equation (2.3) has a unique solution (see Predoiu, Shaikhet and Shreve [12, Appendix A]). Similarly,

$$-E_t^Y=-Y_t+\int_0^t \widetilde{h}(E_{u-}^Y)\,du=X_t'-\int_0^t h(-E_{u-}^Y)\,du,$$

from which it follows that $E^{X'}=-E^Y$. In view of the observation that $\widetilde{\psi}_a$ and $\widetilde{F}_a$ satisfy the defining relation given by (3.1), we calculate that

$$\begin{aligned}
&\mathbb{E}\big[C_a^p\big((X,Y);(\widetilde{h},\widetilde{F}_a)\big)\big]\\
&\quad=\mathbb{E}\bigg[\int_0^T A_t\,dX_t+\int_0^T \widetilde{\psi}_a(E_{t-}^X)\,dX_t^c+\sum_{0\le t\le T}\int_0^{\triangle X_t}\widetilde{\psi}_a(E_{t-}^X+x)\,dx\\
&\qquad+\int_0^T B_t\,dY_t+\int_0^T \widetilde{\psi}_a(E_{t-}^Y)\,dY_t^c+\sum_{0\le t\le T}\int_0^{\triangle Y_t}\widetilde{\psi}_a(E_{t-}^Y+x)\,dx\bigg]\\
&\quad=\mathbb{E}\bigg[-\int_0^T B_t\,dY_t'+\int_0^T \psi_a(E_{t-}^{Y'})\,d(Y_t')^c+\sum_{0\le t\le T}\int_0^{\triangle Y_t'}\psi_a(E_{t-}^{Y'}+x)\,dx\\
&\qquad-\int_0^T B_t\,dX_t'+\int_0^T \psi_a(E_{t-}^{X'})\,d(X_t')^c+\sum_{0\le t\le T}\int_0^{\triangle X_t'}\psi_a(E_{t-}^{X'}+x)\,dx\bigg]\\
&\quad=2B_0\overline{Y}-\mathbb{E}\big[C_a^s\big((X',Y');(h,F_a)\big)\big].
\end{aligned} \tag{6.3}$$

We then have the following result regarding the solution to the optimal liquidation problem for the ALOB model.

Proposition 6.1. *Let $X^* \in \mathcal{P}$ be an optimal strategy for the optimal purchasing problem in the ALOB model given by Definition 3.1 with data $(\widetilde{h}, \widetilde{F}_a)$. Then $Y^* = \overline{Y} - X^* \in \mathcal{S}$ is an optimal strategy for the optimal liquidation problem in the ALOB model given by Definition 3.2 with data (h, F_a).*

Proof. From Prediou, Shaikhet and Shreve [12], we know that there exists $X^* \in \mathcal{P}$ such that

$$\mathbb{E}\big[C_a^p\big((X^*, 0); (\widetilde{h}, \widetilde{F}_a)\big)\big] = \max_{(X,Y)\in\mathcal{A}_p} \mathbb{E}\big[C_a^p\big((X, Y); (\widetilde{h}, \widetilde{F}_a)\big)\big].$$

With reference to (6.3), we therefore obtain

$$\sup_{(X',Y')\in\mathcal{A}_s} \mathbb{E}\big[C_a^s\big((X', Y'); (h, F_a)\big)\big] \le 2B_0\overline{Y} - \mathbb{E}\big[C_a^p\big((X^*, 0); (\widetilde{h}, \widetilde{F}_a)\big)\big],$$

and

$$\mathbb{E}\big[C_a^s\big((0, Y^*); (h, F_a)\big)\big] = 2B_0\overline{Y} - \mathbb{E}\big[C_a^p\big((X^*, 0); (\widetilde{h}, \widetilde{F}_a)\big)\big],$$

from which the result follows. □

7. The solution to the optimal execution problems; the MLOB case

In order to solve the optimal execution problems for the MLOB model, we will first prove that the optimal purchasing strategy involves no sales orders and that the optimal liquidation strategy involves no purchasing orders. We will then show that the optimal purchasing strategy and the optimal liquidation strategy are deterministic. This is the result in this paper which is the hardest to prove, and a key result in order to solve these problems. Once we have proved that the optimal strategies are deterministic, we can use the relationship between the ALOB and MLOB models which are equally calibrated at time 0. It turns out that if the models are equally calibrated at time 0, then the optimal strategies for the execution problems are equal in the ALOB model and the MLOB model, and that the corresponding expected cost/cash for both models coincide.

Lemma 7.1. *It holds that*

$$\inf_{X\in\mathcal{P}} \mathbb{E}\big[C_m^p(X)\big] = \inf_{(X,Y)\in\mathcal{A}_p} \mathbb{E}\big[C_m^p(X, Y)\big], \tag{7.1}$$

i.e., the optimal strategy for the purchasing problem in the MLOB model given by Definition 4.1 involves no sales orders, and

$$\sup_{Y\in\mathcal{S}} \mathbb{E}\big[C_m^s(Y)\big] = \sup_{(X,Y)\in\mathcal{A}_s} \mathbb{E}\big[C_m^s(X, Y)\big], \tag{7.2}$$

i.e., the optimal strategy for the liquidation problem in the MLOB model given by Definition 4.2 involves no purchasing orders.

Proof. Let us first prove that the optimal strategy for the liquidation problem given by Definition 4.2 involves no purchasing orders. Let $(X,Y)\in\mathcal{A}_s$, and define $Y_t^+=\max\{Y_t,0\}$ and $Y_t^-=\min\{Y_t,0\}$. Then $Y=Y^++Y^-$, $Y_T^+=0$ and $Y_T^-=-X_T$. We calculate that

$$\begin{aligned}
C_m^s(X,Y) &\le -\int_0^T B_t \mathrm{e}^{\psi_m(E_{t-}^Y)}\,dY_t^c - \sum_{0\le t\le T}\int_0^{\triangle Y_t} B_t\,\mathrm{e}^{\psi_m(E_{t-}^Y+y)}\,dy - \int_0^T B_t\,dX_t \\
&= -\int_0^T B_t\mathrm{e}^{\psi_m(E_{t-}^{Y^+})}\,d(Y^+)_t^c - \sum_{0\le t\le T}\int_0^{\triangle Y_t^+} B_t\mathrm{e}^{\psi_m(E_{t-}^{Y^+}+y)}\,dy \\
&\quad -\int_0^T B_t\mathrm{e}^{\psi_m(E_{t-}^Y)}\,d(Y^-)_t^c - \sum_{0\le t\le T}\int_0^{\triangle Y_t^-} B_t\mathrm{e}^{\psi_m(E_{t-}^Y+y)}\,dy - \int_0^T B_{t-}\,dX_t \\
&\le C_m(Y^+) - \int_0^T B_t\,dY_t^- - \int_0^T B_t\,dX_t \\
&= C_m(Y^+) + \int_0^T \big(X_{t-}+Y_{t-}^-\big)\,dB_t,
\end{aligned}$$

Hence for every $(X,Y)\in\mathcal{A}_s$, the strategy $Y^+\in\mathcal{S}$ satisfy

$$\mathbb{E}\big[C_m^s(X,Y)\big]\le\mathbb{E}\big[C_m^s(Y^+)\big],$$

from which (7.2) follows. In order to verify (7.1), let $(X,Y)\in\mathcal{A}_p$ and define $X^+=\min\{\overline{Y},X\}$. Then $X^+\in\mathcal{P}$, and similar arguments as above verify that

$$\mathbb{E}\big[C_m^p(X,Y)\big]\ge\mathbb{E}\big[C_m^s(X^+)\big],$$

from which the result follows. □

The next result states that the optimal purchasing strategy and the optimal liquidation strategy are deterministic. Due to the multiplicative relationship between the randomness in the MLOB model, represented by I, and the impact of trading, represented by $\exp\big(\psi_m(E_\cdot^Y)\big)$ and $\exp\big(\psi_m(E_\cdot^X)\big)$, it is not as obvious as in the ALOB model. In fact, in view of equation 5.3, which shows that the MLOB model can be written in the ALOB form, but with a limit order book structure which vary stochastically in time, it may seem rather surprising that the optimal strategy for the MLOB problems can be taken to be deterministic.

Proposition 7.2. *Let $\mathcal{P}_d$ and $\mathcal{S}_d$ denote the set of deterministic strategies in $\mathcal{P}$ and $\mathcal{S}$, respectively. Then*

$$\inf_{X\in\mathcal{P}}\mathbb{E}\big[C_m^p(X)\big]=\inf_{X\in\mathcal{P}_d}\mathbb{E}\big[C_m^p(X)\big] \tag{7.3}$$

and

$$\sup_{Y\in\mathcal{S}}\mathbb{E}\big[C_m^s(Y)\big]=\sup_{Y\in\mathcal{S}_d}\mathbb{E}\big[C_m^s(Y)\big]. \tag{7.4}$$

Proof. We start with the proof of equation (7.4). Let δ be a non-negative $C^\infty(\mathbb{R})$ function with support in $[0,1]$ satisfying $\int_0^1 \delta(x)\,dx = 1$, and define a sequence of functions $\{\delta^{(n)}\}_{n=1}^\infty$ by

$$\delta^{(n)}(x) = n\delta(nx), \qquad x \in \mathbb{R}.$$

Further, define two sequences of functions $\{\psi^{(n)}\}_{n=1}^\infty$ and $\{\Phi^{(n)}\}_{n=1}^\infty$ by

$$\psi^{(n)}(x) = \int_0^1 \psi_m(x+s)\delta^{(n)}(s)\,ds, \qquad x \in \mathbb{R},$$

and

$$\Phi^{(n)}(x) = \int_0^x \exp\bigl(\psi^{(n)}(u)\bigr)\,du, \qquad x \in \mathbb{R}.$$

Then, $\psi^{(n)}$ and $\Phi^{(n)}$ are continuously differentiable functions, for every $n \in \mathbb{N}$. Since $\psi_m(x)$ is right-continuous for every $x \le 0$ and continuous at $x = 0$, it follows that, for every $x \le 0$, $\psi^{(n)}(x)$ converges pointwise to $\psi_m(x)$ and $\Phi^{(n)}(x)$ converges pointwise to $\Phi(x)$ given by

$$\Phi(x) = \int_0^x \mathrm{e}^{\psi_m(u)}\,du, \qquad x \in \mathbb{R},$$

as n tends to infinity. Observe that

$$0 \le -\int_0^T B_t\,\mathrm{e}^{\psi^{(n)}(E^Y_{t-})}\,dY^c_t - \sum_{0\le t\le T}\int_0^{\triangle Y_t} B_t\,\mathrm{e}^{\psi^{(n)}(E^Y_{t-}+y)}\,dy \le \int_0^T B_t\,dY_t.$$

From the previous inequality and the dominated convergence theorem, it follows that

$$\mathbb{E}\bigl[C^s_m(Y)\bigr] = \lim_{n\to\infty}\mathbb{E}\Bigl[-\int_0^T B_t\,\mathrm{e}^{\psi^{(n)}(E^Y_{t-})}\,dY^c_t - \sum_{0\le t\le T}\int_0^{\triangle Y_t} B_t\,\mathrm{e}^{\psi^{(n)}(E^Y_{t-}+y)}\,dy\Bigr].$$

To ease notation, denote by $B^{(n)}$ the process given by

$$B^{(n)}_t = B_t\,\mathrm{e}^{\psi^{(n)}(E^Y_t)}, \qquad t \ge 0,$$

and set

$$C_n(Y) = -\int_0^T B_t\,\mathrm{e}^{\psi^{(n)}(E^Y_{t-})}\,dY^c_t - \sum_{0\le t\le T}\int_0^{\triangle Y_t} B_t\,\mathrm{e}^{\psi^{(n)}(E^Y_{t-}+y)}\,dy.$$

We calculate

$$C_n(Y) = -\int_0^T B^{(n)}_{t-}\,dY^c_t - \sum_{0\le t\le T} B^{(n)}_{t-}\triangle Y_t - \sum_{0\le t\le T} B^{(n)}_{t-}\int_0^{\triangle Y_t}\Bigl\{\mathrm{e}^{\psi^{(n)}(E^Y_{t-}+y)-\psi^{(n)}(E^Y_{t-})} - 1\Bigr\}\,dy$$

(7.5)

$$= -\int_0^T B^{(n)}_{t-}\,dY_t - \sum_{0\le t\le T} B^{(n)}_{t-}\int_0^{\triangle Y_t}\Bigl\{\mathrm{e}^{\psi^{(n)}(E^Y_{t-}+y)-\psi^{(n)}(E^Y_{t-})} - 1\Bigr\}\,dy.$$

Observe that the quadratic co-variation $[B^{(n)}, Y]$ between $B^{(n)}$ and Y is

$$[B^{(n)}, Y]_T = \sum_{0\le t\le T} B^{(n)}_{t-}\left\{ \mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}+\triangle Y_t\right)-\psi^{(n)}\left(E^Y_{t-}\right)} - 1\right\}\triangle Y_t,$$

since Y has finite variation and hence the quadratic co-variation is the sum of the products of the jumps (see e.g. Protter [13]). Therefore,

$$\begin{aligned} B^{(n)}_T Y_T - B^{(n)}_{0-}Y_{0-} = &\int_0^T B^{(n)}_{t-}\, dY_t + \int_0^T Y_{t-}\, dB^{(n)}_t \\ &+ \sum_{0\le t\le T} B^{(n)}_{t-}\left\{ \mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}+\triangle Y_t\right)-\psi^{(n)}\left(E^Y_{t-}\right)} - 1\right\}\triangle Y_t. \end{aligned} \tag{7.6}$$

With reference to the definition of $B^{(n)}$, we calculate that

$$\begin{aligned} \int_0^T Y_{t-}\, dB^{(n)}_t = &\int_0^T Y_{t-}B^{(n)}_{t-}\, dM_t - \int_0^T h\big(E^Y_{t-}\big)(\psi^{(n)})'\big(E^Y_{t-}\big)Y_{t-}B^{(n)}_{t-}\, dt \\ &+ \int_0^T (\psi^{(n)})'\big(E^Y_{t-}\big)Y_{t-}B^{(n)}_{t-}\, dY^c_t \\ &+ \sum_{0\le t\le T} Y_{t-}B^{(n)}_{t-}\big\{ \mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}+\triangle Y_t\right)-\psi^{(n)}\left(E^Y_{t-}\right)} - 1\big\}. \end{aligned} \tag{7.7}$$

Equation (7.6) provides an expression for $\int_0^T B^{(n)}_{t-}\, dY_t$, which combined with (7.5) and (7.7), imply that

$$\begin{aligned} C_n(Y) = &-B^{(n)}_T Y_T + \int_0^T Y_{t-}B^{(n)}_{t-}\, dM_t - \int_0^T h\big(E^Y_{t-}\big)(\psi^{(n)})'\big(E^Y_{t-}\big)Y_{t-}B^{(n)}_{t-}\, dt \\ &+ \int_0^T (\psi^{(n)})'\big(E^Y_{t-}\big)Y_{t-}B^{(n)}_{t-}\, dY^c_t + \sum_{0\le t\le T} Y_{t-}B^{(n)}_{t-}\left\{ \mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}+\triangle Y_t\right)-\psi^{(n)}\left(E^Y_{t-}\right)} - 1\right\} \\ &- \sum_{0\le t\le T} B^{(n)}_{t-}\int_0^{\triangle Y_t}\left\{ \mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}+y\right)-\psi^{(n)}\left(E^Y_{t-}\right)} - 1\right\} dy \\ &+ \sum_{0\le t\le T} B^{(n)}_{t-}\left\{ \mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}+\triangle Y_t\right)-\psi^{(n)}\left(E^Y_{t-}\right)} - 1\right\}\triangle Y_t. \end{aligned} \tag{7.8}$$

Introduce a sequence of functions $\{G_n\}_{n=1}^\infty$, where $G_n : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$G_n(x, y, z) = x\left(y\mathrm{e}^{\psi^{(n)}(z)} + \int_z^{z-y} \mathrm{e}^{\psi^{(n)}(s)}\, ds\right).$$

Then by Itô's formula,

$$G_n\big(B_T, Y_T, E^Y_T\big) = G_n\big(B_0, Y_{0-}, E^Y_{0-}\big) + \int_0^T \left\{ Y_{t-}\mathrm{e}^{\psi^{(n)}(E^Y_{t-})} + \int_{E^Y_{t-}}^{E^Y_{t-}-Y_{t-}} \mathrm{e}^{\psi^{(n)}(s)}\, ds\right\} B_t\, dM_t$$

$$
\begin{aligned}
&+\int_0^T (\psi^{(n)})'\big(E^Y_{t-}\big)Y_{t-}B^{(n)}_{t-}\,dY^c_t - \int_0^T h\big(E^Y_{t-}\big)(\psi^{(n)})'\big(E^Y_{t-}\big)Y_{t-}B^{(n)}_{t-}\,dt \\
&-\int_0^T h\big(E^Y_{t-}\big)\Big\{\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}-Y_{t-}\right)-\psi^{(n)}\left(E^Y_{t-}\right)}-1\Big\}B^{(n)}_{t-}\,dt \\
&+\sum_{0\le t\le T} B^{(n)}_{t-}Y_{t-}\Big\{\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}+\triangle Y_t\right)-\psi^{(n)}\left(E^Y_{t-}\right)}-1\Big\} \\
&+\sum_{0\le t\le T}\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}+\triangle Y_t\right)}B^{(n)}_{t-}\triangle Y_t + B_t\sum_{0\le t\le T}\int_{E^Y_{t-}+\triangle Y_t}^{E^Y_{t-}}\mathrm{e}^{\psi^{(n)}(s)}\,ds.
\end{aligned}
$$

This provides an expression for $\int_0^T (\psi^{(n)})'(E^Y_{t-})Y_tB^{(n)}_{t-}\,dY^c_t$, which inserted in (7.8) implies that

$$
\begin{aligned}
C_n(Y) &= -B^{(n)}_T Y_T + G_n\big(B_T,Y_T,E^Y_T\big) - G_n\big(B_0,Y_{0-},E^Y_{0-}\big) \\
&\quad -\int_0^T\bigg(\int_{E^Y_{t-}}^{E^Y_{t-}-Y_{t-}}\mathrm{e}^{\psi^{(n)}(s)}\,ds\bigg)B_t\,dM_t \\
&\quad +\int_0^T h\big(E^Y_{t-}\big)\Big\{\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}-Y_{t-}\right)}-\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}\right)}\Big\}B_t\,dt \\
&= -B_0\Phi^{(n)}\big(-\overline{Y}\big) + \int_0^T\Big(\Phi^{(n)}\big(E^Y_{t-}\big)-\Phi^{(n)}\big(E^Y_{t-}-Y_{t-}\big)\Big)B_t\,dM_t \\
&\quad +\int_0^T h\big(E^Y_{t-}\big)\Big\{\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}-Y_{t-}\right)}-\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}\right)}\Big\}B_t\,dt.
\end{aligned}
$$

Now introduce a measure $\widetilde{\mathbb{P}}$ given by

$$
\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \frac{B_t}{B_0},
$$

which in view of (2.1) is a probability measure. Then

$$
\begin{aligned}
\mathbb{E}\big[C^s_m(Y)\big] &= \lim_{n\to\infty}\mathbb{E}\big[C_n(Y)\big] \\
&= -B_0\lim_{n\to\infty}\Phi^{(n)}\big(-\overline{Y}\big) \\
&\quad + B_0\lim_{n\to\infty}\int_0^T\widetilde{\mathbb{E}}\Big[h\big(E^Y_{t-}\big)\Big\{\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}-Y_{t-}\right)}-\mathrm{e}^{\psi^{(n)}\left(E^Y_{t-}\right)}\Big\}\Big]\,dt \\
&= -B_0\Phi\big(-\overline{Y}\big) + B_0\widetilde{\mathbb{E}}\Big[\int_0^T h\big(E^Y_{t-}\big)\Big\{\mathrm{e}^{\psi_m\left(E^Y_{t-}-Y_{t-}\right)}-\mathrm{e}^{\psi_m\left(E^Y_{t-}\right)}\Big\}\,dt\Big].
\end{aligned}
$$

Since

$$
Y\mapsto\int_0^T h\big(E^Y_{t-}\big)\Big\{\mathrm{e}^{\psi_m\left(E^Y_{t-}-Y_{t-}\right)}-\mathrm{e}^{\psi_m\left(E^Y_{t-}\right)}\Big\}\,dt
$$

is a deterministic function of Y, equation (7.4) follows.

By similar arguments as in the proof of equation (7.4), one can show that

$$\mathbb{E}\big[C^p_m(X)\big] = B_0\Phi(\overline{Y}) - B_0\widetilde{\mathbb{E}}\bigg[\int_0^T h\big(E^X_{t-}\big)\Big\{\mathrm{e}^{\psi_m\left(E^X_{t-}-X_{t-}+\overline{Y}\right)} - \mathrm{e}^{\psi_m\left(E^X_{t-}\right)}\Big\}\,dt,$$

from which equation (7.3) follows. □

We can now proceed by deriving a relationship between the optimal liquidation problem for the MLOB model and the optimal liquidation for the ALOB problem. Let $X \in \mathcal{P}$ be a deterministic purchasing strategy. If the ALOB and the MLOB models are equally calibrated at time 0, then with reference to equation (5.4), we calculate that

$$\begin{aligned}\mathbb{E}\big[C^p_m(X)\big] &= \int_0^T A_0\mathrm{e}^{\psi_m(E^X_{t-})}\,dX^c_t + \sum_{0\le t\le T}\int_0^{\triangle X_t} A_0\mathrm{e}^{\psi_m(E^X_{t-}+x)}\,dx\\ &= \int_0^T \big(A_0+\psi_a(E^X_{t-})\big)\,dX^c_t + \sum_{0\le t\le T}\int_0^T \big(A_0+\psi_a(E^X_{t-}+x)\big)\,dx\\ &= \mathbb{E}\big[C^p_a(X)\big]. \qquad (7.9)\end{aligned}$$

Similarly, if $Y \in \mathcal{S}$ is a deterministic liquidation strategy and the ALOB and the MLOB models are equally calibrated at time 0, then

$$\begin{aligned}\mathbb{E}\big[C^s_m(Y)\big] &= -\int_0^T B_0\mathrm{e}^{\psi_m(E^Y_{t-})}\,dY^c_t - \sum_{0\le t\le T}\int_0^{\triangle Y_t} B_0\mathrm{e}^{\psi_m(E^Y_{t-}+y)}\,dy\\ &= -\int_0^T \big(B_0+\psi_a(E^Y_{t-})\big)\,dY^c_t - \sum_{0\le t\le T}\int_0^T \big(B_0+\psi_a(E^Y_{t-}+y)\big)\,dy\\ &= \mathbb{E}\big[C^s_a(Y)\big]. \qquad (7.10)\end{aligned}$$

We can now state the main result of the paper.

Theorem 7.3. *Suppose that the ALOB model and the MLOB model are equally calibrated at time* 0. *Then an optimal purchasing strategy $X^* \in \mathcal{P}$ and an optimal liquidation strategy $Y^* \in \mathcal{S}$ for the MLOB execution problems exist, and the following holds.*

(i) $X^* \in \mathcal{P}$ is an optimal purchasing strategy for the ALOB problem formulated in Definition 3.1 if and only if X^* is an optimal liquidation strategy for the MLOB problem formulated in Definition 4.1. Moreover, the optimal expected purchasing cost $\mathbb{E}\big[C^p_a(X^*)\big]$ for the ALOB problem is equal to the optimal expected purchasing cost $\mathbb{E}\big[C_m(Y^*)\big]$ for the MLOB problem.

(ii) $Y^* \in \mathcal{S}$ is an optimal liquidation strategy for the ALOB problem formulated in Definition 3.2 if and only if Y^* is an optimal liquidation strategy for the MLOB problem formulated in Definition 4.2. Moreover, the optimal expected cash amount $\mathbb{E}\big[C^s_a(Y^*)\big]$ for the ALOB problem is equal to the optimal expected cash amount $\mathbb{E}\big[C^s_m(Y^*)\big]$ for the MLOB problem.

Proof. The proof for the optimal purchasing case follows from Lemma 7.1, Proposition 7.2 and equation (7.9). The proof for the optimal liquidation case follows from Proposition 6.1, Lemma 7.1, Proposition 7.2 and equation (7.10). ☐

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