OPTIMAL LIQUIDATION IN A LIMIT ORDER BOOK FOR A RISK AVERSE INVESTOR

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Abstract. In a limit order book model with exponential resilience, general shape function, and an unaffected stock price following the Bachelier model, we consider the problem of optimal liquidation for an investor with constant absolute risk aversion. We show that the problem can be reduced to a two dimensional deterministic problem which involves no buy orders. We derive an explicit expression for the value function and the optimal liquidation strategy. The analysis is complicated by the fact that the intervention boundary, which determines the optimal liquidation strategy, is discontinuous if there are levels in the limit order book with relatively little market depth. Despite this complication, the equation for the intervention boundary is fairly simple. We show that the optimal liquidation strategy possess the natural properties one would expect, and provide an explicit example for the case where the limit order book has a constant shape function.

1. Introduction

The growing popularity of algorithmic execution has resulted in an increasing interest in asset price models incorporating illiquidity and optimal execution of large orders. A brief history of the growth of algorithmic execution can be found in Aldridge [1], which also provides an overview of current market practise and common models. These models tend to be based on the investor's trading rate, like the Almgren and Chriss [2] model, or to be variations of the limit order book model of Obizhaeva and Wang [17].

Bertsimas and Lo [9] introduced a class of discrete time models for asset prices incorporating illiquidity effects, and used dynamic programming to derive the strategy which minimize the expected cost of trading. Almgren and Chriss [2],[3] addressed the problem of maximizing the expected revenue of trading, taking into account its variance. Almgren [4] generalized the model of Almgren and Chriss [2],[3] to incorporate non-linear impact functions. Gatheral [13] further generalized the Almgren model [4] to incorporate a decay kernel and provide conditions for existence and absence of price manipulation strategies. Based on the model of Almgren [4] with a linear impact factor, Schied and Schöneborn [21] study the problem of optimal liquidation of a large stock position for an investor aiming to maximize the expected utility of their cash position at the end of time, and compare the qualitative behaviour of the optimal strategy for different utility functions in case of

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increasing and decreasing prices. They relate their results to the strategies corresponding to the various criteria discussed in Kissel and Malamut [16], which involves different benchmark price targets. The models listed above have in common that the underlying price process is based on the Bachelier model, with an additional impact depending on the transactions made by a large trader who wants to purchase or sell a large number of shares. There is a permanent effect depending on the size of the trade, and a temporary effect on the stock price depending on the time derivative of the large trader's position in the stock. The problem of finding the optimal liquidation or purchase strategy typically takes the form as the solution to an Euler-Lagrange equation.

Obizhaeva and Wang [17] introduced a limit order book model, which specifies the stochastic dynamics of limit orders and the best offered bid and ask price. The cost of purchasing or selling shares thus depend on the limit orders and the behaviour of the best offered bid and ask price as orders eat into the limit order book, as well as the resilience of the limit order book, which specifies how the best offered bid and ask prices recover after orders have been executed. Alfonsi, Fruth and Schied [5] generalize the limit order book model of Obizhaeva and Wang to allow more general shape functions, and provide two slightly different ways to model resilience. Predoiu, Shaikhet and Shreve [18] introduce a one-sided limit order book model where the shape of the limit orders are given in terms of a measure. Alfonsi and Schied [6] and Alfonsi, Schied and Slynko [7] provide conditions for the absence of price manipulation strategies in limit order books and models related to that of Gatheral [13]. Gatheral, Schied and Slynko [14] introduce a limit order book with a general resilience function and provide a comparison between such models and that of Gatheral [13]. Schied, Schöneborn and Tehranchi [22] provide a connection between the maximization criterion of Almgren and Chriss [2],[3] and maximization of expected CARA utility. They also show that if the risk of the stock price is given in terms of a Lévy process and if there is a certain additive structure between the risk and the trading cost, then the optimal strategy for an investor with CARA utility is deterministic.

In this paper we adopt a limit order book model with a general shape function and exponential resilience rate as in Alfonsi, Fruth and Schied [5], where the stochastic dynamics of the non-affected stock price follows the Bachelier model, and consider a large investor with constant absolute risk aversion who wants to liquidate his share position without time constraints. This model for the dynamics of the risky asset can be viewed as the limit order book equivalent to the models of Almgren and Chriss [2], Almgren [4] and Gatheral [13]. We do not explicitly model the ask orders and best ask price, but assume that the best bid price and bid orders are unaffected by the investor's buy orders and that the unaffected bid price process provides a lower bound for the best offered ask price. The utility maximization problem that we consider can be viewed as a limit order book version of the problem considered by Schied and Schöneborn [21], provided that the investor has constant absolute risk aversion. Based on the ideas in Schied, Schöneborn and Tehranchi [22], we prove that the optimal strategy is deterministic and involves no buy orders. Moreover, the maximum expected utility is strictly negative and can be expressed in terms of the value function of a certain two dimensional optimal control problem.

Under fairly mild condition on the shape function of the limit order book, we derive an explicit solution to the optimal liquidation problem for a general shape function. The optimal strategy turns out to differ from the corresponding optimal strategy in the Almgren [4] model. Depending on the investor's past trading history, the optimal liquidation strategy involves an initial block trade and then continuously sell shares, or first wait for a certain amount of time to let the limit order book recover, and then continuously sell shares. If there are levels in the limit order book with relatively little market depth, there are periods where it is optimal to wait in order for the best bid price to recover from the level with relatively little depth. To the best of the author's knowledge, this Markovian dependence on the investor's past trading history and corresponding state of the limit order book has not previously been explicitly explored, and in the Almgren model [4] there is no such dependence since the returns in this case do not depend on the investor's past trading history. Also, the optimal liquidation strategy in models based on the trading rate, like the Almgren [4] model, does not involve any block trades. It is therefore interesting to note that the optimal liquidation strategy for the large investor in a limit order book model typically consists of an initial block trade.

In order to derive the optimal liquidation strategy for the large investor, we derive an explicit solution to the Hamilton-Jacobi-Bellman equation corresponding to the value function of the reduced deterministic optimization problem. The Hamilton-Jacobi-Bellman equation takes the form of a free boundary problem, where it turns out that the boundary can be discontinuous. The discontinuities happen when there are levels in the limit order book with relatively little market depth. While the discontinuities of the intervention boundary complicates the analysis and proofs, the equation for the boundary takes a fairly simple form. The optimal strategy then essentially consist of trading in such a way that the state process, which consists of the number of shares held and the current state of the limit order book, remain on the boundary at all times. We also show that the optimal strategy possess the natural properties one would expect, e.g. that an increased depth implies faster liquidation, increased volatility of the unaffected stock price implies faster liquidation, and an increased risk aversion implies faster liquidation.

At the end, we provide an example of a limit order book with a constant shape function, and compare the optimal liquidation strategy to that provided in Schied and Schöneborn [21] for the Almgren model with a linear impact function.

2. Model specification and problem formulation

We adopt the limit order book model of Alfonsi, Fruth and Schied [5] with an unaffected best bid price process following the Bachelier model. However, instead of modelling the full limit order book, we explicitly model the bid order book and assume that the unaffected bid price provides a lower bound for the best ask price and that the best bid price and bid prices are unaffected by the large investor's buy orders. These assumptions are satisfied in the full limit order book model. We show that under these assumptions, the optimal liquidation strategy does not involve any buy orders.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions and supporting a one-dimensional Brownian motion W, and set

$$\mathbb{R}^+ = [0, \infty) \quad \text{and} \quad \mathbb{R}^- = (-\infty, 0].$$

We consider the following set of admissible liquidation strategies for the large investor.

Definition 2.1. For $y \in \mathbb{R}^+$, let $\mathcal{A}(y)$ denote the set of all pairs (X, Y), where X and Y are $(\mathcal{F}_t)$-adapted, càdlàg processes, X is non-decreasing and Y is non-increasing, $X_{0-} = 0$ and $Y_{0-} = y$ and

$$\int_0^\infty \|X_t + Y_t\|^2_{L^\infty(\Omega)}\, dt < \infty. \tag{2.1}$$

The non-negative quantity X_t represents the number of shares bought over the time interval $[0, t]$, $Y_t - y$ represents the number of shares sold over the time interval $[0, t]$ and $X_t + Y_t$ is the net position in shares held at time t. Denote by $\mathcal{A}_D^-(y)$ the set of all deterministic strategies $(X, Y) \in \mathcal{A}(y)$ with $X = 0$. Thus, we can identify $\mathcal{A}_D^-(y)$ with the set of deterministic càdlàg, non-increasing processes with values in $[0, y]$ satisfying $Y_{0-} = y$ and

$$\int_0^\infty |Y_t|^2\, dt < \infty. \tag{2.2}$$

The unaffected bid process B^0 is assumed to follow the Bachelier model, that is

$$B_t^0 = b + \sigma W_t, \qquad t \geq 0,$$

where $b > 0$ is the bid price at time 0 and $\sigma > 0$ is the volatility. The interpretation of the unaffected bid process B^0 is that if the large investor make no trades, then the best offered bid price at time t is B_t^0. The Bachelier model may seem simplistic, but the Bachelier model is widely used in the optimal liquidation literature (see e.g. Almgren and Criss [2], Kissel and Malamut [16], Schied and Schöneborn [21] and Gatheral [13]). We do not explicitly model the ask prices, but make the following assumption

Assumption 2.2. The best bid prices are unaffected by the large investor's buy orders and the best unaffected bid price provides a lower bound for the best ask price.

In particular we note that the full limit order book in Obizhaeva and Wang [17], Alfonsi, Fruth and Schied [5] and Gatheral, Schied and Slynko [14], all satisfy this assumption.

There are two other components which together specify the limit order book model. One is the shape function, which we denote by ϕ, and the other is the resilience rate, which describes how the market recover. The shape function ϕ is static, and the connection between the shape function ϕ and the bid prices in the limit order book is that, at time t, the number of bids at price $B_t + x$ is equal to $\phi(x)\, dx$, where $x \leq 0$, provided the investor has not made any large trades before time t. We impose the following condition on the shape function.

Assumption 2.3. The shape function $\phi : \mathbb{R}^- \to (0, \phi_{\max}]$ is continuous.

Let B_t^Y denote the best bid price offered at time t if the large investor follows a liquidation strategy $(X, Y) \in \mathcal{A}(y)$. As the notation suggests, the best offered bid price depends on the past history of the strategy Y, but according to Assumption 2.2, does not depend on the buy strategy X. We assume that B^Y is a càdlàg process. Denote by D^Y the spread process given by

$$D_t^Y = B_t^Y - B_t^0, \qquad t \geq 0,$$

i.e. the spread between the best offered bid price and the unaffected bid price if the large investor adopts a liquidation strategy Y. If the large trader adopts a strategy Y which consists of selling a number $\triangle Y_t = Y_t - Y_{t-}$ of shares at time t, then the effect of this on the best offered bid price is that the new spread changes from D_{t-}^Y to D_t^Y, where D_t^Y satisfies

$$\int_{D_{t-}^Y}^{D_t^Y} \phi(u)\, du = \triangle Y_t.$$

This corresponds to the best bid orders being executed in order to match the large trader's sales order of $-\triangle Y_t$ number of shares. In order to ease notation, introduce the functions $\Phi : \mathbb{R}^- \to \mathbb{R}^-$ and $\psi : \mathbb{R}^- \to \mathbb{R}^-$ by

$$\Phi(x) = \int_0^x \phi(u)\, du \quad \text{and} \quad \psi(z) = \Phi^{-1}(z). \tag{2.3}$$

The inverse of Φ is well defined since Φ is strictly increasing, due to the assumption that ϕ takes strictly positive values. From the assumptions made on the properties of ϕ in Assumption 2.3, it follows that $\psi : \mathbb{R}^- \to \mathbb{R}^-$ is an increasing $C^1(\mathbb{R}^-)$ function satisfying

$$\psi(0) = 0, \tag{2.4}$$

$$\text{there exists } \delta > 0 \text{ such that } \psi'(z) \geq \delta, \text{ for all } z \in \mathbb{R}^-, \tag{2.5}$$

$$\text{there exists } C > 0 \text{ and } \epsilon > 0 \text{ such that } \psi'(z) \leq C, \text{ for all } z \in (-\epsilon, 0]. \tag{2.6}$$

As in Alfonsi, Fruth and Schied [5] and Obizhaeva and Wang [17], we assume that the limit order book has an exponential resilience rate, which means that the limit order book recovers at an exponential rate. Introduce the process Z^Y given by

$$Z_t^Y = z\mathrm{e}^{-\lambda t} + \int_0^t \mathrm{e}^{-\lambda(t-s)}\, dY_s, \qquad t \geq 0, \tag{2.7}$$

where $z \leq 0$ is the initial value of Z^Y at time 0 and $\lambda > 0$ is the resilience speed. For future reference, note that Z^Y is the unique càdlàg solution to

$$dZ_t^Y = -\lambda Z_{t-}^Y\, dt + dY_t, \qquad Z_0^Y = z \in \mathbb{R}^-. \tag{2.8}$$

The process Z^Y captures how the large investor's implementation of the liquidation strategy Y affects the best offered bid price and how this recovers over time through the relation

$$Z_t^Y = \Phi(D_t^Y),$$

where D_t^Y is the spread at time t. The initial state z therefore provides the initial state of the limit order book, which takes into account the past trading history of the large investor. In the literature the letter E is often used rather than Z to denote the process Z^Y, since it represent the part of the order book which is "eaten up". With reference to the definition of the spread process D^Y and (2.3), the best offered bid price B_t^Y, at time t, is given by

$$B_t^Y = B_t^0 + \psi(Z_t^Y), \tag{2.9}$$

if the large investor use the liquidation strategy Y.

So far we have described how the large investor's trading affect the best offered bid price, but not how this affects the large investor's cash position. Suppose that the large investor's initial cash position is c and that he implements a strategy $(X, Y) \in \mathcal{A}(y)$ which consists of a number of block sales, i.e. Y is a decreasing step function and X is zero. Then the large investor's cash position at time $T > 0$ is

$$C_T(X,Y) = c - \sum_{0 \le t \le T} \int_0^{\triangle Y_t} \big\{B_t^0 + \psi\big(Z_{t-}^Y + x\big)\big\}\, dx, \tag{2.10}$$

which corresponds to the best bids offered at all times being executed first so as to match the large trader's sales orders. Let Y^c denote the continuous part of Y, as defined in e.g. Protter [19]. If the large trader implements a continuous sales strategy $Y = Y^c$ with no buy orders, then his cash position at time $T > 0$ is given by the Lebesgue-Stieltjes integral

$$\int_0^T B_{t-}^Y \, dY_t. \tag{2.11}$$

By assumption the best ask prices are greater than or equal to the unaffected bid prices, from which it follows that the cost of purchasing the shares at the best offered ask prices is greater than or equal to purchasing the shares at the unaffected bid prices. In view of (2.10) and (2.11), we therefore conclude that if the large investor's initial cash position is c and he use a liquidation strategy $(X, Y) \in \mathcal{A}(y)$, his cash position at time $T > 0$ satisfies

$$C_T(X,Y) \le c - \int_0^T B_{t-}^Y \, dY_t^c - \sum_{0 \le t \le T} \int_0^{\triangle Y_t} \big\{B_t^0 + \psi\big(Z_{t-}^Y + x\big)\big\}\, dx - \int_0^T B_t^0 \, dX_t, \tag{2.12}$$

where we have equality in (2.12) if $X = 0$. We note that (2.12) can be viewed as a version of Alfonsi and Schied [6, Proposition 2.22].

We assume that the large trader has constant absolute risk aversion, an initial cash position c and an initial position in the stock consisting of y number of shares. We further assume that the large trader wants to maximize the expected utility of his cash position at the end of time. In mathematical terms, the large investor's optimal liquidation problem is

$$\sup_{(X,Y) \in \mathcal{A}(y)} \mathbb{E}\big[U\big(C_\infty(X,Y)\big)\big], \tag{2.13}$$

where the utility function U is given by

$$U(c) = -\mathrm{e}^{-Ac}, \qquad A > 0.$$

This can be seen as the limit order book equivalent formulation of the optimal liquidation problem studied by Schied and Schöneborn [21], but restricted to the case of large investors with constant absolute risk aversion.

3. Preliminary observations and problem simplification

Our approach to solving the utility maximization problem (2.13) is to first show that the problem can be reduced to a deterministic optimization problem involving only liquidation strategies in $\mathcal{A}_D^-(y)$. This reduction of the problem is based on the ideas in Schied, Schöneborn and Tehranchi [22], who proved that if the market has a certain structure and the investor has a constant absolute risk aversion, the optimal strategy is deterministic.

Let $(X,Y) \in \mathcal{A}(y)$ be an admissible liquidation strategy. Then it follows from (2.12) that

$$C_T(X,Y) \le c + by - B_T^0(X_T+Y_T) + \int_0^T \big(X_{t-}+Y_{t-}\big)\sigma\, dW_t - F_T(Y), \tag{3.1}$$

where F is given by

$$F_T(Y) = \int_0^T \psi\big(Z_{t-}^Y\big)\, dY_t^c + \sum_{0\le t\le T} \int_0^{\triangle Y_t} \psi\big(Z_{t-}^Y + x\big)\, dx, \tag{3.2}$$

and where we have equality in (3.1) if $X=0$. It follows from (2.1) that $B_T^0(X_T+Y_T)$ tends to 0 in $L^1(\mathbb{P})$ as $T\to\infty$, and that

$$\int_0^\infty (X_{t-}+Y_{t-})\sigma\, dW_t$$

is a well defined random variable with expectation 0 and finite variance. Also note that $F_T(Y)$ is an increasing function of T, and therefore $F_\infty(Y)$ is well defined, possibly being equal to $+\infty$. Thus F_∞ is a function from the set of càdlàg, non-increasing functions into the extended non-negative real numbers. We conclude that

$$C_\infty(X,Y) \le c + by + \int_0^\infty \big(X_{t-}+Y_{t-}\big)\sigma\, dW_t - F_\infty(Y), \tag{3.3}$$

where we have equality in (3.3) if $X=0$. Also note that

$$C_\infty(X,Y) \le c + by + \int_0^\infty \big(X_{t-}+Y_{t-}\big)\sigma\, dW_t,$$

from which it follows that the market is arbitrage free. In particular there does not exist any price manipulation strategies (see Huberman and Stanzl [15], Gatheral [13] or Alfonsi and Schied [6]). We have the following monotonicity result for the function F and deterministic strategies.

Lemma 3.1. *Let F be given by (3.2) and $(X,Y)\in\mathcal{A}(y)$ be any deterministic liquidation strategy. Then there exists a strategy $\widetilde{Y}\in\mathcal{A}_D^-(y)$ such that*

$$\int_0^\infty \big(X_{t-}+Y_{t-}\big)^2\, dt \ge \int_0^\infty \widetilde{Y}_{t-}^2\, dt \quad \textit{and} \quad F_\infty(Y) \ge F_\infty(\widetilde{Y}).$$

Proof. Let ξ be a càdlàg, non-increasing function satisfying $\xi_{0-}=0$. Then

$$Z_t^{Y+\xi}-Z_t^Y=-\lambda\int_0^t\big(Z_{u-}^{Y+\xi}-Z_{u-}^Y\big)\,du+\xi_t,$$

from which it follows that $Z_t^{Y+\xi}\le Z_t^Y$, for all $t\ge 0$. Therefore

$$\begin{aligned}F_\infty(Y+\xi)&=\int_0^\infty\psi\big(Z_{t-}^{Y+\xi}\big)\,dY_t^c+\int_0^\infty\psi\big(Z_{t-}^{Y+\xi}\big)\,d\xi_t^c\\&\qquad+\sum_{t\ge0}\int_0^{\triangle Y_t}\psi\big(Z_{t-}^{Y+\xi}+x\big)\,dx+\sum_{t\ge0}\int_{\triangle Y_t}^{\triangle Y_t+\triangle\xi_t}\psi\big(Z_{t-}^{Y+\xi}+x\big)\,dx\\&\ge\int_0^\infty\psi\big(Z_{t-}^Y\big)\,dY_t^c+\sum_{t\ge0}\int_0^{\triangle Y_t}\psi\big(Z_{t-}^Y+x\big)\,dx\\&=F_\infty(Y),\end{aligned}$$

by the monotonicity of ψ. Thus if $Y\le\widetilde{Y}$ then $F_\infty(Y)\ge F_\infty(\widetilde{Y})$. Let $(X,Y)\in\mathcal{A}(y)$ be a deterministic strategy, and define $\widetilde{Y}_t=\max\{0,Y_t\}$ and $\widetilde{X}=0$. Then the strategy $(\widetilde{X},\widetilde{Y})$ is in $\mathcal{A}_D^-(y)$, and since $Y\le\widetilde{Y}$ and

$$\int_0^\infty\big(X_{t-}+Y_{t-}\big)^2\,dt\ge\int_0^\infty\widetilde{Y}_{t-}^2\,dt,$$

the result follows. □

Let $(X,Y)\in\mathcal{A}(y)$ and define the process M by

$$M_t=\exp\bigg(-\sigma A\int_0^t\big(X_{s-}+Y_{s-}\big)\,dW_s-\frac{1}{2}\sigma^2A^2\int_0^t\big(X_{s-}+Y_{s-}\big)^2\,ds\bigg).$$

From the assumption (2.1) it follows that M is a martingale closed by M_∞ (see e.g. Protter [19]). We can therefore define a probability measure $\widetilde{\mathbb{P}}$ by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}=M_\infty.$$

Based on the ideas of Schied, Schöneborn and Tehranchi [22, Theorem 2.8], (3.3) and Lemma 3.1, we calculate

$$\begin{aligned}&\sup_{(X,Y)\in\mathcal{A}(y)}\mathbb{E}\big[U\big(C_\infty(X,Y)\big)\big]\\&\qquad\overset{(*)}{\le}-\mathrm{e}^{-A(c+by)}\inf_{(X,Y)\in\mathcal{A}(y)}\mathbb{E}\bigg[\exp\bigg(-A\int_0^\infty\big(X_{t-}+Y_{t-}\big)\sigma\,dW_t+AF_\infty(Y)\bigg)\bigg]\\&\qquad=-\mathrm{e}^{-A(c+by)}\inf_{(X,Y)\in\mathcal{A}(y)}\mathbb{E}\bigg[M_\infty\exp\bigg(\frac{1}{2}\sigma^2A^2\int_0^\infty\big(X_{t-}+Y_{t-}\big)^2\,dt+AF_\infty(Y)\bigg)\bigg]\\&\qquad=-\mathrm{e}^{-A(c+by)}\inf_{(X,Y)\in\mathcal{A}(y)}\widetilde{\mathbb{E}}\bigg[\exp\bigg(\frac{1}{2}\sigma^2A^2\int_0^\infty\big(X_{t-}+Y_{t-}\big)^2\,dt+AF_\infty(Y)\bigg)\bigg]\end{aligned}$$

(3.4)
$$= -\mathrm{e}^{-A(c+by)} \exp\left(\inf_{Y \in \mathcal{A}_D^-(y)} \left\{ \frac{1}{2}\sigma^2 A^2 \int_0^\infty \big(X_{t-} + Y_{t-}\big)^2 \, dt + AF_\infty(Y) \right\} \right),$$

where $(*)$ holds with equality for strategies $(X, Y) \in \mathcal{A}(y)$ with $X = 0$. In particular $(*)$ in (3.4) holds with equality for strategies $Y \in \mathcal{A}_D^-(y)$. The last step in (3.4) follows from Jensen's inequality and Lemma 3.1. We have therefore reduced the utility maximization problem (2.13) to an optimization problem involving only deterministic liquidation strategies which involve no buy orders.

In order to derive a Hamilton-Jacobi-Bellman equation for the value function of the optimization problem, we want to obtain an expression for F_∞ which is more convenient for this purpose. As a first step towards this, the next result express the cash position at a time $T > 0$ corresponding to a liquidation strategy which involves no buy orders.

Lemma 3.2. *For every initial cash position c and liquidation strategy $(X, Y) \in \mathcal{A}(y)$ satisfying $X = 0$, the large investor's cash position at time $T > 0$ is given by*

$$\begin{aligned} C_T(Y) = c + B_{0-}^0 Y_{0-} - \int_{Z_{0-}^Y}^{Z_{0-}^Y - Y_{0-}} \psi(s)\, ds - B_T^0 Y_T + \int_{Z_T^Y}^{Z_T^Y - Y_T} \psi(s)\, ds \\ + \int_0^T Y_{t-}\, dB_t^0 + \int_0^T \lambda Z_{t-}^Y \{\psi(Z_{t-}^Y - Y_{t-}) - \psi(Z_{t-}^Y)\}\, dt. \end{aligned} \tag{3.5}$$

The expression for the cash position at time T in Lemma 3.2 should be of independent interest. It can be used to derive the Hamilton-Jacobi-Bellman equation for the optimal liquidation problem with a finite time-horizon and a general utility function. Our idea is similar to the idea in Schied and Schöneborn [21], which consists of rewriting the expression for the cash position at the end of time in such a way that you make use of the assumption that the stock position at the end of time is zero. The next result makes use of Lemma 3.2 and provides an expression for the cash position at the end of time in accordance with this idea.

Lemma 3.3. *Assume that the large investor's initial cash position is c and that the large investor use a liquidation strategy $(X, Y) \in \mathcal{A}(y)$, with $X = 0$. Then the large investor's cash position at the end of time is*

$$\begin{aligned} C_\infty(Y) = c + by - \int_z^{z-y} \psi(s)\, ds \\ + \int_0^\infty \sigma Y_{t-}\, dW_t + \int_0^\infty \lambda Z_{t-}^Y \{\psi\big(Z_{t-}^Y - Y_{t-}\big) - \psi\big(Z_{t-}^Y\big)\}\, dt, \end{aligned} \tag{3.6}$$

where $b = B_0^0$ and $z = Z_{0-}^Y$. Moreover,

$$0 \le \int_0^\infty \lambda Z_{t-}^Y \{\psi\big(Z_{t-}^Y - Y_{t-}\big) - \psi\big(Z_{t-}^Y\big)\}\, dt \le \int_z^{z-y} \psi(s)\, ds. \tag{3.7}$$

The various terms in (3.6), for the cash position at the end of time, all have economic interpretations. The term

$$c + by - \int_z^{z-y} \psi(s)\, ds$$

corresponds to the cash position of the large investor after immediate liquidation of the entire position in risky assets. The term

$$\int_0^\infty \sigma Y_{t-}\, dW_t$$

represent the risk of the large investor's cash position at the end of time if the stock position is not liquidated immediately, and

$$\int_0^\infty \lambda Z_{t-}^Y \{\psi(Z_{t-}^Y - Y_{t-}) - \psi(Z_{t-}^Y)\}\, dt$$

represents the gain to the large investor's cash position for not liquidating immediately.

Remark 3.4. From the bounds established in the proof of Lemma 3.6, it follows that if Y^n is a sequence of liquidation strategies in $\mathcal{A}_D^-(y)$ such that Y^n converges to $Y \in \mathcal{A}_D^-(y)$ in total variation, then $C_\infty(Y^n)$ converges to $C_\infty(Y)$ in $L^1(\mathbb{P})$. This shows that the limit order book model exhibits a certain robustness. For instance, given a sequence of absolutely continuous liquidation strategies converging to a liquidation strategy consisting of block trades, the corresponding cash position of the absolutely continuous strategy will converge to the cash position corresponding to that consisting of the block trades. This differs completely from the situation in the supply curve models of Cetin, Jarrow and Protter [10] and Baum and Bank [8].

For strategies $Y \in \mathcal{A}_D^-(y)$, (3.3) takes the form

$$C_\infty(Y) = c + by + \int_0^\infty \sigma Y_{t-}\, dW_t - F_\infty(Y),$$

and by comparing this expression for $C_\infty(Y)$ with the expression for $C_\infty(Y)$ in Lemma 3.3, we conclude that

$$F_\infty(Y) = \int_z^{z-y} \psi(s)\, ds + \int_0^\infty \lambda Z_{t-}^Y \{\psi(Z_{t-}^Y) - \psi(Z_{t-}^Y - Y_{t-})\}\, dt. \tag{3.8}$$

From the calculations in (3.4) and equation (3.8), it follows that the optimization problem (2.13) takes the form

$$\sup_{(X,Y)\in\mathcal{A}(y)} \mathbb{E}\big[U\big(C_\infty(X,Y)\big)\big] = -\exp\left(-A(c+by) + A\int_z^{z-y} \psi(s)\, ds\right) \exp\big(AV(y,z)\big), \tag{3.9}$$

where

$$V(y,z) = \inf_{Y\in\mathcal{A}_D^-(y)} \int_0^\infty \lambda Y_{t-} \left(aY_{t-} + Z_{t-}^Y \frac{\psi(Z_{t-}^Y) - \psi(Z_{t-}^Y - Y_{t-})}{Y_{t-}} \right) dt, \tag{3.10}$$

with $a = \frac{\sigma^2 A}{2\lambda}$ and $z = Z^Y_{0-}$. Thus we have reduced the original utility optimization problem to (3.10). Moreover, from assumption (2.2) and Lemma 3.3, it follows that V given by (3.10) is well defined and real valued.

Remark 3.5. Observe that from Lemma 3.3, it follows that for a liquidation strategy $Y \in \mathcal{A}^-_D(y)$,

$$\mathbb{E}\big[C_\infty(Y)\big] = c + by - \int_z^{z-y} \psi(s)\, ds + \int_0^\infty \lambda Z^Y_{t-}\big\{\psi\big(Z^Y_{t-} - Y_{t-}\big) - \psi\big(Z^Y_{t-}\big)\big\}\, dt$$

and

$$\mathrm{Var}\big(C_\infty(Y)\big) = \int_0^\infty \sigma^2 Y^2_{t-}\, dt.$$

Therefore the right-hand side of (3.10), with $a = \frac{\sigma^2 A}{2\lambda}$, can be written

$$c + by - \int_z^{z-y} \psi(s)\, ds + \inf_{Y \in \mathcal{A}^-_D(y)} \left\{ \frac{A}{2} \mathrm{Var}\big(C_\infty(Y)\big) - \mathbb{E}\big[C_\infty(Y)\big] \right\},$$

which corresponds to the Almgren and Chriss [3] criterion. Schied, Schöneborn and Tehranchi [22] proved that this relationship between the Almgren and Chriss criterion and CARA utility holds also in the more general setting where the unaffected price process follows a Lévy process.

4. The solution to the optimization problem

Our next aim is to derive an explicit solution to the optimization problem (3.10), which will be based on the principle of dynamic programming. With reference to the general theory of optimal control (see e.g. Fleming and Soner [11]), the Hamilton-Jacobi-Bellman equation corresponding to V given by (3.10) takes the form

$$\max\left\{ zv_z(y,z) - ay^2 - z\big(\psi(z) - \psi(z-y)\big),\ \max_{0 \le \triangle \le y} v(y,z) - v(y-\triangle, z-\triangle) \right\} = 0, \tag{4.1}$$

with associated boundary condition $v(0,z) = 0$, for all $z \le 0$. Formally, we can obtain equation (4.1) as follows. As we are optimizing over deterministic selling strategies (no buy orders), there are only two choices; to sell a number $\triangle > 0$ of shares or to wait. Given a state (y,z), it may or may not be optimal to sell a number $\triangle$ of shares, hence

$$v(y,z) \le v\big(y - \triangle, z - \triangle\big), \tag{4.2}$$

as the sale of $\triangle$ number of shares decrease the number of shares held from y to $y - \triangle$ and the state of the limit order book from z to $z - \triangle$. The inequality (4.2) should hold for all $0 \le \triangle \le y$, which implies that

$$\max_{0 \le \triangle \le y} \big\{ v(y,z) - v(y-\triangle, z-\triangle) \big\} \le 0. \tag{4.3}$$

On the other hand, it may or may not be optimal to wait for a period of time $\triangle t > 0$, hence

$$
\begin{aligned}
v(y,z) &\le v\big(y, Z_{\triangle t}\big) + \int_0^{\triangle t} \lambda y\bigg(ay + Z_{u-}\frac{\psi\big(Z_{u-}\big) - \psi\big(Z_{u-} - y\big)}{y}\bigg)\,du \\
&= v(y,z) + \int_0^{\triangle t} \bigg\{\lambda y\bigg(ay + Z_{u-}\frac{\psi\big(Z_{u-}\big) - \psi\big(Z_{u-} - y\big)}{y}\bigg) - v_z\big(y, Z_{u-}\big)\lambda Z_{u-}\bigg\}\,du,
\end{aligned} \tag{4.4}
$$

where $Z_u = z\mathrm{e}^{-\lambda u}$, for $0 \le u \le \triangle t$. By multiplying inequality (4.4) by $(\triangle t)^{-1}$ and letting $\triangle t$ tend to 0, we obtain

$$
z v_z(y,z) - ay^2 - z\big\{\psi(z) - \psi(z-y)\big\} \le 0. \tag{4.5}
$$

Since it is optimal to either sell a certain number of shares or wait, we must have equality in either (4.3) or (4.5), and we obtain (4.1).

For this type of singular optimal control problem, the optimal strategy can be characterized by two disjoint sets $\mathcal{D}_s$ and $\mathcal{D}_w$, where the union of $\mathcal{D}_s$ and $\mathcal{D}_w$ is equal to the state space $\mathbb{R}^+ \times \mathbb{R}^-$. These sets satisfy

$$
\begin{aligned}
\max_{0 \le \triangle \le y} \big\{v(y,z) - v(y - \triangle, z - \triangle)\big\} &= 0, \qquad \text{for } (y,z) \in \mathcal{D}_s, \\
z v_z(y,z) - ay^2 - z\big\{\psi(z) - \psi(z-y)\big\} &= 0, \qquad \text{for } (y,z) \in \mathcal{D}_w.
\end{aligned} \tag{4.6}
$$

If $(y,z) \in \mathcal{D}_s$, then the optimal strategy consists of making an immediate sale of $\triangle$ number of shares, where $\triangle$ is such that $(y - \triangle, z - \triangle)$ is on the nearest boundary between $\mathcal{D}_s$ and $\mathcal{D}_w$ (or the line $y = 0$). If $(y,z) \in \mathcal{D}_w$, then the optimal strategy consists of waiting until the first time $\big(y, z\mathrm{e}^{-\lambda t}\big)$ is on the boundary between $\mathcal{D}_s$ and $\mathcal{D}_w$, as the limit order book recovers at an exponential rate λ. If (y,z) is on the boundary between $\mathcal{D}_s$ and $\mathcal{D}_w$, then the optimal strategy consists of taking minimal action to ensure that the state process (Y_t, Z_t^Y) remain on the boundary.

The simplest case is when $\mathcal{D}_s$ and $\mathcal{D}_w$ are separated by a function h, i.e. $(y,z) \in \mathcal{D}_s$ if $z \ge h(y)$ and $(y,z) \in \mathcal{D}_w$ if $z < h(y)$ (or vice versa). Since

$$
\lambda y\bigg(ay + z\frac{\psi(z) - \psi(z-y)}{y}\bigg) \tag{4.7}
$$

is positive if z is large (i.e. close to zero) compared to y, and increasingly negative for small values of z, it is natural that $(y,z) \in \mathcal{D}_s$ if $z \ge h(y)$ and $(y,z) \in \mathcal{D}_w$ if $z < h(y)$. It turns out that this indeed is the case, and that the boundary between $\mathcal{D}_s$ and $\mathcal{D}_w$ can be described by a strictly decreasing càdlàg function $h : \mathbb{R}^+ \to \mathbb{R}^-$ satisfying $h(0) = 0$ and $\lim_{y\to\infty} h(y) = -\infty$, and that we should be looking for a function v satisfying

$$
v_y(y,z) + v_z(y,z) = 0, \qquad \text{for } z \ge h(y), \tag{4.8}
$$

$$
z v_z(y,z) - ay^2 - z\big(\psi(z) - \psi(z-y)\big) \le 0, \qquad \text{for } z > h(y), \tag{4.9}
$$

and

$$
z v_z(y,z) - ay^2 - z\big(\psi(z) - \psi(z-y)\big) = 0, \qquad \text{for } z \le h(y), \tag{4.10}
$$

$$D_y^+ v(y,z) + v_z(y,z) \le 0, \qquad \text{for } z < h(y), \tag{4.11}$$

where

$$D_y^+ v(y,z) = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\Big(v(y+\epsilon,z) - v(y,z)\Big). \tag{4.12}$$

Let us examine in more detail the strategy corresponding to the case where $\mathcal{D}_s$ and $\mathcal{D}_w$ are described by an intervention boundary function h. Given a strictly decreasing càdlàg function $h : \mathbb{R}^+ \to \mathbb{R}^-$ satisfying $h(0) = 0$ and $\lim_{y\to\infty} h(y) = -\infty$, and corresponding sets $\mathcal{D}_s$ and $\mathcal{D}_w$ given by $(y,z) \in \mathcal{D}_s$ if $z \ge h(y)$ and $(y,z) \in \mathcal{D}_w$ if $z < h(y)$, we might ask whether the corresponding liquidation strategy, denoted by Y^h, exists. For future reference, introduce the following functions related to h:

$$\gamma_h(y) = h(y) - y, \qquad \text{for } y \in \mathbb{R}^+, \tag{4.13}$$

$$\rho_h(z) = z - h^{-1}(z), \qquad \text{for } z \in \mathbb{R}^-, \tag{4.14}$$

$$h^{-1}(z) = \sup\big\{y \in \mathbb{R}^+ \,:\, h(y) \ge z\big\}, \qquad \text{for } z \in \mathbb{R}^-, \tag{4.15}$$

$$\gamma_h^{-1}(z) = \sup\big\{y \in \mathbb{R}^+ \,:\, \gamma_h(y) \ge z\big\}, \qquad \text{for } z \in \mathbb{R}^-, \tag{4.16}$$

and let ρ_h^{-1} denote the inverse of ρ_h. We can then observe that if $(y,z) \in \mathcal{D}_s$, then the strategy Y^h corresponding to the intervention boundary described by h, will consist of making an initial sale of $\triangle$ number of shares such that $(y-\triangle, z-\triangle)$ is on the graph of h (see Figure 1). Hence, we want $(y-\triangle, z-\triangle) = (h^{-1}(z-\triangle), z-\triangle)$. With $Z_{0-}^{Y^h} = z$ and $Z_0^{Y^h} = z - \triangle$, we see that this equation is equivalent to

$$\rho_h\big(Z_0^{Y^h}\big) = Z_0^{Y^h} - h^{-1}(Z_0^{Y^h}) = z - y,$$

from which it follows that $Z_0^{Y^h} = \rho_h^{-1}(z-y)$ and $\triangle = z - \rho_h^{-1}(z-y)$. For future reference, we can also observe that $\rho_h^{-1}(x) = x + \gamma_h^{-1}(x)$, for $x \in \mathbb{R}^-$. Hence, the number $\triangle$ of shares, can also be expressed by $\triangle = y - \gamma_h^{-1}(z-y)$. If $(y,z) \in \mathcal{D}_w$, i.e. $z < h(y)$, then the strategy Y^h consists in waiting until the state process $(Y_t^h, Z_t^{Y^h})$ is on the graph of h (see Figure 1). While no action is taken, $Y_t^h = y$ and $Z_t^{Y^h} = z\mathrm{e}^{-\lambda t}$, from which it follows that the first time t_w that the state process is on the graph of h is given by the equation $z\mathrm{e}^{-\lambda t_w} = h(y)$. Once the state process (Y^h, Z^{Y^h}) is on the graph of h, the strategy Y^h consists of taking minimal action such that the state process remains on the graph of h (see Figure 1). This implies that $\big(Y_t^h, Z_t^{Y^h}\big) = \big(h^{-1}(Z_t^{Y^h}), Z_t^{Y^h}\big)$. With reference to (2.8), this implies that Z^{Y^h} should solve

$$dZ_t^{Y^h} = -\lambda Z_{t-}^{Y^h}\, dt + dh^{-1}\big(Z_t^{Y^h}\big),$$

which is equivalent to

$$d\rho_h\big(Z_t^{Y^h}\big) = -\lambda Z_{t-}^{Y^h}\, dt.$$

The following result establish the existence and uniqueness of such a strategy Y^h for a given intervention boundary function h.

Lemma 4.1. *Let $(y, z) \in \mathbb{R}^+ \times \mathbb{R}^-$ and let $h : \mathbb{R}^+ \to \mathbb{R}^-$ be a strictly decreasing càdlàg function satisfying $h(0) = 0$ and $\lim_{y\to\infty} h(y) = -\infty$, let h^{-1} and γ_h^{-1} be given by (4.15) and (4.16). Set*

$$t_w = \begin{cases} 0, & \text{if } z \geq h(y), \\ \lambda^{-1}\big\{\ln(-z) - \ln(-h(y))\big\}, & \text{if } z < h(y), \end{cases} \tag{4.17}$$

and let Y^h denote the decreasing càdlàg liquidation strategy with the following description:

(a) *If $z \geq h(y)$, then immediately sell $y - \gamma_h^{-1}(z - y)$ number of shares. This block trade ensures that $Y_0^h = h^{-1}\big(Z_0^{Y^h}\big)$. Then continuously sell shares so that $Y_t^h = h^{-1}\big(Z_t^{Y^h}\big)$, for all $t \geq 0$.*

(b) *If $z < h(y)$, then do nothing until time t_w. The time t_w has the property that $y = h^{-1}\big(Z_{t_w}^{Y^h}\big)$. Then continuously sell shares so that $Y_t^h = h^{-1}\big(Z_t^{Y^h}\big)$, for all $t \geq t_w$.*

Such a strategy Y^h exists and is unique. In particular,

$$Y_t^h = h^{-1}(Z_t^{Y^h}), \quad \text{for } t \geq t_w, \tag{4.18}$$

and Z^{Y^h} is the unique solution to

$$Z_t^{Y^h} = Z_{t_w}^{Y^h} - h^{-1}(Z_{t_w}^{Y^h}) - \int_{t_w}^{t} \lambda Z_u^{Y^h}\, du + h^{-1}(Z_t^{Y^h}), \tag{4.19}$$

where

$$Z_{t_w}^{Y_h} = h(y), \ \text{if } z < h(y), \quad \text{and} \quad Z_{t_w}^{Y^h} = z - y + \gamma_h^{-1}(z - y), \ \text{if } z \geq h(y). \tag{4.20}$$

If $t_w > 0$, then $Y_t^h = y$ and $Z_t^{Y^h} = z\mathrm{e}^{-\lambda t}$, for $0 \leq t \leq t_w$.

The next result concerns the relative speed of liquidation corresponding to two different intervention boundaries. In particular, we will need this result later in order to prove that our candidate for the optimal strategy is admissible.

Lemma 4.2. *Let $h_1, h_2 : \mathbb{R}^+ \to \mathbb{R}^-$ be strictly decreasing càdlàg functions satisfying $h_1(0) = h_2(0) = 0$ and $\lim_{y\to\infty} h_1(y) = \lim_{y\to\infty} h_1(y) = -\infty$. Denote by Y^{h_1} and Y^{h_2} the strategies defined in Lemma 4.1 by (4.18)–(4.19) corresponding to h_1 and h_2, respectively. If $h_1 \leq h_2$ then $Y^{h_1} \leq Y^{h_2}$. In particular, if there exist $C > 0$ and $\epsilon > 0$ such that $h(y) \leq -Cy$ for all $y < \epsilon$, then $Y^h \in \mathcal{A}_D^-(y)$.*

In order to obtain an explicit expression for the value function of our problem, we progress by deriving an explicit expression for the performance associated with a strategy Y^h, given an arbitrary intervention boundary function h. For an initial state (y, z) and strategy Y^h, with associated bid order book state process Z^{Y^h}, define the performance function J_h by

$$J_h(y, z) = \int_0^\infty \lambda Y_{t-}^h \left(a Y_{t-}^h + Z_{t-}^{Y^h} \frac{\psi\big(Z_{t-}^{Y^h}\big) - \psi\big(Z_{t-}^{Y^h} - Y_{t-}^h\big)}{Y_{t-}^h} \right) dt, \tag{4.21}$$

where $Y^h_{0-} = y$ and $Z^{Y^h}_{0-} = z$. Also, define

$$I_h(z) = J_h\big(h^{-1}(z), z\big), \qquad \text{for } z \in \mathbb{R}^-,$$

which corresponds to the performance of the strategy Y^h if the initial state is on the graph of h. If the initial state (y, z) is such that $z \geq h(y)$ then the strategy Y^h consists of an initial sale of $y - \gamma_h^{-1}(z - y) = z - \rho_h^{-1}(z - y)$ number of shares, and the state after the block sale is $\big(Y^h_0, Z^{Y^h}_0\big) = \big(h^{-1}\big(\rho_h^{-1}(z - y)\big), \rho_h^{-1}(z - y)\big)$, where γ_h^{-1} and ρ_h are given by (4.16) and (4.14), respectively. Hence,

$$J_h(y, z) = I_h\big(\rho_h^{-1}(z - y)\big), \qquad \text{for } z \geq h(y). \tag{4.22}$$

Thus, if we obtain an explicit expression for I_h, we have an explicit expression for $J_h(y, z)$, for all $z \geq h(y)$. For an initial state $(y, z) = (h^{-1}(z), z)$, the liquidation strategy Y^h satisfies $(Y^h_t, Z^{Y^h}_t) = \big(h^{-1}(Z^{Y^h}_t), Z^{Y^h}_t\big)$, for all $t \geq 0$. Therefore,

$$I_h(z) = \int_0^\infty \left(a\lambda\big(h^{-1}(Z^{Y^h}_{t-})\big)^2 + \lambda Z^{Y^h}_{t-}\big\{\psi\big(Z^{Y^h}_{t-}\big) - \psi\big(\rho_h(Z^{Y^h}_{t-})\big)\big\}\right) dt, \qquad Z^{Y^h}_{0-} = z.$$

With reference to (2.8), we note that formally,

$$dt = -\frac{d\rho_h(Z^{Y^h}_t)}{\lambda Z^{Y^h}_{t-}},$$

and hence

$$\begin{aligned}
I_h(z) &= -\int_0^\infty \frac{a\big(h^{-1}(Z^{Y^h}_{t-})\big)^2}{Z^{Y^h}_{t-}}\, d\rho_h\big(Z^{Y^h}_t\big) + \int_0^\infty \psi\big(\rho_h(Z^{Y^h}_{t-})\big)\, d\rho_h\big(Z^{Y^h}_t\big) \\
&\quad - \int_0^\infty \psi\big(Z^{Y^h}_{t-}\big)\, d\rho_h\big(Z^{Y^h}_t\big) \\
&= -\int_0^\infty \frac{a\big(\gamma_h^{-1}\big(\rho_h(Z^{Y^h}_{t-})\big)\big)^2}{\rho_h^{-1}\big(\rho_h(Z^{Y^h}_{t-})\big)}\, d\rho_h\big(Z^{Y^h}_t\big) + \int_0^\infty \psi\big(\rho_h(Z^{Y^h}_{t-})\big)\, d\rho_h\big(Z^{Y^h}_t\big) \\
&\quad - \int_0^\infty \psi\big(\rho_h^{-1}\big(\rho_h(Z^{Y^h}_{t-})\big)\big)\, d\rho_h\big(Z^{Y^h}_t\big) \\
&= \int_0^{\rho_h(z)} \left(\frac{a\big(\gamma_h^{-1}(u)\big)^2}{\rho_h^{-1}(u)} + \psi\big(\rho_h^{-1}(u)\big) - \psi(u)\right) du,
\end{aligned}$$

where we have used that $\gamma_h^{-1}\big(\rho_h^{-1}(x)\big) = h^{-1}(x)$, for $x \in \mathbb{R}^-$. With reference to (4.22) we conclude that

$$J_h(y, z) = \int_0^{z-y} \left(\frac{a\big(\gamma_h^{-1}(u)\big)^2}{\rho_h^{-1}(u)} + \psi\big(\rho_h^{-1}(u)\big) - \psi(u)\right) du, \qquad \text{for } z \geq h(y). \tag{4.23}$$

In order to obtain an expression for $J_h(y, z)$ for $z < h(y)$, we note that the strategy Y^h consists of waiting until the first time t_w for which $(Y^h_{t_w}, Z^{Y^h}_{t_w})$ is on the graph of h, where

$Y_t^h = y$ and $Z_t^{Y^h} = z\mathrm{e}^{-\lambda t}$, for $0 \le t \le t_w$. Therefore, $t_w = \lambda^{-1} \ln\big(z/h(y)\big)$, and

$$\begin{aligned} J_h(y,z) &= \int_0^{t_w} \lambda y \left(ay + z\mathrm{e}^{-\lambda t} \frac{\psi\big(z\mathrm{e}^{-\lambda t}\big) - \psi\big(z\mathrm{e}^{-\lambda t} - y\big)}{y} \right) dt + I_h\big(h(y)\big) \\ &= ay^2 \ln\left(\frac{z}{h(y)}\right) + \int_{z-y}^{h(y)-y} \psi(u)\, du - \int_z^{h(y)} \psi(u)\, du \\ &\quad + \int_0^{h(y)-y} \left(\frac{a\big(\gamma_h^{-1}(u)\big)^2}{\rho_h^{-1}(u)} + \psi\big(\rho_h^{-1}(u)\big) - \psi(u) \right) du, \qquad \text{for } z < h(y). \end{aligned}$$

While this provides an explicit expression for $J_h(y,z)$, it is not obvious from this expression that it is continuous in y (and has a one-sided derivative with respect to y), as h is only a càdlàg function. However, we can calculate that

$$\begin{aligned} \int_0^{\gamma_h(y)} \left(\frac{a\big(\gamma_h^{-1}(u)\big)^2}{\rho_h^{-1}(u)} + \psi\big(\rho_h^{-1}(u)\big) \right) du &= \int_0^y \left(\frac{au^2}{h(u-)} + \psi\big(h(u-)\big) \right) d\gamma_h^c(u) \\ &\quad + \sum_{0\le u\le y} au^2 \ln\left(\frac{h(u)}{h(u-)}\right) + \sum_{0\le u\le y} \int_{h(u-)}^{h(u)} \psi(s)\, ds. \end{aligned}$$

From this expression, as well as

$$\int_0^{h(y)} \psi(u)\, du = \int_0^y \psi(u)\, dh^c(u) + \sum_{0\le u\le y} \int_{h(u-)}^{h(u)} \psi(s)\, ds,$$

and

$$ay^2 \ln\big(-h(y)\big) = \int_0^y 2au \ln\big(h(u-)\big)\, du + \int_0^y \frac{au^2}{h(u-)}\, dh^c(u) + \sum_{0\le u\le y} au^2 \ln\left(\frac{h(u)}{h(u-)}\right),$$

it follows that the performance function $J_h(y,z)$ admits the expression

$$\begin{aligned} J_h(y,z) &= ay^2 \ln\big(-z\big) + \int_{z-y}^{z} \psi(u)\, du \\ &\quad - \int_0^y \left(\frac{au^2}{h(u)} + \psi\big(h(u)\big) + 2au \ln\big(-h(u)\big) \right) du. \qquad \text{for } z < h(y), \end{aligned}$$

The next result provides an explicit equation for the intervention boundary function h and the corresponding value function which solves equation (4.1) with associated boundary condition $v(0,z) = 0$, for all $z \in \mathbb{R}^-$.

Proposition 4.3. *For $y \in \mathbb{R}^+$, define the function $\Gamma(\cdot; y) : \mathbb{R}^- \to \mathbb{R}$ by*

$$\Gamma(x;y) = \psi(x) + \frac{ay^2}{x} + 2ay \ln(-x),$$

and let $h = h(y)$ be the smallest $h \in \mathbb{R}^-$ satisfying

$$\max_{x\le 0} \Gamma(x;y) = \Gamma\big(h(y); y\big). \tag{4.24}$$

This defines a unique strictly decreasing càdlàg function $h : \mathbb{R}^+ \to \mathbb{R}^-$ satisfying $h(0) = 0$ and $\lim_{y\to\infty} h(y) = -\infty$, and in particular

$$\psi'\big(h(y)\big)h(y)^2 + 2ayh(y) - ay^2 = 0, \qquad \text{for all } y \in \mathbb{R}^+. \tag{4.25}$$

Let γ_h^{-1}, ρ_h and ρ_h^{-1} be the functions defined in (4.16) and (4.14). Then $v : \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}^-$ given by

$$v(y,z) = \int_0^{z-y} \left(\frac{a\big(\gamma_h^{-1}(s)\big)^2}{\rho_h^{-1}(s)} + \psi\big(\rho_h^{-1}(s)\big) - \psi(s) \right) ds, \qquad \text{for } z \geq h(y), \tag{4.26}$$

and

$$\begin{aligned} v(y,z) = {} & ay^2 \ln\big(-z\big) + \int_{z-y}^{z} \psi(s)\,ds \\ & - \int_0^y \left(\frac{as^2}{h(s)} + \psi\big(h(s)\big) + 2as\ln\big(-h(s)\big) \right) ds, \qquad \text{for } z < h(y), \end{aligned} \tag{4.27}$$

is a $C^{0,1}(\mathbb{R}^+ \times \mathbb{R}^-)$ function which solves equation (4.1) with the boundary condition $v(0,z) = 0$, for all $z \in \mathbb{R}^-$. In particular, v satisfies (4.8)–(4.11). Moreover, $D_y^+ v : \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}$ is continuous in z and càdlàg in y, and v is continuously differentiable with respect to y, for $z \geq h(y)$.

Based on the expression for the performance function $J_h(y,z)$ and the principle of smooth fit, it follows that the intervention boundary function h should satisfy (4.25). However, (4.25) does not necessarily have a unique solution and the value function does in general not satisfy the smoothness principle. Based on the observation that

$$D_y^+ v(y,z) + v_z(y,z) = \Gamma(z;y) - \Gamma\big(h(y);y\big), \qquad \text{for } z < h(y),$$

and with reference to (4.11), equation (4.24) is a natural candidate.

The following result states that the function v given by (4.26) and (4.27) is equal to the value function V given by (3.10), that the strategy Y^h corresponding to h given by (4.24) is an optimal liquidation strategy, and hence provides the solution to the utility maximization problem (2.13).

Theorem 4.4. *Let the large investor's risk aversion be A, the volatility of the non-affected asset price be σ and let the resilience rate be λ. Set $a = \frac{\sigma^2 A}{2\lambda}$ and let h denote the smallest solution to (4.24), let v be given by (4.26) and (4.27) and let V be given by (3.10). Then $v = V$ and*

$$\sup_{(X,Y)\in\mathcal{A}(y)} \mathbb{E}\big[U\big(C_\infty(X,Y)\big)\big] = -\exp\left(-A(c+by) + A\int_z^{z-y} \psi(s)\,ds\right)\exp\big(Av(y,z)\big),$$

where $z = Z_{0-}^Y$ is the initial state of the bid order book. The optimal strategy Y^* is equal to $Y^h \in \mathcal{A}_D^-(y)$, where Y^h is the strategy given by (4.18)–(4.20) in Lemma 4.1 corresponding to h, with $Y_{0-}^h = y$.

As a corollary, we obtain that the optimal liquidation strategy possess all the properties one would expect, like increased limit order depth implies faster liquidation, increased volatility of the unaffected stock price implies faster liquidation, and increased risk aversion implies faster liquidation.

Corollary 4.5. *Let $Y_1^* \in \mathcal{A}_D^-(y)$ be the strategy which attains the optimality in (2.13) given a shape function ϕ_1, and let $Y_2^* \in \mathcal{A}_D^-(y)$ denote the strategy which attains the optimality in (2.13) given a shape function ϕ_2. Then $\phi_1 \leq \phi_2$ implies $Y_1^* \geq Y_2^*$, provided the volatility σ, the resilience rate λ and the large investor's risk aversion A is the same.*

Let $Y_1^* \in \mathcal{A}_D^-(y)$ be the strategy which attains the optimality in (2.13) given volatility σ_1 and risk aversion A_1, and let $Y_2^* \in \mathcal{A}_D^-(y)$ denote the strategy which attains the optimality in (2.13) given volatility σ_2 and risk aversion A_2. Then $\sigma_1^2 A_1 \leq \sigma_2^2 A_2$ implies that $Y_1^* \geq Y_2^*$, provided that the shape function ϕ and the resilience rate λ is the same.

Proof. Observe that $\phi_1 \leq \phi_2$ implies that $\psi_1' \geq \psi_2'$, and hence $\psi_1 \leq \psi_2$. It follows that $h_1 \geq h_2$, where h_1 and h_2 denote smallest solution to (4.24) corresponding to ψ_1 and ψ_2, respectively. The result then follows from Lemma 4.2.

Notice that $a_1 \leq a_2$ implies that $h_1 \geq h_2$, where h_1 and h_2 denote the smallest solution to (4.24) corresponding to a_1 and a_2, respectively. The result then follows from Lemma 4.2. □

The next example shows that if the shape function ϕ is constant, and hence ψ is a linear function, the solution to equation (4.1) is a linear function, and the corresponding optimal strategy takes an even simpler form. In this case the impact of the large trader's strategy is linear, so it is natural to compare the results with the corresponding strategy for the Almgren and Chriss [2] model with an infinite horizon as in Schied and Schöneborn [21].

Example. Suppose that the shape function ϕ of the limit order book is constant, i.e. $\phi = c$, for some $c > 0$. Let the large investor's risk aversion be A, the volatility of the unaffected stock price be σ, let the resilience rate be λ and set $a = \frac{\sigma^2 A}{2\lambda}$. Observe that for all $y \in \mathbb{R}^+$, equation (4.25) has a unique solution $h = h(y)$ given by

$$h(y) = -\kappa y,$$

where $\kappa = ac + \sqrt{a^2c^2 + ac}$. It follows that this function h is the unique solution to equation (4.24), and therefore defines the optimal intervention boundary. Moreover, we can observe that

$$h^{-1}(z) = -\frac{1}{\kappa} z, \qquad \gamma_h^{-1}(z) = -\frac{1}{\kappa+1} z \quad \text{and} \quad \rho_h^{-1}(z) = \frac{\kappa}{\kappa+1} z.$$

With reference to (4.26) and (4.27), it follows from Proposition 4.3 that

$$v(y,z) = \frac{ac-\kappa}{2c\kappa(\kappa+1)}(y-z)^2, \qquad \text{if } z \geq -\kappa y,$$

and

$$v(y,z) = \frac{ac(\kappa+1)+\kappa(\kappa-1)}{2c\kappa} y^2 + \frac{zy}{c} + ay^2 \ln\left(\frac{-z}{\kappa y}\right), \qquad \text{if } z < -\kappa y,$$

is a $C^{1,1}(\mathbb{R}^+ \times \mathbb{R}^-)$ solution to equation (4.1) with boundary condition $v(0,z) = 0$, for all $z \leq 0$. Equation (4.19) takes the form

$$Z_t^{Y^h} = Z_{t_w}^{Y^h} + \frac{1}{\kappa} Z_{t_w}^{Y^h} - \int_{t_w}^t \lambda Z_{u-}\, du - \frac{1}{\kappa} Z_t^{Y^h}, \qquad \text{for } t \geq t_w.$$

from which it follows that

$$Z_t^{Y^h} = Z_{t_w}^{Y^h} \exp\left(-t\frac{\lambda\kappa}{\kappa+1}\right), \qquad \text{for } t \geq t_w.$$

Therefore, the strategy $Y^* \in \mathcal{A}_D^-(y)$ which attains the optimality in (2.13) is as follows:

(a) if $z \geq -\kappa y$, then immediately sell $\frac{\kappa y+z}{1+\kappa}$ number of shares, i.e. $Y_0^* - y = -\frac{\kappa y+z}{1+\kappa}$, and then continuously sell shares according to

$$Y_t^* = \frac{y-z}{\kappa+1} \exp\left(-t\frac{\lambda\kappa}{1+\kappa}\right), \qquad \text{for } t \geq 0;$$

(b) if $z < -\kappa y$, then do nothing until time $t_w = \lambda^{-1}\big\{\ln(-z) - \ln(\kappa y)\big\}$, and then continuously sell shares according to

$$Y_t^* = y \exp\left(-(t-t_w)\frac{\lambda\kappa}{1+\kappa}\right), \qquad \text{for } t \geq t_w.$$

It is natural to compare our result for the limit order book with constant shape function with the optimal liquidation strategy in the Almgren and Chriss model [2]. In this model, the stock price dynamics are

$$P_t = P_0 + \sigma W_t + \alpha(Y_t - Y_0) + \beta \dot{Y}_t, \tag{4.28}$$

where Y_t denotes the number of shares held by the large investor at time t, and where the process Y is absolutely continuous with density $\dot{Y}$, i.e.

$$Y_t = y + \int_0^t \dot{Y}_u\, du.$$

The parameter $\alpha \geq 0$ is a parameter for the level of permanent impact of the large investor's trading, and the parameter $\beta \geq 0$ describes the temporary impact of the large investor's trading. The optimal liquidation strategy for a large investor with an initial position of y number of shares is

$$Y_t^* = y \exp\left(-t\sqrt{\frac{\sigma^2 A}{2\beta}}\right), \qquad t \geq 0,$$

(see Schied and Schöneborn [21]) if the large investor has constant absolute risk aversion A and aim to maximize his cash position at the end of time. We can observe that the optimal strategy in the limit order book model and the optimal strategy in the Almgren and Chriss model [2] look similar, as in both models liquidation follows an exponential function. Yet, there are two aspects which make the strategies different. In the limit order book model, the optimal strategy depends on the past history of the large investor, while in the Almgren model there is no such dependence since future returns are unaffected by the large investor's past trades. Also, in the limit order book model, the optimal strategy typically consist of an initial block trade, while in the Almgren and Chriss model the optimal strategy is absolutely continuous. However, as pointed out by an anonymous referee, we can recover the optimal liquidation strategy for the Almgren and Chriss model by taking the limit as $\lambda \to \infty$ with $c = c^{(\lambda)} = \frac{1}{\beta\lambda}$. Thus as the resilience rate tends to infinity, the quantity available in the limit order book decreases such that the cost of trading has a finite limit. More specifically, since

$$\lim_{x \to 0^+} \frac{x + \sqrt{x^2 + x}}{\sqrt{x}} = 1,$$

we calculate that

$$\lim_{\lambda \to \infty} \lambda \kappa^{(\lambda)} = \lim_{\lambda \to \infty} \lambda \sqrt{\frac{\sigma^2 A}{2\lambda} c^{(\lambda)}} = \sqrt{\frac{\sigma^2 A}{2\beta}}.$$

In the limit as $\lambda \to \infty$, the stateprocess Z is identically equal to 0. Therefore, if we let $Y^{*,\lambda}$ denote the optimal strategy described in (a) and (b) corresponding to λ, we conclude that

$$\lim_{\lambda \to \infty} Y_t^{*,\lambda} = \lim_{\lambda \to \infty} \frac{y}{\kappa^{(\lambda)} + 1} \exp\left(-t \frac{\lambda \kappa^{(\lambda)}}{1 + \kappa^{(\lambda)}}\right) = y \exp\left(-t \sqrt{\frac{\sigma^2 A}{2\beta}}\right),$$

which is the optimal strategy in the Almgren and Chriss model. To explain this result, we can note that if Y is an admissible liquidation strategy which has the form

$$\dot{Y}_t = \sum_{n=1}^{m} q_n \mathbf{1}_{[\tau_n, \tau_{n+1})}(t),$$

for stopping times $0 \leq \tau_1 < \tau_2 \cdots < \tau_{m+1}$ and random variables $q_1, \dots, q_{m+1}$, where q_n is $\mathcal{F}_{\tau_n}$-measurable, then

$$\begin{aligned}
\lim_{\lambda \to \infty} B_t^Y &= B_t^0 + \lim_{\lambda \to \infty} \frac{Z_t^{Y,\lambda}}{c^{(\lambda)}} \\
&= B_t^0 + \lim_{\lambda \to \infty} \beta \lambda \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s} \dot{Y}_s \, ds \\
&= B_t^0 + \sum_{n=1}^{m} \beta q_n \lim_{\lambda \to \infty} \left(\mathrm{e}^{\lambda(\min\{\tau_{n+1}, t\} - t)} - \mathrm{e}^{\lambda(\min\{\tau_n, t\} - t)} \right) \\
&= B_t^0 + \beta \dot{Y}_t,
\end{aligned}$$

which is the Almgren and Chriss model with permanent impact factor α equal to zero.

5. Proofs of results

Proof. (of Lemma 3.2.) With reference to equation (2.12) we calculate

$$
\begin{aligned}
C_T(Y) &= c - \int_0^T B^Y_{t-}\, dY^c_t - \sum_{0\le t\le T} \int_0^{\triangle Y_t} \big\{ B^0_t + \psi\big(Z^Y_{t-} + x\big)\big\}\, dx \\
&= c - \int_0^T B^Y_{t-}\, dY^c_t - \sum_{0\le t\le T} B^Y_{t-}\triangle Y_t - \sum_{0\le t\le T} \int_0^{\triangle Y_t} \big\{ \psi(Z^Y_{t-} + x) - \psi(Z^Y_{t-})\big\}\, dx \\
&= c - \int_0^T B^Y_{t-}\, dY_t - \sum_{0\le t\le T} \int_0^{\triangle Y_t} \big\{ \psi(Z^Y_{t-} + x) - \psi(Z^Y_{t-})\big\}\, dx. \qquad (5.1)
\end{aligned}
$$

Since Y is a càdlàg process of finite variation, it follows from Protter [19, Theorems II.26 and II.28] that the quadratic co-variation $[B^Y, Y]$ between B^Y and Y is

$$
[B^Y, Y]_T = \sum_{0\le t\le T} \big\{ \psi\big(Z^Y_{t-} + \triangle Y_t\big) - \psi\big(Z^Y_{t-}\big)\big\} \triangle Y_t.
$$

Hence,

$$
\begin{aligned}
B^Y_T Y_T - B^Y_{0-} Y_{0-} = \int_0^T B^Y_{t-}\, dY_t + \int_0^T Y_{t-}\, dB^Y_t& \\
+ \sum_{0\le t\le T} \big\{ \psi\big(Z^Y_{t-} + \triangle Y_t\big) - \psi\big(Z^Y_{t-}\big)\big\} \triangle Y_t.& \qquad (5.2)
\end{aligned}
$$

With reference to the dynamics of B^Y given by (2.9),

$$
\begin{aligned}
\int_0^T Y_{t-}\, dB^Y_t = \int_0^T \sigma Y_{t-}\, dW_t - \int_0^T \lambda Z^Y_{t-} \psi'\big(Z^Y_{t-}\big) Y_{t-}\, dt& \\
+ \int_0^T \psi'\big(Z^Y_{t-}\big) Y_{t-}\, dY^c_t + \sum_{0\le t\le T} Y_{t-} \big\{ \psi\big(Z^Y_{t-} + \triangle Y_t\big) - \psi\big(Z^Y_{t-}\big)\big\}.& \qquad (5.3)
\end{aligned}
$$

Equation (5.2) provides an expression for $\int_0^T B^Y_{t-}\, dY_t$, which combined with (5.1) and (5.3) imply

$$
\begin{aligned}
C_T(Y) = c - B^Y_T Y_T + B^Y_{0-} Y_{0-} + \int_0^T \sigma Y_{t-}\, dW_t - \int_0^T \lambda Z^Y_{t-} \psi'\big(Z^Y_{t-}\big) Y_{t-}\, dt& \\
+ \int_0^T \psi'\big(Z^Y_{t-}\big) Y_{t-}\, dY^c_t + \sum_{0\le t\le T} Y_{t-} \big\{ \psi\big(Z^Y_{t-} + \triangle Y_t\big) - \psi\big(Z^Y_{t-}\big)\big\}& \\
- \sum_{0\le t\le T} \int_0^{\triangle Y_t} \big\{ \psi\big(Z^Y_{t-} + x\big) - \psi\big(Z^Y_{t-}\big)\big\}\, dx& \\
+ \sum_{0\le t\le T} \big\{ \psi\big(Z^Y_{t-} + \triangle Y_t\big) - \psi\big(Z^Y_{t-}\big)\big\} \triangle Y_t.& \qquad (5.4)
\end{aligned}
$$

Define the function $f : \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}^2$ by

$$f(y,z) = y\psi(z) + \int_z^{z-y} \psi(s)\,ds.$$

Then by Itô's formula,

$$\begin{aligned} f\big(Y_T, Z_T^Y\big) &= f\big(Y_{0-}, Z_{0-}^Y\big) + \int_0^T \psi'\big(Z_{t-}^Y\big) Y_{t-}\,dY_t^c \\ &\quad - \int_0^T \lambda Z_{t-}^Y \psi'\big(Z_{t-}^Y\big) Y_{t-}\,dt - \int_0^T \lambda Z_{t-}^Y \big\{\psi\big(Z_{t-}^Y - Y_{t-}\big) - \psi\big(Z_{t-}^Y\big)\big\}\,dt \\ &\quad + \sum_{0\le t\le T} Y_{t-}\big\{\psi\big(Z_{t-}^Y + \triangle Y_t\big) - \psi\big(Z_{t-}^Y\big)\big\} + \sum_{0\le t\le T} \psi\big(Z_{t-}^Y + \triangle Y_t\big)\triangle Y_t \\ &\quad + \sum_{0\le t\le T} \int_{Z_{t-}^Y + \triangle Y_t}^{Z_{t-}^Y} \psi(s)\,ds. \end{aligned}$$

This provides an expression for $\int_0^T \psi'(Z_{t-}^Y) Y_t\,dY_t^c$, which inserted in (5.4) imply that

$$\begin{aligned} C_T(Y) &= c - B_T^Y Y_T + B_{0-}^Y Y_{0-} + f\big(Y_T, Z_T^Y\big) - f\big(Y_{0-}, Z_{0-}^Y\big) \\ &\quad + \int_0^T \sigma Y_{t-}\,dW_t + \int_0^T \lambda Z_{t-}^Y \big\{\psi\big(Z_{t-}^Y - Y_{t-}\big) - \psi\big(Z_{t-}^Y\big)\big\}\,dt, \end{aligned}$$

from which the result follows. □

Proof. (of Lemma 3.3.) With reference to the expression for the large investor's cash position at time $T > 0$ obtained in Lemma 3.2, we first note that $z - y \le Z_t^Y \le 0$, for all $t \ge 0$. Therefore

$$\lim_{T\to\infty} \left(Y_T Z_T^Y + \int_{Z_T^Y}^{Z_T^Y - Y_T} \psi(s)\,ds \right) = 0,$$

almost surely and in $L^1(\mathbb{P})$. By the Cauchy-Schwartz inequality, we calculate that

$$\lim_{T\to\infty} \mathbb{E}\big[\,\big|\,B_T^Y Y_T\,\big|\,\big] \le \sigma \left(\lim_{T\to\infty} \mathbb{E}\big[T Y_T^2\big] \right)^{1/2} = 0, \tag{5.5}$$

since (2.2) implies

$$\lim_{t\to\infty} \mathbb{E}\big[t Y_t^2\big] = 0. \tag{5.6}$$

The convergence of $\int_0^\infty \sigma Y_{t-}\,dW_t$ follows from (2.2) and the Itô isometry. In order to establish the inequalities (3.7), observe that $B^0 \ge B^Y$, for every $(X,Y) \in \mathcal{A}(y)$. Since Y is decreasing, it follows that

$$C_T(Y) \le c - \int_0^T B_t^0\,dY_t = c + B_{0-}^0 Y_{0-} - B_T^0 Y_T + \int_0^T \sigma Y_{t-}\,dW_t.$$

By similar arguments as in (5.5), $B_T^0 Y_T$ converges to 0 in $L^1(\mathbb{P})$ as $T \to \infty$. Since

$$\lambda Z_{t-}^Y \big\{\psi\big(Z_{t-}^Y - Y_{t-}\big) - \psi\big(Z_{t-}^Y\big)\big\} \geq 0, \qquad t \geq 0,$$

we conclude that the inequalities (3.7) hold and that the integral

$$\int_0^\infty \lambda Z_{t-}^Y \big\{\psi\big(Z_{t-}^Y - Y_{t-}\big) - \psi\big(Z_{t-}^Y\big)\big\}\, dt$$

is almost surely convergent. □

Proof. (of Lemma 4.1.) Let $\mathbb{Z}(z)$ be the set of continuous functions $Z : \mathbb{R}^+ \to [z - h^{-1}(z), 0]$ satisfying $Z_0 = z$ and $\lim_{t\to\infty} Z_t = 0$, where $z \in \mathbb{R}^-$. We consider $\mathbb{Z}(z)$ as a subset of $C_0(\mathbb{R})$, i.e. the space of all real-valued continuous functions vanishing at infinity, equipped with the uniform topology. Observe that $\mathbb{Z}(z)$ is a convex set, and by the Ascoli-Arzelà theorem (see e.g. Folland [12]) it follows that $\mathbb{Z}(z)$ is compact. Introduce the function Ψ_z given by

$$\big(\Psi_z(Z)\big)_t = \begin{cases} z - h^{-1}(z) - \int_0^t \lambda Z_u\, du + h^{-1}(Z_t), & \text{for } 0 \leq t \leq \bar{t}, \\ 0, & \text{for } t > \bar{t}, \end{cases}$$

where

$$\bar{t} = \inf\left\{ t \geq 0 \,:\, z - h^{-1}(z) - \int_0^t \lambda Z_u\, du + h^{-1}(Z_t) = 0 \right\}.$$

Since h^{-1} is continuous and decreasing, it follows that $\Psi_z : \mathbb{Z}(z) \to \mathbb{Z}(z)$ is continuous. The Schauder-Tychonoff fixed point theorem (see e.g. Rudin [20]) therefore guarantees the existence of a $Z \in \mathbb{Z}(z)$ such that $Z = \Psi_z(Z)$. We want to show that such a Z is unique. Assume that $Z^{(1)} = \Psi_z\big(Z^{(1)}\big)$ and $Z^{(2)} = \Psi_z\big(Z^{(2)}\big)$, where $Z^{(1)}, Z^{(2)} \in \mathbb{Z}(z)$ and $Z_t^{(1)} = Z_t^{(2)}$ for $0 \leq t \leq t_1$, and $Z_t^{(1)} < Z_t^{(2)}$ for $t_1 < t < t_2$. Then for $t_1 < t < t_2$,

$$\begin{aligned} Z_t^{(1)} &= z - h^{-1}(z) - \int_0^t \lambda Z_u^{(1)} + h^{-1}\big(Z_t^{(1)}\big) \\ &> z - h^{-1}(z) - \int_0^t \lambda Z_u^{(2)} + h^{-1}\big(Z_t^{(2)}\big) \\ &= Z_t^{(2)}, \end{aligned}$$

which contradicts the assumption that $Z_t^{(1)} < Z_t^{(2)}$ for $t_1 < t < t_2$. We conclude that there exists a unique $Z \in \mathbb{Z}(z)$ such that $Z = \Psi_z(Z)$. Moreover, since the function $z \mapsto z - h^{-1}(z)$ is strictly increasing for $z \in \mathbb{R}^-$ and $t \mapsto z - h^{-1}(z) - \int_0^t \lambda Z_u\, du$ is strictly increasing as long as $Z_t < 0$, it follows that the solution to $Z = \Psi_z(Z)$ is strictly increasing while $Z_t < 0$.

Suppose that $z \geq h(y)$, and let Z^{Y^h} be given by (4.19)–(4.20). The existence and uniqueness of such a Z^{Y^h} follows from the previous part of the proof. We calculate that

$$h^{-1}\big(\xi + \gamma_h^{-1}(\xi)\big) = \gamma_h^{-1}(\xi), \quad \text{for all } \xi \in \mathbb{R}^-,$$

and therefore

$$h^{-1}\big(Z_0^{Y^h}\big) = h^{-1}\big(z - y + \gamma_h^{-1}(z-y)\big) = \gamma_h^{-1}(z-y) = Y_0^h,$$

as required. With reference to (4.18) and (4.19) we have that

$$
\begin{aligned}
Z_t^{Y^h} &= Z_0^{Y^h} - h^{-1}(Z_0^{Y^h}) - \int_0^t \lambda Z_u^{Y^h}\,du + h^{-1}(Z_t^{Y^h}) \\
&= Z_0^{Y^h} - \int_0^t \lambda Z_u^{Y^h}\,du + Y_t^h - Y_0^h,
\end{aligned}
\tag{5.7}
$$

since $z \geq h(Y)$ is equivalent to $t_w = 0$, and $Y_t^h = h^{-1}(Z_t^{Y^h})$. Equation (5.7) shows that Z^{Y^h} satisfies (2.8), and we conclude that Y^h is the unique process with the property that $Y_t^h = h^{-1}(Z_t^{Y^h})$, for all $t \geq 0$. Also note that from the first part of the proof, $t \mapsto Z_t^{Y^h}$ is continuous and decreasing for $t > 0$, which combined with the monotonicity and continuity of h^{-1} imply that Y^h is càdlàg and decreasing. Hence Y^h is the unique decreasing càdlàg process satisfying the description given in part (a).

Suppose that $z < h(y)$, and let Y^h be given by (4.18)–(4.20). Then

$$
Z_t^{Y^h} = z\mathrm{e}^{-\lambda t}, \qquad \text{for } 0 \leq t \leq t_w, \tag{5.8}
$$

and $Z_{t_w}^{Y^h} = h(y) = \lim_{t \to t_w} Z_t^{Y^h}$, which correspond to the description given in part (b). With reference to (4.18)–(4.20), we have that

$$
\begin{aligned}
Z_t^{Y^h} &= h(y) - y - \int_{t_w}^t \lambda Z_u^{Y^h}\,du + h^{-1}(Z_t^{Y^h}) \\
&= h(y) - \int_{t_w}^t \lambda Z_u^{Y^h}\,du + Y_t^h - Y_0^h,
\end{aligned}
$$

for $t \geq t_w$. Since $Y_t^h = y = h^{-1}\big(h(y)\big)$ and Z^{Y^h} is given by (5.8) for $0 \leq t < t_w$, it follows that

$$
Z_t^{Y^h} = z - \int_0^t \lambda Z_u^{Y^h}\,du + Y_t^h - Y_0^h,
$$

for $t \geq 0$, which verifies that Z^{Y^h} satisfies (2.8), and $Y_t^h = h^{-1}\big(Z_t^{Y^h}\big)$, for all $t \geq t_w$. We conclude that Y^h is the unique decreasing càdlàg process as described in part (b). □

Proof. (of Lemma 4.2.) Let $t_w^{h_1}$ and $t_w^{h_2}$ be given by (4.17), corresponding to h_1 and h_2, respectively. If $z < h_1(y)$ then $Y_t^{h_1} = Y_t^{h_2}$ for $0 \leq t \leq t_w^{h_1}$ and $Y_t^{h_1} \leq Y_t^{h_2}$ for $t_w^{h_1} \leq t \leq t_w^{h_2}$. If $h_1(y) \leq z < h_2(y)$ then $Y_t^{h_1} \leq Y_t^{h_2}$ for $0 = t_w^{h_1} \leq t \leq t_w^{h_2}$. Also, if $z \geq h_2(y)$, then $t_w^{h_1} = t_w^{h_2} = 0$ and $Y_0^{h_1} \leq Y_0^{h_2}$. We want to show that $\{t \geq 0 \,:\, Y_t^{h_1} > Y_t^{h_2}\} = \emptyset$. In order to get a contradiction, suppose that

$$
t_1 = \inf\big\{t \geq 0 \,:\, Y_t^{h_1} > Y_t^{h_2}\big\} < \infty \qquad \text{and define } t_2 = \inf\big\{t \wedge \infty \geq t_1 \,:\, Y_t^{h_1} \leq Y_t^{h_2}\big\}.
$$

By the previous observations $t_1 > t_w$, and by continuity of $Y_t^{h_1}$ and $Y_t^{h_2}$, for $t > 0$, it follows that $t_1 < t_2$. The monotonicity of h_1 and h_2 imply that if $Y_t^{h_1} > Y_t^{h_2}$ then

$$
Z_t^{Y^{h_1}} \leq h_1(Y_t^{h_1}-) < h_1(Y_t^{h_2}) \leq h_2(Y_t^{h_2}) \leq Z_t^{Y^{h_2}}, \tag{5.9}
$$

and if $Z_t^{Y^{h_1}} < Z_t^{Y^{h_2}}$ then

$$Y_t^{h_1} = h_1^{-1}\big(Z_t^{Y^{h_1}}\big) > h_1^{-1}\big(Z_t^{Y^{h_2}}\big) \geq h_2^{-1}\big(Z_t^{Y^{h_2}}\big) = Y_t^{h_2}. \tag{5.10}$$

With reference to (4.18) and (4.19), we have that for $t_1 < t < t_2$,

$$\begin{aligned} Z_t^{Y^{h_1}} &= Z_{t_1}^{Y^{h_1}} - Y_{t_1}^{h_1} - \int_{t_1}^t \lambda Z_u^{Y^{h_1}}\, du + h_1^{-1}(Z_t^{Y^{h_1}}) \\ &\geq Z_{t_1}^{Y^{h_1}} - Y_{t_1}^{h_1} - \int_{t_1}^t \lambda Z_u^{Y^{h_1}}\, du + h_2^{-1}(Z_t^{Y^{h_1}}) \\ &> Z_{t_1}^{Y^{h_1}} - Y_{t_1}^{h_1} - \int_{t_1}^t \lambda Z_u^{Y^{h_2}}\, du + h_2^{-1}(Z_t^{Y^{h_2}}) \\ &\geq Z_{t_1}^{Y^{h_2}} - Y_{t_1}^{h_2} - \int_{t_1}^t \lambda Z_u^{Y^{h_2}}\, du + h_2^{-1}(Z_t^{Y^{h_2}}) \\ &= Z_t^{Y^{h_2}}, \end{aligned} \tag{5.11}$$

where the last equality is due to $Y_{t_1}^{h_1} = Y_{t_1}^{h_2}$, (5.9) and (5.10), which imply that $Z_{t_1}^{Y^{h_1}} \geq Z_{t_1}^{Y^{h_2}}$. However, in view of (5.9) and (5.10), the inequality (5.11) contradicts the definition of t_1. Thus we conclude that $Y^{h_1} \leqq Y^{h_2}$.

Let $\bar{h} : \mathbb{R}^+ \to \mathbb{R}^-$ be given by $\bar{h}(y) = -Cy$, for $C > 0$, and let $Y^{\bar{h}}$ denote the corresponding strategy given by (4.18)–(4.20) in Lemma 4.1. Then (4.19) takes the form

$$Z_t^{Y^{\bar{h}}} = Z_{t_w}^{Y^{\bar{h}}} + \frac{1}{C} Z_{t_w}^{Y^{\bar{h}}} - \int_{t_w}^t \lambda Z_u^{Y^{\bar{h}}}\, du - \frac{1}{C} Z_t^{Y^{\bar{h}}},$$

which has a unique solution

$$Z_t^{Y^{\bar{h}}} = Z_{t_w}^{Y^{\bar{h}}} \exp\bigg(-t\frac{\lambda C}{1+C}\bigg), \qquad \text{for } t \geq t_w.$$

Also, (4.20) takes the form

$$Z_{t_w}^{Y^{\bar{h}}} = -Cy, \text{ if } z < -Cy \quad \text{and} \quad Z_{t_w}^{Y^{\bar{h}}} = (z-y)\frac{C}{1+C}, \text{ if } z \geq -Cy.$$

Therefore the strategy $Y^{\bar{h}}$ is given by

$$Y_t^{\bar{h}} = -\frac{z-y}{1+C} \exp\bigg(-t\frac{\lambda C}{1+C}\bigg), \qquad \text{for } t \geq 0, \qquad \text{if } z \geq -Cy, \tag{5.12}$$

and

$$Y_t^{\bar{h}} = y \exp\bigg(-(t-t_w)\frac{\lambda C}{1+C}\bigg), \qquad \text{for } t \geq t_w, \qquad \text{if } z < -Cy, \tag{5.13}$$

where $t_w = \lambda^{-1}\big\{\ln(-z) - \ln(Cy)\big\}$, if $z < -Cy$. Since $t_w < \infty$, for all $(y,z) \in \mathbb{R}^+ \times \mathbb{R}^-$ and the right-hand side of (5.12) and (5.13) are square integrable, it follows that $Y^{\bar{h}} \in \mathcal{A}_D^-(y)$, for all initial positions $Y_{0-}^{\bar{h}} = y$.

Assume that there exist $C > 0$ and $\epsilon > 0$ such that $h(y) \leq -Cy$, for $0 \leq y < \epsilon$. Then for every $y_0 \in \mathbb{R}^+$, there exist $C_{y_0} > 0$ such that $h(y) \leq -C_{y_0} y$, for all $0 \leq y \leq y_0$. Also observe that the strategy Y^h given by (4.18)–(4.20) in Lemma 4.1, with initial position $Y^h_{0-} = y_0 \in \mathbb{R}^+$, is completely determined by the values of $h(y)$ for $0 \leq y \leq y_0$. Therefore, if h_1 and h_2 are two functions satisfying $h_1(y) = h_2(y)$, for $0 \leq y \leq y_0$, then $Y^{h_1} = Y^{h_2}$ if the initial position $Y^{h_1}_{0-} = Y^{h_2}_{0-} = y$ is less than or equal to y_0. With reference to the previous parts of the proof, we therefore conclude that $Y^h \in \mathcal{A}^-_D(y)$ for all initial positions $Y^h_{0-} = y$. □

Proof. (of Proposition 4.3.) First note that for $y > 0$, the properties of ψ given in (2.4)–(2.6) imply that

$$\lim_{x \to -\infty} \Gamma(x; y) = -\infty \qquad \text{and} \qquad \lim_{x \to 0} \Gamma(x; y) = -\infty.$$

Moreover, $\Gamma(x; y)$ is continuously differentiable in x and y, for all $x < 0$ and $y > 0$. We conclude that $h = h(y)$ defined as the smallest solution to (4.24) is well defined and that h must satisfy (4.25). Moreover, ψ is strictly increasing and $\Gamma(x; 0) = \psi(x)$, which imply that $h(0) = 0$. Let $\bar{h} = \bar{h}(y)$ denote the largest solution to (4.25), and define $L : \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}^+$ by $L(h, y) = ay^2 - 2ayh$ and $H : \mathbb{R}^- \to \mathbb{R}^+$ by $H(h) = \psi'(h)h^2$. Since H is continuous and $\lim_{y\to\infty} L(h, y) = \infty$, for all $h \in \mathbb{R}^-$, it follows that $\lim_{y\to\infty} \bar{h}(y) = -\infty$. Since $h \leq \bar{h}$, we conclude that $\lim_{y\to\infty} h(y) = -\infty$.

For $\triangle > 0$, $y \in \mathbb{R}^+$ and $x \in \mathbb{R}^-$, we calculate that

$$\Gamma(x; y + \triangle) - \Gamma(x; y) = \int_y^{y+\triangle} 2a \left\{ \frac{u}{x} + \ln(-x) \right\} du,$$

and

$$\frac{d}{dx} \Big[\Gamma(x; y + \triangle) - \Gamma(x; y) \Big] = \int_y^{y+\triangle} 2a \left\{ \frac{1}{x} - \frac{u}{x^2} \right\} du < 0. \tag{5.14}$$

We want to show that $h(y)$ is strictly decreasing as a function of y. In order to get a contradiction, suppose that there exists $y \in \mathbb{R}^+$ and $\triangle > 0$ such that $h(y + \triangle) \geq h(y)$. With reference to (5.14), this implies that

$$\Gamma\big(h(y + \triangle); y + \triangle\big) - \Gamma\big(h(y + \triangle); y\big) < \Gamma\big(h(y); y + \triangle\big) - \Gamma\big(h(y); y\big).$$

However, this contradicts the definition of h, which implies that

$$\Gamma\big(h(y + \triangle); y + \triangle\big) \geq \Gamma\big(h(y); y + \triangle\big) \qquad \text{and} \qquad \Gamma\big(h(y); y\big) \geq \Gamma\big(h(y + \triangle); y\big).$$

We conclude that h is strictly decreasing. The definition of $h = h(y)$ as the smallest solution to (4.24) implies that h is càdlàg.

Introduce the function $Q : \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}$ given by

$$\begin{aligned} Q(y, z) = &\int_0^{z-y} \left(\frac{a\big(\gamma_h^{-1}(s)\big)^2}{\rho_h^{-1}(s)} + \psi\big(\rho_h^{-1}(s)\big) - \psi(s) \right) ds \\ &- \left\{ ay^2 \ln\big(-z\big) + \int_{z-y}^{z} \psi(s)\, ds - \int_0^y \left(\frac{as^2}{h(s)} + \psi\big(h(s)\big) + 2as \ln\big(-h(s)\big) \right) ds \right\}, \end{aligned}$$

which is the difference between the expression for v given by (4.26) and the expression for v given by (4.27). We calculate that

$$Q_z(y,z) = \frac{a\big(\gamma_h^{-1}(z-y)\big)^2}{\rho_h^{-1}(z-y)} + \psi\big(\rho_h^{-1}(z-y)\big) - \psi(z-y) - \left\{\frac{ay^2}{z} + \psi(z) - \psi(z-y)\right\},$$

which we observe is a continuous function of (y,z). Moreover, in view of the observation

$$\gamma_h^{-1}\big(z - h^{-1}(z)\big) = \sup\big\{y \geq 0 \,:\, \gamma_h(y) \geq z - h^{-1}(z)\big\} = h^{-1}(z),$$

it follows that $Q_z\big(h^{-1}(z), z\big) = 0$, for all $z \in \mathbb{R}^-$. We further calculate that

$$\begin{aligned} D_y^+ Q(y,z) = &-\frac{a\big(\gamma_h^{-1}(z-y)\big)^2}{\rho_h^{-1}(z-y)} - \psi\big(\rho_h^{-1}(z-y)\big) + \psi(z-y) \\ &- \left\{-\frac{ay^2}{h(y)} + \psi(z-y) - \psi\big(h(y)\big) + 2ay\big\{\ln(-z) - \ln\big(-h(y)\big)\big\}\right\}, \end{aligned}$$

and observe that the function $D_y^+ Q(y,z)$ is continuous in z and càdlàg in y. Since $\rho_h^{-1}\big(h(y) - y\big) = h(y)$, it follows that $D_y^+ Q\big(y, h(y)\big) = 0$, for all $y \in \mathbb{R}^+$. Moreover, $Q(0,0) = 0$ and hence

$$Q\big(y, h(y)\big) = \int_0^y Q_z\big(u, h(u)\big)\, dh(u) + \int_0^y D_y^+ Q\big(u, h(u)\big)\, du = 0, \qquad \text{for all } y \in \mathbb{R}^+,$$

and we conclude that

$$Q\big(h^{-1}(z), z\big) = 0, \qquad \text{for all } z \in \mathbb{R}^-,$$

since $Q_z\big(h^{-1}(z), z\big) = 0$, for all $z \in \mathbb{R}^-$ implies that $Q_z\big(y, h(y)\big) = 0$, for all $y \in \mathbb{R}^+$. From the properties of the function Q given above, We conclude that $v \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R}^-)$ and that $D_y^+ v(y,z)$ is continuous in z and càdlàg in y. Moreover, straightforward calculations show that v is continuously differentiable with respect to y for $z \geq h(y)$.

Standard calculations show that v satisfy (4.8) and (4.10). In order to verify that v satisfies (4.9), we calculate that

$$\begin{aligned} &zv_z(y,z) - ay^2 - z\big\{\psi(z) - \psi(z-y)\big\} \\ &= -ay^2 + z\left\{\frac{a\big(\gamma_h^{-1}(z-y)\big)^2}{\rho_h^{-1}(z-y)\big)} + \psi\big(\rho_h^{-1}(z-y)\big) - \psi(z)\right\}, \end{aligned} \tag{5.15}$$

for $z \geq h(y)$. Set $s = \rho_h^{-1}(z-y)$, which is equivalent to $z - y = \rho_h(s)$. As ρ_h^{-1} is an increasing function and $y \geq h^{-1}(z)$ is equivalent to $z - y \leq \rho_h(z)$, it follows that $y \geq h^{-1}(z)$ if and only if $s \leq z$. Since $\gamma_h^{-1}(z-y) = \gamma_h^{-1}\big(\rho_h(s)\big) = h^{-1}(s)$, it follows from (5.15) that

$$\sup_{y \geq h^{-1}(z)} \left\{zv_z(y,z) - ay^2 - z\big\{\psi(z) - \psi(z-y)\big\}\right\} = z \inf_{s \leq z} G(s;z), \tag{5.16}$$

where

$$G(s;z) = \frac{a\big(h^{-1}(s)\big)^2}{s} - \frac{a\big(z - \rho_h(s)\big)^2}{z} + \psi(s) - \psi(z).$$

In particular, observe that $G(z;z)=0$. We calculate that

$$
\begin{aligned}
0 &= G(s;z) + \int_s^z -\frac{a\big(h^{-1}(u)\big)^2}{u^2}\,du + \int_s^z \frac{2ah^{-1}(u)}{u}\,dh^{-1}(u) \\
&\quad - \int_s^z \frac{2a}{z}\big\{z-u+h^{-1}(u)\big\}\,d\big(h^{-1}(u)-u\big) - \int_s^z \psi'(u)\,du \\
&= G(s;z) - \int_s^z \left(\psi'(u) + \frac{a\big(h^{-1}(u)\big)^2}{u^2} - \frac{2ah^{-1}(u)}{u}\right) du \\
&\quad + \int_s^z 2a\left\{h^{-1}(u)\left(\frac{1}{u}-\frac{1}{z}\right) + \left(\frac{u}{z}-1\right)\right\} d\big(h^{-1}(u)-u\big) \\
&= G(s;z) - \int_s^{h(h^{-1}(s)-)} \left(\psi'(u) + \frac{a\big(h^{-1}(s)\big)^2}{u^2} - \frac{2ah^{-1}(s)}{u}\right) du \\
&\quad - \int_{h(h^{-1}(z))}^z \left(\psi'(u) + \frac{a\big(h^{-1}(z)\big)^2}{u^2} - \frac{2ah^{-1}(z)}{u}\right) du \\
&\quad + \int_s^z 2a\left\{h^{-1}(u)\left(\frac{1}{u}-\frac{1}{z}\right) + \left(\frac{u}{z}-1\right)\right\} d\big(h^{-1}(u)-u\big), \qquad (5.17)
\end{aligned}
$$

since $h^{-1}(u)$ is constant for $h(h^{-1}(\xi)) \le u \le h(h^{-1}(\xi)-)$, for any $\xi \in \mathbb{R}^-$ and

$$
\int_{h(h^{-1}(s)-)}^{h(h^{-1}(z))} \left(\psi'(u) + \frac{a\big(h^{-1}(s)\big)^2}{u^2} - \frac{2ah^{-1}(s)}{u}\right) du = 0,
$$

for any $s \le z \le 0$, by the definition of h and the continuity of Γ, which implies that $\Gamma\big(h(h^{-1}(s)-);h^{-1}(s)\big) = \Gamma\big(h(h^{-1}(s));h^{-1}(s)\big)$, for every $s \in \mathbb{R}^-$. Also observe that

$$
-\int_s^{h(h^{-1}(s)-)} \left(\psi'(u) + \frac{a\big(h^{-1}(s)\big)^2}{u^2} - \frac{2ah^{-1}(s)}{u}\right) du = \Gamma\big(s;h^{-1}(s)\big) - \Gamma\big(h(h^{-1}(s));h^{-1}(s)\big),
$$

which is negative by the optimality of h. Since $s \le z$ and $u \mapsto h^{-1}(u)-u$ is strictly decreasing, it follows from (5.17) that

$$
\begin{aligned}
0 &\le G(s;z) - \int_{h(h^{-1}(z))}^z \left(\psi'(u) + \frac{a\big(h^{-1}(z)\big)^2}{u^2} - \frac{2ah^{-1}(z)}{u}\right) du \\
&\quad + \int_{h(h^{-1}(z))}^z 2a\left\{h^{-1}(u)\left(\frac{1}{u}-\frac{1}{z}\right) + \left(\frac{u}{z}-1\right)\right\} d\big(h^{-1}(u)-u\big) \\
&= G(s;z) + \psi\big(h(h^{-1}(z))\big) - \psi(z) \\
&\quad + \int_{h(h^{-1}(z))}^z \left\{2a\left(\frac{h^{-1}(z)}{z}+1\right) - \frac{a\big(h^{-1}(z)\big)^2}{u^2} - \frac{2au}{z}\right\} du \\
&\le G(s;z), \qquad (5.18)
\end{aligned}
$$

since ψ is strictly increasing, $h\big(h^{-1}(z)\big) \le z$ and

$$2a\left(\frac{h^{-1}(z)}{z}+1\right)-\frac{a\big(h^{-1}(z)\big)^2}{u^2}-\frac{2au}{z}\le\frac{2ah^{-1}(z)}{z}<0,\qquad \text{for } u\le z.$$

With reference to (5.16) and (5.18), we conclude that v satisfies (4.9). Finally, we need to show that v satisfies (4.11). We calculate that

$$\begin{aligned}D_y^+v(y,z)+v_z(y,z)&=ay^2\left(\frac{1}{z}-\frac{1}{h(y)}\right)+\psi(z)-\psi\big(h(y)\big)+2ay\ln\left(\frac{z}{h(y)}\right)\\&=\Gamma(z;y)-\Gamma\big(h(y);y\big)\\&\le 0,\end{aligned}\tag{5.19}$$

by the definition of h. □

Proof. (of Theorem 4.4.) Let δ be a non-negative $C^\infty(\mathbb{R})$ function with support in $[0,1]$ satisfying $\int_0^1\delta(x)\,dx=1$, and define a sequence of functions $\{\delta_n\}_{n=1}^\infty$ by

$$\delta_n(s)=n\,\delta(ns),\quad s\ge 0.$$

We mollify v to obtain a sequence of function $\{v^{(n)}\}_{n=1}^\infty$, given by

$$v^{(n)}(y,z)=\int_0^1 v(y+s,z)\,\delta_n(s)\,ds$$

Then $v^{(n)}\in C^{1,1}(\mathbb{R}^+\times\mathbb{R}^-)$, for all $n\in\mathbb{N}$, and

$$\begin{aligned}v(y,z)&=\lim_{n\to\infty}v^{(n)}(y,z),\\ v_z(y,z)&=\lim_{n\to\infty}v_z^{(n)}(y,z),\\ D_y^+v(y,z)&=\lim_{n\to\infty}v_y^{(n)}(y,z),\end{aligned}$$

where the last equality is due to $D_y^+v(y,z)$ being càdlàg in y. Moreover, for every $(y_0,z_0)\in\mathbb{R}^+\times\mathbb{R}^-$ there exists a $K>0$ such that

$$\big|\,v^{(n)}(y,z)\,\big|\le K,\quad 0\le y\le y_0,\ z_0-y_0\le z\le 0\text{ and }n\in\mathbb{N},\tag{5.20}$$

$$\big|\,v_z^{(n)}(y,z)\,\big|\le K,\quad 0\le y\le y_0,\ z_0-y_0\le z\le 0\text{ and }n\in\mathbb{N},\tag{5.21}$$

$$\big|\,v_y^{(n)}(y,z)\,\big|\le K,\quad 0\le y\le y_0,\ z_0-y_0\le z\le 0\text{ and }n\in\mathbb{N}.\tag{5.22}$$

Then

$$\begin{aligned}&v^{(n)}\big(Y_T,Z_T^Y\big)+\int_0^T\Big\{\lambda aY_{t-}^2+\lambda Z_{t-}^Y\{\psi\big(Z_{t-}^Y\big)-\psi\big(Z_{t-}^Y-Y_{t-}\big)\}\Big\}\,dt\\&=v^{(n)}(y,z)+\int_0^T\Big\{v_y^{(n)}(Y_{t-},Z_{t-}^Y)+v_z^{(n)}(Y_{t-},Z_{t-}^Y)\Big\}\,dY_t^c\\&\quad+\sum_{0\le t\le T}\Big\{v^{(n)}\big(Y_{t-}+\triangle Y_t,Z_{t-}^Y+\triangle Y_t\big)-v^{(n)}\big(Y_{t-},Z_{t-}^Y\big)\Big\}\end{aligned}$$

$$-\lambda\int_0^T\left\{Z^Y_{t-}v^{(n)}_z(Y_{t-},Z^Y_{t-})-aY^2_{t-}-Z^Y_{t-}\big\{\psi\big(Z^Y_{t-}\big)-\psi\big(Z^Y_{t-}-Y_{t-}\big)\big\}\right\}dt, \tag{5.23}$$

for all $Y\in\mathcal{A}^-_D(y)$. For every $\epsilon>0$ and $Y\in\mathcal{A}^-_D(y)$ there exists $t_0\in\mathbb{R}^+$ such that $Y_t\le\epsilon$, for all $t\ge t_0$. Therefore

$$|Z^Y_t|\le\ \big|\,Z^Y_{t_0}\mathrm{e}^{-\lambda(t-t_0)}\,\big|\ +\ \left|\int_{t_0}^t\mathrm{e}^{-\lambda(t-s)}\,dY_s\right|\ \le|Z^Y_{t_0}|\,\mathrm{e}^{-\lambda(t-t_0)}+\epsilon,$$

from which it follows that Z^Y_t tends to 0 as $t\to\infty$. With reference to (2.8), we therefore conclude that

$$\int_0^\infty|Z^Y_t|\,dt=\frac{|z-y|}{\lambda}. \tag{5.24}$$

With reference to (5.21), (5.22) and (5.24), we calculate that

$$\int_0^\infty\sup_{n\in\mathbb{N}}\left|\,Z^Y_{t-}\left(v^{(n)}_z(Y_{t-},Z^Y_{t-})-\big\{\psi\big(Z^Y_{t-}\big)-\psi\big(Z^Y_{t-}-Y_{t-}\big)\big\}\right)\,\right|\,dt\le\frac{K+C_1}{\lambda}|z-y|,$$

for some constant $C_1>0$ which may depend on the initial conditions y and z. Similarly,

$$\int_0^\infty\sup_{n\in\mathbb{N}}\left|\,v^{(n)}_y\big(Y_{t-},Z^Y_{t-}\big)+v^{(n)}_z\big(Y_{t-},Z^Y_{t-}\big)\,\right|\,d(-Y^c_t)\le 2Ky$$

and

$$\sum_{0\le t\le\infty}\sup_{n\in\mathbb{N}}\left|\,v^{(n)}\big(Y_{t-}+\triangle Y_t,Z^Y_{t-}+\triangle Y_t\big)-v^{(n)}\big(Y_{t-},Z^Y_{t-}\big)\,\right|\,\le 2Ky.$$

Hence, by (5.23) and the dominated convergence theorem, we obtain that

$$\begin{aligned}\int_0^\infty&\left\{\lambda aY^2_{t-}+\lambda Z^Y_{t-}\big\{\psi\big(Z^Y_{t-}\big)-\psi\big(Z^Y_{t-}-Y_{t-}\big)\big\}\right\}dt\\&=v(y,z)+\int_0^\infty\left\{D^+_y v\big(Y_{t-},Z^Y_{t-}\big)+v_z\big(Y_{t-},Z^Y_{t-}\big)\right\}dY^c_t\\&\quad+\sum_{t\ge0}\left\{v\big(Y_{t-}+\triangle Y_t,Z^Y_{t-}+\triangle Y_t\big)-v\big(Y_{t-},Z^Y_{t-}\big)\right\}\\&\quad-\lambda\int_0^\infty\left\{Z^Y_{t-}v_z\big(Y_{t-},Z^Y_{t-}\big)-aY^2_{t-}-Z^Y_{t-}\big\{\psi\big(Z^Y_{t-}\big)-\psi\big(Z^Y_{t-}-Y_{t-}\big)\big\}\right\}dt,\end{aligned} \tag{5.25}$$

for any $Y\in\mathcal{A}^-_D(y)$, by taking the limits as $n\to\infty$ and $T\to\infty$, and noting that $v(Y_T,Z^Y_T)$ tends to 0 as $T\to\infty$ due to the boundary condition $v(0,z)=0$. Since according to Proposition 4.3, v satisfy (4.8)–(4.11), it follows that

$$\int_0^\infty\left\{\lambda aY^2_{t-}+\lambda Z^Y_{t-}\big\{\psi\big(Z^Y_{t-}\big)-\psi\big(Z^Y_{t-}-Y_{t-}\big)\big\}\right\}dt\ge v(y,z), \tag{5.26}$$

and thus $V\ge v$.

With reference to Assumption 2.3 and (2.6), it follows that there exists $C > 0$ and $\epsilon > 0$ such that $h(y) \geq -Cy$, for $-\epsilon \leq y \leq 0$, where h denotes the smallest solution to (4.24). According to Lemma 4.2, we therefore have $Y^* = Y^h \in \mathcal{A}_D^-(y)$, for all initial positions $Y^*_{0-} = y$. We want to show that (5.26) holds with equality for Y^*. Observe that $\triangle Y^*_t < 0$ only if $t = 0$ and $z > h(y)$, in which case $\triangle Y^*_0$ is such that after the jump, $Y^*_0 = h^{-1}(Z_0^{Y^*})$. With reference to (4.8) and Proposition 4.3, we have that $v_y(y,z) + v_z(y,z) = 0$, for $z \geq h(y)$. Therefore

$$\sum_{t\geq 0} \Big\{ v\big(Y^*_{t-} + \triangle Y^*_t, Z^{Y^*}_{t-} + \triangle Y^*_t\big) - v\big(Y^*_{t-}, Z^{Y^*}_{t-}\big) \Big\} = 0. \tag{5.27}$$

If $z < h(y)$ then $Y^*_t = y$, for $0 \leq t \leq t_w$, where t_w is as defined in Lemma 4.1. Also

$$Z^{Y^*}_t = z\mathrm{e}^{-\lambda t}, \qquad \text{for } 0 \leq t \leq t_w.$$

Moreover $Z^{Y^*}_t < h(y)$, for $t < t_w$, and $Z^{Y^*}_{t_w} = h(y)$. With reference to (4.10) and Proposition 4.3, it follows that

$$\int_0^{t_w} \Big\{ Z^{Y^*}_{t-} v_z\big(Y^*_{t-}, Z^{Y^*}_{t-}\big) - a(Y^*_{t-})^2 - Z^{Y^*}_{t-} \big\{ \psi\big(Z^{Y^*}_{t-}\big) - \psi\big(Z^{Y^*}_{t-} - Y^*_{t-}\big) \big\} \Big\}\, dt = 0.$$

By definition $Y^*_t = h^{-1}(Z^{Y^*}_t)$, for $t \geq t_w$. According to Proposition 4.3, v satisfies (4.8), and therefore

$$\int_{t_w}^{\infty} \Big\{ D^+_y v\big(Y^*_{t-}, Z^{Y^*}_{t-}\big) + v_z\big(Y^*_{t-}, Z^{Y^*}_{t-}\big) \Big\}\, d(Y^*_t)^c = 0.$$

Moreover, since v satisfies (4.10), and therefore

$$\int_{t_w}^{\infty} \Big\{ Z^{Y^*}_{t-} v_z\big(Y^*_{t-}, Z^{Y^*}_{t-}\big) - a(Y^*_{t-})^2 - Z^{Y^*}_{t-} \big\{ \psi\big(Z^{Y^*}_{t-}\big) - \psi\big(Z^{Y^*}_{t-} - Y^*_{t-}\big) \big\} \Big\}\, dt = 0.$$

With reference to (5.25), we therefore conclude that $v = V$ and that $Y^* = Y^h \in \mathcal{A}_D^-(y)$ is an admissible optimal liquidation strategy for the optimization problem (3.10). The result then follows from (3.9). ☐

References

[1] I. Aldridge (2010), High-Frequency Trading, *John Wiley & Sons*.

[2] R. Almgren and N. Chriss (1999), Value under liquidation, *Risk* **12**(12), 61–63.

[3] R. Almgren and N. Chriss (2001), Optimal execution of portfolio transactions, *J. Risk* **3**(2), 5–39.

[4] R. F. Almgren (2003), Optimal execution with nonlinear impact functions and trading-enhanced risk, *Applied Mathematical Finance* **10**, 1–18.

[5] A. Alfonsi, A. Fruth and A. Schied (2010), Optimal execution strategies in limit order books with general shape functions, *Quantitative Finance* **10**, 143–157.

[6] A. Alfonsi and A. Schied (2010), Optimal trade execution and absence of price manipulations in limit order book models, *SIAM J. Financial Math.* **1**, 490–522.

[7] A. Alfonsi, A. Schied and A. Slynko (2011), Order book resilience, price manipulation, and the positive portfolio problem, *Preprint*.

[8] P. Bank and D. Baum (2004), Hedging and portfolio optimization in financial markets with a large trader, *Mathematical Finance* **14** (1), 1–18.

[9] D. Bertsimas and A. W. Lo (1998), Optimal control of execution costs, *Journal of Financial Markets* **1**, 1–50.

[10] U. Cetin, R. A. Jarrow and P. Protter (2004), Liquidity risk and arbitrage pricing theory, *Finance and Stochast.* **8**, 1–31.

[11] W. H. Fleming and H. M. Soner (1993), Controlled Markov Processes and Viscosity Solutions, *Springer-Verlag.*

[12] G. B. Folland (1984), Real Analysis, *John Wiley & Sons.*

[13] J. Gatheral (2010), No-dynamic-arbitrage and market impact, *Quantitative Finance* **10**(7), 749–759.

[14] J. Gatheral, A. Schied and A. Slynko (2011), Exponential resilience and decay of market impact, *In: Econophysics of Order-driven Markets, F. Abergel, B. K. Chakrabarti, A. Chakraborti, M. Mitra (Eds.), 225–236, Springer.*

[15] G. Huberman and W. Stanzl (2004), Price manipulation and quasi-arbitrage, *Econometrica* **72** (4), 1247–1275.

[16] R. Kissell and R. Malamut (2005), Understanding the profit and loss distribution of trading algorithms, *Institutional Investor*, Guide to Algorithmic Trading, Spring 2005.

[17] A. Obizhaeva and J. Wang (2005), Optimal trading strategy and supply/demand dynamics, *Working paper.*

[18] S. Predoiu, G. Shaikhet and S. Shreve (2011), Optimal execution in a general one-sided limit-order book, *SIAM J. Finan. Math.* **2**, 183–212.

[19] P. Protter (1990), Stochastic Integration and Differential Equations, *Springer.*

[20] W. Rudin (1991), Functional Analysis, *McGraw-Hill.*

[21] A. Schied and T. Schöneborn (2009), Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets, *Finance Stoch* **13**, 181–204.

[22] A. Schied, T. Schöneborn and M. Tehranchi (2010), Optimal basket liquidation for CARA investors is deterministic, *Applied Mathematical Finance* **17**(6), 471–489.

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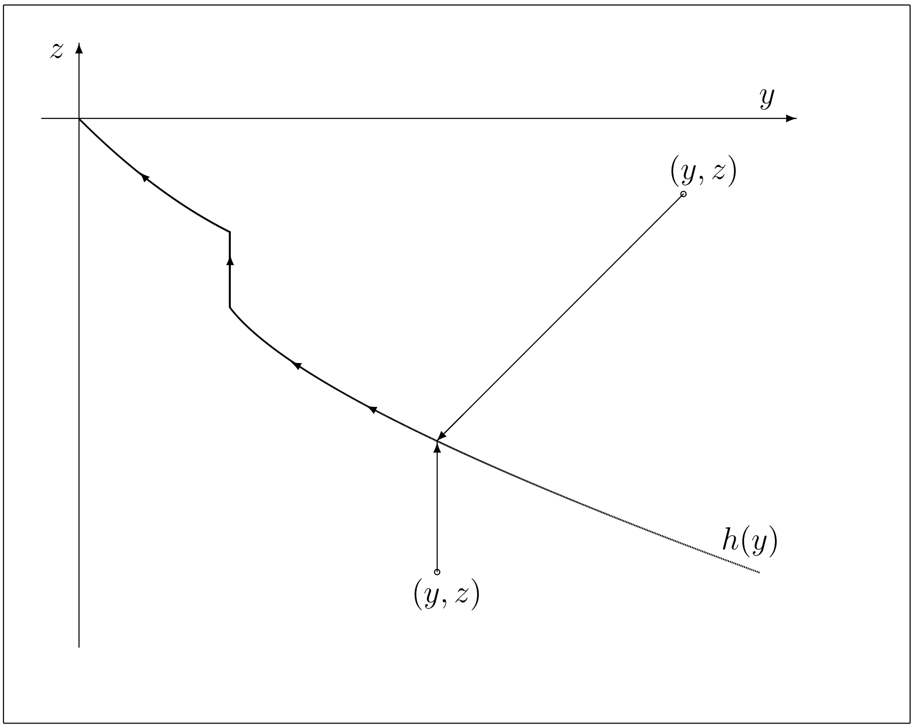

Figure 1. Illustration of the strategy Y^h corresponding to h.