432018 PHILOSOPHY OF PHYSICS (Spring 2002)

Lecture 7: Determinism and the problem of hidden variables

Preliminary reading: Sklar, pp. 200-12.

We now turn to the second group of theories which deny the collapse postulate, i.e. those theories which deny that the states we have been using to represent quantum systems give us a *complete* description of such systems. As such, these theories postulate the existence of 'hidden variables', i.e. variables which are relevant to the description of the system but are not accounted for by the state. Our discussion of these theories will take two lectures. In this lecture we consider why one might want to utilise hidden variables and the arguments which purported to show that such hidden variable theories were *impossible*. Then, in the next lecture, we shall consider Bohm's hidden variable theory.

1 Determinism and hidden variables

Philosophically, a precise account of what is meant by the term 'determinism' has proved to be particularly elusive. However, in spite of this, it seems quite clear that most classical phenomena are deterministic in the simple sense that: 'given the state of a system and the interactions which affect it, we can determine the state of the system at a later time'. And, this is generally believed to be true despite the fact that:

- it could give rise to philosophical problems concerning such concepts as 'free will' and 'fatalism'.
- there are some purported counterexamples to this thesis.[1]

But, these philosophical and 'classically'-oriented problems with determinism are not our concern here.

So, getting back to the point, what about the world described by QM? Should we be tempted to say that it is indeterministic? Well,

- on the one hand, if we have a system which is in some state and isolated from certain external interactions (in particular, from measurements), then the temporal evolution of this state is governed by the Schrödinger equation, and this, as we know, is deterministic in the sense that it allows us to determine the state of the system at a later time.
- whereas, on the other, given an observable quantity that we are trying to measure, there is a fixed set of possible outcomes and the quantum state only allows us to infer that one of these outcomes will arise with a certain probability, i.e. QM *appears* to be indeterministic since there is no way of determining the post-measurement state of the system based on our pre-measurement information.[2]

On the basis of this latter fact, it is normal to claim that QM is indeterministic.

However, this latter fact need not commit us to indeterminism in QM since it is possible that there is something else, a 'hidden variable' quantity say, which would allow us to determine the outcome of the measurement in the appropriate way. That is, if we were to claim that the description of reality given by the quantum states was *incomplete* we could, perhaps, develop a quantum mechanical theory that was deterministic. Consequently, we can see that we need to investigate whether such hidden variable theories are possible, and then, if they are and we can find one, we may be in a position to reassess the claim that QM is indeterministic.

[1] Although, in my opinion, these are quite dodgy.

[2] Although, recall that if the system is in an eigenstate of the operator corresponding to the observable, then we can predict the outcome of the measurement with certainty.

2 Arguments against hidden variable theories

For a long time the majority of physicists thought that a hidden variable theory, such as the one that we will consider in the next lecture, was impossible. This was mainly due to the 'impossibility proof' which von Neumann gave to rule out such theories. This 'proof' and the reason why it is inadequate will be our starting point. We will then briefly consider the Kochen-Specker theorem, which rather than ruling hidden variable theories out, places some *restrictions* on the form that such a theory could take.

2.1 Von Neumann — the impossibility of hidden variable theories

Let's consider how von Neumann's 'impossibility proof' works in a specific case.[3] Once we have done this, we will be in a position to see why it fails to rule out the theory which we shall consider in the next lecture.

Setting up

Assume that we have a quantum system, and that we are going to make a measurement of the observable quantity which is represented by the operator $\hat{O}$. Furthermore, let's assume that prior to the measurement the state of the system is $|\psi\rangle$ and the hidden variable (whatever it is) is X. Now, the result of the experiment, let's call it R, is obviously going to depend on: *what* we are measuring, i.e. $\hat{O}$; the *state* of the system prior to the measurement, i.e. $|\psi\rangle$; and the *value* of the hidden variable, i.e. X. That is, the result we find will be determined by these three things and so we represent the results of a measurement as a function of them, i.e. $R(\hat{O}, |\psi\rangle, X)$. Now, the vital assumption in von Neumann's proof is that, given operators $\hat{O}$, $\hat{P}$ and $\hat{Q}$ which are linearly related, i.e.

$$\hat{O} = \alpha\hat{P} + \beta\hat{Q},$$

for some $\alpha, \beta \in \mathbb{R}$, the results must be similarly related, i.e.

$$R(\hat{O}, |\psi\rangle, X) = \alpha R(\hat{P}, |\psi\rangle, X) + \beta R(\hat{Q}, |\psi\rangle, X),$$

where, of course, the results of each of these measurements[4] must be eigenvalues of the appropriate operator.

A specific case

Now, consider the [non-commuting] operators $\hat{S}_x$ and $\hat{S}_y$ which are the operators that represent measurements of the x and y-spin of an electron (say). That is, depending on the value of X (and in units of $\hbar/2$), the result of a

- measurement of the electron's x-spin will be either 1 or -1, and
- measurement of the electron's y-spin will be either 1 or -1,

depending on whether the outcome of the experiment is either 'spin-up' or 'spin-down' in the appropriate direction. Further, consider a measurement of the electron's spin in the direction $\mathbf{n}$ which is at 45° to the x and y-axes in the xy-plane (see Figure 1). If $\hat{S}_{\mathbf{n}}$ is the operator corresponding to this measurement, we find that

$$\hat{S}_{\mathbf{n}} = \frac{1}{\sqrt{2}}(\hat{S}_x + \hat{S}_y),$$

and (in the same units) the result of a

- measurement of the electron's $\mathbf{n}$-spin will be either 1 or -1,

[3]Obviously, due to the technical nature of von Neumann's work, we are in no position to consider the general case here.

[4]That is, the appropriate values of the function R.

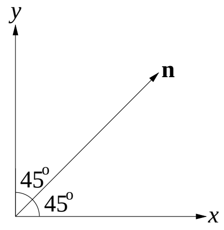

Figure 1: *The x, y and $\mathbf{n}$-directions in our example of von Neumann's 'impossibility proof'.*

But here, we have an operator $\hat{S}_{\mathbf{n}}$ which is linearly related to the operators $\hat{S}_x$ and $\hat{S}_y$, and so von Neumann's assumption demands that we should get results which are such that

$$R(\hat{S}_{\mathbf{n}}, |\psi\rangle, X) = \frac{1}{\sqrt{2}}\Big[R(\hat{S}_x, |\psi\rangle, X) + R(\hat{S}_y, |\psi\rangle, X)\Big],$$

or, on rearranging, we have

$$\sqrt{2}\, R(\hat{S}_{\mathbf{n}}, |\psi\rangle, X) = R(\hat{S}_x, |\psi\rangle, X) + R(\hat{S}_y, |\psi\rangle, X), \tag{A}$$

which is clearly impossible since the right-hand-side of (A) can only take the values given by:

$R(\hat{S}_x, \lvert\psi\rangle, X)$	$R(\hat{S}_y, \lvert\psi\rangle, X)$	RHS
1	1	2
1	-1	0
-1	1	0
-1	-1	-2

whereas the left-hand-side of (A) can only take the values $\sqrt{2}$ and $-\sqrt{2}$. Consequently, if von Neumann is right in insisting that (A) should hold in this case, since this equality can *not* hold for any of the allowed values of the function R, there can be no way of determining the results of a measurement by using a function like R which relies on a hidden variable such as X.

Why the 'proof' doesn't work

Now, this may seem pretty convincing, and it did in fact convince a lot of people. But there is a problem, namely that von Neumann's vital assumption is essentially that

> (VN) the eigenvalues of linearly related operators are themselves linearly related.

And, although this must certainly hold once we have averaged over X to get the quantum 'averages', this cannot possibly hold prior to 'averaging' since, as a matter of *mathematical* fact, the eigenvalues of three linearly related non-commuting operators are *not* themselves linearly related. Thus, von Neumann's impossibility proof works by demanding that a hidden variable theory must do something which is *mathematically* impossible for *any* theory to do.[5]

Or, to put it in more physical terms, von Neumann has made an assumption which effectively demands that an arbitrary linear relation must hold between the results of three *incompatible* measurements. That is, three measurements which *could* have been made on a given occasion, but only one of which *can*, in fact, be made in any given experiment.

2.2 Kochen and Specker — a restriction on the form of a hidden variable theory

The Kochen-Specker theorem essentially tells us that we cannot consistently ascribe a 'full set' of classical properties to a quantum system. That is, it is impossible for us to have a hidden variable theory which would allow us to give such an ascription of properties to the system under consideration.

[5]Just to make this point explicit, this includes QM *itself*.

Again, to see how the Kochen-Specker theorem works, we shall look at a specific example.[6] Once we have done this, we shall consider the kind of restriction it places on hidden variable theories.

The basic idea

The basic idea behind the Kochen-Specker theorem can be illustrated by considering the situation in Figure 2. Here we have a quantum system which can possess a certain quantity which takes the

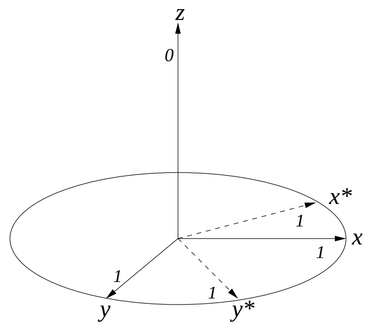

Figure 2: *Illustrating the basic idea behind the Kochen-Specker theorem.*

values 0 and 1 (say) in three mutually perpendicular directions in such a way that one of these directions must take the value 0 and the other two directions must take the value 1. The question we are asking is then: Can we have some kind of hidden variable that will allow us to determine which values this quantity should take in all possible sets of mutually perpendicular directions? The argument that establishes the Kochen-Specker theorem tells us that the answer to this question is 'no'. And, this argument relies on the fact that any given direction is a member of an infinite number of different sets of mutually perpendicular directions. (For example, returning to Figure 2, we see that the z-direction belongs to the sets of mutually perpendicular directions given by x, y, z and x^*, y^*, z.) Indeed, one proceeds by showing that it is impossible to assign the values 0 and 1 to all possible sets of mutually perpendicular directions in the required way.

The upshot of this is that, although it may be possible for us to 'simultaneously measure' the values of this quantity in three mutually perpendicular directions, it is not generally possible for us to simultaneously measure its values in directions which are not perpendicular to each other (For example, in the directions given by x and x^*.) As such, if we want a theory that incorporates hidden variables which determine the values of this quantity, the values that *are* determined may have to vary with our choice of mutually perpendicular directions when we make our simultaneous measurements. That is, the experimental *context*, exactly how we orientate our apparatus in these directions when we make our measurements, will have to be 'factored in' to our hidden variable theory.

A specific example[7]

Let us consider a pair of electrons (labelled '1' and '2') which are in a 'singlet state', i.e. a state which looks like:

$$\frac{1}{\sqrt{2}}\Big(|\uparrow_x\rangle_1|\downarrow_x\rangle_2 - |\downarrow_x\rangle_1|\uparrow_x\rangle_2\Big),$$

and we denote the outcome of a measurement of the x-spin of electron 1, i.e. $\hat{S}_{1,x}$, by x_1 and a measurement of the x-component of electron 2, i.e. $\hat{S}_{2,x}$, by x_2. Now,

i. we can measure the product of these two operators, i.e. $\hat{S}_{1,x}\hat{S}_{2,x}$, by measuring $\hat{S}_{1,x}$ and $\hat{S}_{2,x}$ separately.[8]

[6]Obviously, since the Kochen-Specker theorem is a technical result, we are in no position to consider the general case here.

[7]This is taken from: A. Peres, *Incompatible Results of Quantum Measurements*, Physics Letters A, **151** (1990), p. 107.

[8]Or, alternatively, using a single non-local procedure. (See later.)

And, we find that the expected value of $\hat{S}_{1,x}\hat{S}_{2,x}$ is -1, i.e. we have $x_1x_2 = -1$. Likewise, we would find that $y_1y_2 = -1$ and $z_1z_2 = -1$ if we were doing a similar thing for the y and z-components of the electrons.

Now, let's consider

ii. a measurement of $\hat{S}_{1,x}\hat{S}_{2,y}$ where we measure $\hat{S}_{1,x}$ and $\hat{S}_{2,y}$ separately.

and *assume* that

(A) The outcomes x_1 and y_2 obtained when measuring $\hat{S}_{1,x}\hat{S}_{2,x}$ are the same as the ones we found above.

That is, we are assuming that the results of these measurements do *not* depend on whether the measurement on electron 2 is[9] a measurement of $\hat{S}_{2,x}$ (as in i.) *or* $\hat{S}_{2,y}$ (as in ii.).

But, this leads to trouble since the products $\hat{S}_{1,y}\hat{S}_{2,x}$ and $\hat{S}_{1,x}\hat{S}_{2,y}$ commute, i.e. we can measure them both without disturbing the respective outcomes,[10] and using (A) we can see that the result of measuring $\hat{S}_{1,y}\hat{S}_{2,x}$ is y_1x_2. But, the product of these two commuting operators yields

$$\hat{S}_{1,x}\hat{S}_{2,y}\hat{S}_{1,y}\hat{S}_{2,x} = \hat{S}_{1,x}\hat{S}_{1,y}\hat{S}_{2,y}\hat{S}_{2,x} = \hat{S}_{1,z}\hat{S}_{2,z},$$

and so, from above, we can see that the result of these measurements would be such that

$$x_1y_2y_1x_2 = z_1z_2 = -1,$$

But, this *contradicts* our earlier claim that $x_1x_2 = y_1y_2 = -1$.[11] Thus, we can see that our *assumption*, i.e. (A), must be false. And, as such, the state of the system *must also depend on which other measurements we choose to perform.*

NOTE: Although, like in von Neumann's 'impossibility proof', the Kochen-Specker theorem establishes that something is impossible. The latter impossibility, unlike the former, is an impossibility that is a direct consequence of the quantum mechanical formalism. As such, any theory that wants to give us a deeper understanding *of quantum mechanics* is constrained by it.

The consequences of the Kochen-Specker theorem for hidden variable theories

The result of the Kochen-Specker theorem shouldn't really surprise you, especially since we have seen this sort of idea before. You will recall, for instance, that the Copenhagen interpretation tells us that:

- there is no 'fact of the matter' about what properties a quantum system has prior to measurement, and
- given a choice of complementary properties which may be used to describe a quantum system, the one which is relevant for a description of the system in question depends on the measurement that we choose to make.

So, although we may not have really understood what Bohr was saying, it is nice to see that some of his ideas are firmly rooted in the quantum mechanical formalism.

To use some more relevant terms, we say that we cannot attribute properties to quantum systems without a *context*. That is, a *non*-contextual attribution of properties, i.e. an attribution of properties without due regard for the measurements that are going to be made, will lead to Kochen-Specker-like contradictions. But, this seems to go against hidden variable theories since one would *suppose* that these variables would attribute states to the system regardless of what measurements we are going to perform. But, this is not the case and, in particular, we will see that Bohr's insistence that we must 'consider the experimental set-up as a whole' is as important in Bohm's hidden variable theory as it is in other interpretations of QM.

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[9] Or was, or will be.

[10] Although, we can't measure $\hat{S}_{1,y}\hat{S}_{2,x}$ *and* measure both $\hat{S}_{1,x}$ and $\hat{S}_{2,y}$ separately since these three operators don't commute. That is, these three operators represent 'incompatible' or 'complementary' measurements.

[11] As this implies that $x_1y_2y_1x_2 = x_1(-1)x_2 = (-1)x_1x_2 = (-1)(-1) = 1$.