If a matrix is not diagonalisable, we say that it is *deficient*. But, as we shall now see, given a deficient matrix, say A, we can find an invertible matrix S, such that

$$\mathsf{S}^{-1}\mathsf{A}\mathsf{S} = \mathsf{J},$$

where the matrix J is *almost* diagonal. In particular, we are going to find a matrix S which yields the *Jordan Normal Form* (or JNF) of A, i.e.

$$\mathsf{J} = \begin{bmatrix} \lambda_1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{n-1} & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where the entries are as follows

- along the diagonal of J, i.e. the elements $(\mathsf{J})_{i,i}$ for $1 \leq i \leq n$, we have the eigenvalues of A,
- along the 'upper off-diagonal', i.e. the elements $(\mathsf{J})_{i,i+1}$ for $1 \leq i \leq n-1$, we have the '$*$'s each of which is either a zero or a one,
- every other entry, i.e. $(\mathsf{J})_{i,j}$ for $1 \leq i, j \leq n$ with $j \neq i, i+1$, is zero.

Clearly, this is *almost* diagonal since this matrix would be diagonal if it wasn't for the fact that some (or all) of the '$*$'s could be a one.

Let's consider a 3×3 matrix A and see how this would work. We have the following cases:

- A has three distinct eigenvalues, in which case A is diagonalisable.
- A has two distinct eigenvalues, say λ_1 and λ_2, where $a_{\lambda_1} = 2$. In this case, we could have
  - $g_{\lambda_1} = 2$, in which case A is diagonalisable.
  - $g_{\lambda_1} = 1$, in which case A is not diagonalisable and we seek the JNF of A. Let's call this **Case 1**.
- A has one distinct eigenvalue, say λ_1, where $a_{\lambda_1} = 3$. In this case, we could have
  - $g_{\lambda_1} = 3$, in which case A is diagonalisable.
  - $g_{\lambda_1} = 2$, in which case A is not diagonalisable and we seek the JNF of A. Let's call this **Case 2**.
  - $g_{\lambda_1} = 1$, in which case A is not diagonalisable and we seek the JNF of A. Let's call this **Case 3**.

Note that if A is diagonalisable, then all the '$*$'s will be zero in the JNF. In the other cases we find that...

Case 1: A has two eigenvalues, say $\lambda_1 \neq \lambda_2$, where $a_{\lambda_1} = 2$ and $g_{\lambda_1} = 1$.

Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be the eigenvectors corresponding to λ_1 and λ_2 respectively, i.e.†

$$\mathsf{A}\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1 \qquad \text{and} \qquad \mathsf{A}\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2,$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. We seek a vector $\boldsymbol{v}_1'$ which is related to $\boldsymbol{v}_1$ according to†

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1, \quad \text{or rearranging,} \quad \mathsf{A}\boldsymbol{v}_1' = \boldsymbol{v}_1 + \lambda_1 \boldsymbol{v}_1'.$$

We now construct an invertible matrix S with the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2$ as its columns so that $\mathsf{S}^{-1}\mathsf{AS}$ yields the JNF of A. That is, we want $\mathsf{AS} = \mathsf{SJ}$ where J has the form described above. But, since AS gives

$$\mathsf{A}\underbrace{\begin{bmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_1' & \boldsymbol{v}_2 \\ | & | & | \end{bmatrix}}_{\mathsf{S}} = \begin{bmatrix} | & | & | \\ \mathsf{A}\boldsymbol{v}_1 & \mathsf{A}\boldsymbol{v}_1' & \mathsf{A}\boldsymbol{v}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \lambda_1\boldsymbol{v}_1 & \boldsymbol{v}_1 + \lambda_1\boldsymbol{v}_1' & \lambda_2\boldsymbol{v}_2 \\ | & | & | \end{bmatrix}$$

we take J to be such that

$$\mathsf{SJ} = \begin{bmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_1' & \boldsymbol{v}_2 \\ | & | & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}}_{\text{required JNF!}} = \begin{bmatrix} | & | & | \\ \lambda_1\boldsymbol{v}_1 & \boldsymbol{v}_1 + \lambda_1\boldsymbol{v}_1' & \lambda_2\boldsymbol{v}_2 \\ | & | & | \end{bmatrix}$$

Aside 4.5.1 This means that $\boldsymbol{v}_1$ is in the null space of $\mathsf{A} - \lambda_1\mathsf{I}_3$. Notice that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent since they are eigenvectors corresponding to distinct eigenvalues.

Aside 4.5.2 This means that $\boldsymbol{v}_1'$ is not in the null space of $\mathsf{A} - \lambda_1\mathsf{I}_3$ since

$$(\mathsf{A} - \lambda_1\mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1 \neq \mathbf{0}.$$

As such, $\boldsymbol{v}_1$ and $\boldsymbol{v}_1'$ are linearly independent. We can also see that $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2$ are linearly independent. (Why? See Exercise 1.) Thus we can guarantee the invertibility of the matrix S we are constructing.

Go to Example 1

Case 2: A has one eigenvalue, say λ_1, where $a_{\lambda_1} = 3$ and $g_{\lambda_1} = 2$. †

Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be the eigenvectors corresponding to λ_1, i.e.

$$\mathsf{A}\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1 \qquad \text{and} \qquad \mathsf{A}\boldsymbol{v}_2 = \lambda_1 \boldsymbol{v}_2,$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. We take a vector $\boldsymbol{v}_3 \notin \text{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ and then find a vector, say $\boldsymbol{v}_2'$, such that†

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_3 = \boldsymbol{v}_2', \quad \text{or rearranging,} \quad \mathsf{A}\boldsymbol{v}_3 = \boldsymbol{v}_2' + \lambda_1 \boldsymbol{v}_3.$$

We now construct an invertible matrix S with the vectors $\boldsymbol{v}_i$ (where we take $\boldsymbol{v}_i$ to be whichever of $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$ is linearly independent of $\boldsymbol{v}_2'$), $\boldsymbol{v}_2'$ and $\boldsymbol{v}_3$ as its columns† so that $\mathsf{S}^{-1}\mathsf{A}\mathsf{S}$ yields the JNF of A. That is, we want $\mathsf{A}\mathsf{S} = \mathsf{S}\mathsf{J}$ where J has the form described above. But, since $\mathsf{A}\mathsf{S}$ gives

$$\mathsf{A}\underbrace{\begin{bmatrix} | & | & | \\ \boldsymbol{v}_i & \boldsymbol{v}_2' & \boldsymbol{v}_3 \\ | & | & | \end{bmatrix}}_{\mathsf{S}} = \begin{bmatrix} | & | & | \\ \mathsf{A}\boldsymbol{v}_i & \mathsf{A}\boldsymbol{v}_2' & \mathsf{A}\boldsymbol{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \lambda_1\boldsymbol{v}_i & \lambda_1\boldsymbol{v}_2' & \boldsymbol{v}_2' + \lambda_1\boldsymbol{v}_3 \\ | & | & | \end{bmatrix}$$

and so, we take J to be such that

$$\mathsf{S}\mathsf{J} = \begin{bmatrix} | & | & | \\ \boldsymbol{v}_i & \boldsymbol{v}_2' & \boldsymbol{v}_3 \\ | & | & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}}_{\text{required JNF!}} = \begin{bmatrix} | & | & | \\ \lambda_1\boldsymbol{v}_i & \lambda_1\boldsymbol{v}_2' & \boldsymbol{v}_2' + \lambda_1\boldsymbol{v}_3 \\ | & | & | \end{bmatrix}$$

Aside 4.5.3 This means that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ form a basis for the null space of $\mathsf{A} - \lambda_1 \mathsf{I}_3$.

Aside 4.5.4 Notice that the vector $\boldsymbol{v}_2'$ cannot be $\mathbf{0}$ as that would mean $\boldsymbol{v}_3 \in \text{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$! However, $\boldsymbol{v}_2'$ will be in $\text{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, i.e. it is also an eigenvector of A corresponding to the eigenvalue λ_1. (Why? See Exercise 2.)

Aside 4.5.5 As $\boldsymbol{v}_3 \notin \text{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, we can see that $\boldsymbol{v}_i$, $\boldsymbol{v}_2'$ and $\boldsymbol{v}_3$ are linearly independent. This is what guarantees the invertibility of the matrix S we are constructing.

Go to Example 2

Case 3: A has one eigenvalue, say λ_1, where $a_{\lambda_1} = 3$ and $g_{\lambda_1} = 1$.

Let $\boldsymbol{v}_1$ be the eigenvectors corresponding to λ_1, i.e.†

$$\mathsf{A}\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1,$$

We seek a vector $\boldsymbol{v}_1'$ which is related to $\boldsymbol{v}_1$ according to†

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1, \quad \text{or rearranging,} \quad \mathsf{A}\boldsymbol{v}_1' = \boldsymbol{v}_1 + \lambda_1 \boldsymbol{v}_1',$$

and a vector $\boldsymbol{v}_1''$ which is related to $\boldsymbol{v}_1'$ according to†

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_1'' = \boldsymbol{v}_1', \quad \text{or rearranging,} \quad \mathsf{A}\boldsymbol{v}_1'' = \boldsymbol{v}_1' + \lambda_1 \boldsymbol{v}_1''.$$

We now construct an invertible matrix S with the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_1'$ and $\boldsymbol{v}_1''$ as its columns so that $\mathsf{S}^{-1}\mathsf{A}\mathsf{S}$ yields the JNF of A. That is, we want $\mathsf{A}\mathsf{S} = \mathsf{S}\mathsf{J}$ where J has the form described above. But, since $\mathsf{A}\mathsf{S}$ gives

$$\mathsf{A}\underbrace{\begin{bmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_1' & \boldsymbol{v}_1'' \\ | & | & | \end{bmatrix}}_{\mathsf{S}} = \begin{bmatrix} | & | & | \\ \mathsf{A}\boldsymbol{v}_1 & \mathsf{A}\boldsymbol{v}_1' & \mathsf{A}\boldsymbol{v}_1'' \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \lambda_1\boldsymbol{v}_1 & \boldsymbol{v}_1 + \lambda_1\boldsymbol{v}_1' & \boldsymbol{v}_1' + \lambda_1\boldsymbol{v}_1'' \\ | & | & | \end{bmatrix}$$

we take J to be such that

$$\mathsf{S}\mathsf{J} = \begin{bmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_1' & \boldsymbol{v}_1'' \\ | & | & | \end{bmatrix}\underbrace{\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}}_{\text{required JNF!}} = \begin{bmatrix} | & | & | \\ \lambda_1\boldsymbol{v}_1 & \boldsymbol{v}_1 + \lambda_1\boldsymbol{v}_1' & \boldsymbol{v}_1' + \lambda_1\boldsymbol{v}_1'' \\ | & | & | \end{bmatrix}$$

Aside 4.5.6 This means that $\boldsymbol{v}_1$ is in the null space of $\mathsf{A} - \lambda_1\mathsf{I}_3$.

Aside 4.5.7 This means that $\boldsymbol{v}_1'$ is not in the null space of $\mathsf{A} - \lambda_1\mathsf{I}_3$ as $\boldsymbol{v}_1 \neq \mathbf{0}$. As such, $\boldsymbol{v}_1$ and $\boldsymbol{v}_1'$ are linearly independent. But, notice that $\boldsymbol{v}_1''$ is in the null space of $(\mathsf{A} - \lambda_1\mathsf{I}_3)^2$ as

$$(\mathsf{A} - \lambda_1\mathsf{I}_3)^2\boldsymbol{v}_1' = (\mathsf{A} - \lambda_1\mathsf{I}_3)\boldsymbol{v}_1 = \mathbf{0},$$

since $\boldsymbol{v}_1$ is in the null space of $\mathsf{A} - \lambda_1\mathsf{I}_3$.

Aside 4.5.8 This means that $\boldsymbol{v}_1''$ is not in the null space of $\mathsf{A} - \lambda_1\mathsf{I}_3$ as $\boldsymbol{v}_1 \neq \mathbf{0}$. Also, as

$$(\mathsf{A} - \lambda_1\mathsf{I}_3)^2\boldsymbol{v}_1'' = (\mathsf{A} - \lambda_1\mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1 \neq \mathbf{0},$$

$\boldsymbol{v}_1''$ is not in the null space of $(\mathsf{A} - \lambda_1\mathsf{I}_3)^2$. As such, $\boldsymbol{v}_1$, $\boldsymbol{v}_1'$ and $\boldsymbol{v}_1''$ are linearly independent. This is what guarantees the invertibility of the matrix S we are constructing.

Go to Example 3

For example: To see how **Case 1** works, consider the matrix

$$\mathsf{A} = \begin{bmatrix} 0 & 4 & 4 \\ 1 & 0 & -3 \\ -2 & 4 & 7 \end{bmatrix}$$

which has two eigenvalues $\lambda_1 = 2$ (with $a_2 = 2$) and $\lambda_2 = 3$ (with $a_3 = 1$) and corresponding linearly independent eigenvectors $\boldsymbol{v}_1 = [2, 1, 0]^t$ (i.e. $g_2 = 1$) and $\boldsymbol{v}_2 = [0, -1, 1]^t$ (i.e. $g_3 = 1$) respectively. Following the method above, we seek a vector $\boldsymbol{v}_1'$ such that $(\mathsf{A} - 2\mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1$. The easiest way to find such a vector is to find the components x, y and z of $\boldsymbol{v}_1'$ by solving the matrix equation

$$\begin{bmatrix} -2 & 4 & 4 \\ 1 & -2 & -3 \\ -2 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad x \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} + z \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

which, by inspection, has $\boldsymbol{v}_1' = [x, y, z]^t = [-5, 0, -2]^t$ as a solution.† Thus, we take our invertible matrix, S, and its associated JNF, J, to be

Aside 4.5.9 There are other solutions since the three column vectors in the matrix $(\mathsf{A} - 2\mathsf{I}_3)$ are linearly dependent.

$$\mathsf{S} = \begin{bmatrix} 2 & -5 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{J} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

as you can verify by checking that $\mathsf{AS} = \mathsf{SJ}$. ♣

For example: To see how **Case 2** works, consider the matrix

$$\mathsf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

which has linearly independent eigenvectors $\boldsymbol{v}_1 = [1,1,0]^t$ and $\boldsymbol{v}_2 = [1,0,1]^t$ corresponding to its sole eigenvalue of -1 (i.e. here $a_{-1} = 3$ and $g_{-1} = 2$). Following the method above, We take a vector $\boldsymbol{v}_3$, say $[0,0,1]^t$, which is not in $\text{Lin}\{[1,1,0]^t, [1,0,1]^t\}$ and find a vector $\boldsymbol{v}_2'$ such that $(\mathsf{A}+\mathsf{I}_3)\boldsymbol{v}_3 = \boldsymbol{v}_2'$, i.e.

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

Thus, we take our invertible matrix, S, and its associated JNF, J, to be

$$\mathsf{S} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{J} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

as you can verify by checking that $\mathsf{AS} = \mathsf{SJ}$. ♣

For example: To see how **Case 3** works, consider the matrix

$$\mathsf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$$

which has one linearly independent eigenvector $\boldsymbol{v}_1 = [1, -2, 1]^t$ corresponding to its sole eigenvalue $\lambda_1 = 1$ (i.e. here $a_1 = 3$ and $g_1 = 1$). We seek a vector $\boldsymbol{v}_1' = [x, y, z]^t$ such that $(\mathsf{A} - \mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1$, i.e.

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \implies x \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

which, by inspection, has $\boldsymbol{v}_1' = [x, y, z]^t = [-1, 1, 0]^t$ as a solution.† We then seek a vector $\boldsymbol{v}_1'' = [x, y, z]^t$ such that $(\mathsf{A} - \mathsf{I}_3)\boldsymbol{v}_1'' = \boldsymbol{v}_1'$ such that

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \implies x \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

which, by inspection, has $\boldsymbol{v}_1'' = [x, y, z]^t = [1, 0, 0]^t$ as a solution.† Thus, we take our invertible matrix, S, and its associated JNF, J, to be

$$\mathsf{S} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathsf{J} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

as you can verify by checking that $\mathsf{AS} = \mathsf{SJ}$. ♣

Aside 4.5.10 There are other solutions since the three column vectors in the matrix $(\mathsf{A} - \mathsf{I}_3)$ are linearly dependent.

Aside 4.5.11 There are other solutions since the three column vectors in the matrix $(\mathsf{A} - \mathsf{I}_3)$ are linearly dependent.

Exercise 1: Consider a 3×3 matrix A which has two distinct eigenvalues λ_1 (with $a_{\lambda_1} = 2$) and λ_2 (with $a_{\lambda_2} = 1$) with corresponding eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. If the vector $\boldsymbol{v}_1'$ is given by

Return to where you came from.
Go to the solution.

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1,$$

show that the vectors $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2$ are linearly independent.

Exercise 2: Consider a 3×3 matrix A which has one eigenvalue λ_1 (i.e. $a_{\lambda_1} = 3$) and corresponding linearly independent eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. If the vector $\boldsymbol{v}_2' \neq \boldsymbol{0}$ is given by

Return to where you came from.
Go to the solution.

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_3 = \boldsymbol{v}_2',$$

where $\boldsymbol{v}_3 \notin \text{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, show that $\boldsymbol{v}_2'$ is in $\text{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Solution to Exercise 1: We consider a 3×3 matrix A which has two distinct eigenvalues λ_1 (with $a_{\lambda_1} = 2$) and λ_2 (with $a_{\lambda_2} = 1$) with corresponding eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Given that the vector $\boldsymbol{v}_1'$ is given by

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1,$$

we want to show that the vectors $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2$ are linearly independent.

To show this, we note that

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_1' = \boldsymbol{v}_1 \quad \Longrightarrow \quad \mathsf{A}\boldsymbol{v}_1' = \lambda_1 \boldsymbol{v}_1' + \boldsymbol{v}_1,$$

and we assume that the vectors $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2$ are linearly dependent, i.e. there is some non-zero scalar $\alpha \in \mathbb{R}$ such that $\boldsymbol{v}_1' = \alpha \boldsymbol{v}_2$. If this was the case, we have

$$\mathsf{A}(\alpha \boldsymbol{v}_2) = \lambda_1(\alpha \boldsymbol{v}_2) + \boldsymbol{v}_1 \implies \alpha \mathsf{A}\boldsymbol{v}_2 = \alpha\lambda_1 \boldsymbol{v}_2 + \boldsymbol{v}_1 \implies \alpha\lambda_2 \boldsymbol{v}_2 = \alpha\lambda_1 \boldsymbol{v}_2 + \boldsymbol{v}_1,$$

as $\boldsymbol{v}_2$ is an eigenvector of A corresponding to the eigenvalue λ_2. Thus we have

$$\boldsymbol{v}_1 = \alpha(\lambda_2 - \lambda_1)\boldsymbol{v}_2,$$

i.e. the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent too. But, this contradicts the fact that the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. Consequently, the vectors $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2$ must be linearly independent, as required. ♠

Solution to Exercise 2: We consider a 3×3 matrix A with one eigenvalue λ_1 (i.e. $a_{\lambda_1} = 3$) and corresponding linearly independent eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Given that the vector $\boldsymbol{v}_2' \neq \mathbf{0}$ is given by $(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_3 = \boldsymbol{v}_2'$, where $\boldsymbol{v}_3 \notin \mathrm{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, we want to show that $\boldsymbol{v}_2'$ is in $\mathrm{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

To show this, we note that the set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis of $\mathbb{R}^3$ † and so we can write the vector $\boldsymbol{v}_2' \in \mathbb{R}^3$ as

$$\boldsymbol{v}_2' = \sum_{i=1}^{3} \alpha_i \boldsymbol{v}_i \quad \Longrightarrow \quad \alpha_3 \boldsymbol{v}_3 = \boldsymbol{v}_2' - \sum_{i=1}^{2} \alpha_i \boldsymbol{v}_i,$$

for some scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Multiplying this by A yields

$$\alpha_3 \mathsf{A}\boldsymbol{v}_3 = \mathsf{A}\boldsymbol{v}_2' - \sum_{i=1}^{2} \alpha_i \mathsf{A}\boldsymbol{v}_i = \mathsf{A}\boldsymbol{v}_2' - \sum_{i=1}^{2} \alpha_i \lambda_1 \boldsymbol{v}_i = \mathsf{A}\boldsymbol{v}_2' - \lambda_1 \sum_{i=1}^{2} \alpha_i \boldsymbol{v}_i,$$

since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are eigenvectors corresponding to the eigenvalue λ_1. But,

$$(\mathsf{A} - \lambda_1 \mathsf{I}_3)\boldsymbol{v}_3 = \boldsymbol{v}_2' \implies \mathsf{A}\boldsymbol{v}_3 = \lambda_1 \boldsymbol{v}_3 + \boldsymbol{v}_2' \implies \alpha_3 \mathsf{A}\boldsymbol{v}_3 = \lambda_1 \alpha_3 \boldsymbol{v}_3 + \alpha_3 \boldsymbol{v}_2',$$

and so equating these two expressions for $\alpha_3 \mathsf{A}\boldsymbol{v}_3$ we get

$$\mathsf{A}\boldsymbol{v}_2' - \lambda_1 \sum_{i=1}^{2} \alpha_i \boldsymbol{v}_i = \alpha_3 \lambda_1 \boldsymbol{v}_3 + \alpha_3 \boldsymbol{v}_2' \implies \mathsf{A}\boldsymbol{v}_2' = \alpha_3 \boldsymbol{v}_2' + \lambda_1 \sum_{i=1}^{3} \alpha_i \boldsymbol{v}_i = (\alpha_3 + \lambda_1)\boldsymbol{v}_2'.$$

Thus, α_3 must be zero (or else, as $\boldsymbol{v}_2' \neq \mathbf{0}$, we have *another* eigenvalue given by $\alpha_3 + \lambda_1$) and so $\boldsymbol{v}_2' = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 \in \mathrm{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, as required. ♠

Aside 4.5.12 As the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent, if we take a third vector $\boldsymbol{v}_3 \notin \mathrm{Lin}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, we have three linearly independent vectors in $\mathbb{R}^3$.