MA201: Further Mathematical Methods (Linear Algebra) 2002

General Information — Teaching

This course involves two types of teaching session that you should be attending:

Lectures

This is a half unit course with lectures in the Lent Term only. Lectures take place on Tuesdays 12:00-13:00 and Fridays 13:00-14:00 in the New Theatre (E171). There will also be revision lectures in the Summer Term.

The lecturer for this course is Dr. James Ward. He has office hours on Tuesdays 15:30-16:30 and Fridays 14:30-15:30 in his office, Room B414 (on the fourth floor of Columbia House). If these times are not convenient for you, then you can e-mail him at `j.m.ward@lse.ac.uk` to arrange an appointment.

Classes

Classes start in the second week of the Lent Term and finish in the first week of the Summer Term. Students will be assigned to a particular group and you should attend the meetings of the group to which you have been assigned.

The Class Teachers are Dr. James Ward, Dr. Elizabeth Boardman and Dr. David Cartwright. The object of the classes is to go through the Problem Sheets that will be distributed during lectures. Anyone having difficulties with the course is strongly encouraged to go and see their Class Teacher during their Office Hours.

General Information — Assessment

There are two ways in which you will be assessed during this course:

Problem Sheets

Problems will be set each week. Students are expected to attempt selected questions and hand them in to their Class Teachers by the times that they specify. Work that is handed in will be marked, graded and returned in the next class. The grades gained for this work will provide a means of *informally* assessing the progress of each student.

As with any mathematics course, learning is achieved by doing, and so it is vital that you attempt the Problems. Most of the questions which are set for the classes will be routine, and their purpose is to ensure that you have understood the material covered in the lectures. However, some of the set questions will be harder, and these serve to stretch your understanding. Solution Sheets will be made available in the week following the classes.

The Problem Sheets will also contain 'Other Problems' which you should attempt after the classes to see if you have learnt from your mistakes, and 'Harder Problems' for those of you who find the standard material a bit too easy. These will not be graded, but full answers will be given on the Solution Sheets.

The Examination

The *formal* assessment for this course consists of a two hour examination in the Summer Term. (Note that there are separate examinations for the Calculus and Linear Algebra components of the Further Mathematical Methods course.) The examination will have questions on the theory, methods and applications contained in the course. Consequently, to do well, you will need to have a good grasp of all of these aspects of the material. At the end of the course, a summary of the material which is needed for the examination will be provided.

General Information — The Course

Here is some more specific information about the course:

Prerequisites

Most of the students who enter into this course have already done Mathematical Methods (MA100), and the material we shall cover follows on from this course. Students who have taken Quantitative Methods (MA107) *and* Further Quantitative Methods (MA207) will also be able to take this course, but such students should be aware that some of the assumed theory will be less familiar to them (specifically, the abstract theory of vector spaces and linear transformations). Other students, most notably those following a General Course programme, should see the lecturer if they are unsure whether they have the appropriate background.

Course Objectives

The course has three main objectives and these are to develop an understanding of *theory*, *methods* and *applications* within the context of Linear Algebra. To this end, although we shall mainly be considering *matrices* and their properties, we begin by noting that most of our knowledge about these entities relies on an understanding of *vector spaces*. As such, the theoretical part of this course will examine this mathematical structure and use it as a theme which underlies most of the theoretical material that we shall cover. In particular, we shall see how simple definitions can be combined and our mathematical knowledge extended *via* the use of proofs. With this theory in place, we shall derive some of the more important properties of matrices. However, it is not always convenient to use such basic notions when dealing with matrices and so we shall also derive *methods* that will allow us to analyse matrices without recourse to the underlying theory. These methods, in turn, will allow us to illustrate the utility of matrices as a mathematical tool. In particular, we shall develop some *applications* of matrices to problems in Economics and the Social Sciences. Moreover, we will see that these methods allow us to obtain *qualitative* as well as *quantitative* results.

Textbooks

There are two textbooks that are recommended for this course. For the applications and the earlier parts of the course I recommend H. Anton and C. Rorres, *Elementary Linear Algebra: Applications Version* (7th edition, Wiley, 1994) which was also the course textbook for the Linear Algebra part of Mathematical Methods (MA100). However, for the later parts of the course, the recommended textbook is *Advanced Mathematical Methods* by Adam Ostaszewski, which is also used for the Calculus part of Further Mathematical Methods (MA200). There are many other good Linear Algebra textbooks about and so if the recommended ones are not to your taste, you can also go to the library and browse.

General Information — Course Content

The course is divided into three parts. What follows is a rough summary of the material that we will be covering and how the different topics relate to each other. References to the course textbooks, Ostaszewski (O) and Anton and Rorres (AR) are given in brackets.

1 Vector spaces I

Vector spaces are a type of mathematical structure, and you were introduced to them in Mathematical Methods (MA100). As they are fundamental to many of the topics that will be covered in this course, we shall use them to motivate some of the mathematical ideas that we will encounter. In particular, we shall introduce three fundamentally different vector spaces, examine their properties and then use these in some applications. With this in mind, the first part of the course is a revision (and slight extension) of some things that you should know about vector spaces. (An overview of some of the more basic aspects of this material can be found in Chapters 3 and 4 of Anton and Rorres.)

1.1 Vector spaces, subspaces, bases and dimension

We start by formally introducing the concepts of vector space and subspace. Three examples of vector spaces will be given (real, complex and function space), and some subspaces will be examined. Some properties of linear combinations of vectors will be revised, including the notions of linear independence and linear span. The fact that linear spans are vector spaces will then be explored, and with the concept of linear independence, bases will be introduced. The dimension of a vector space will also be defined. Where possible, the more theoretical material will also be explained in a more intuitive geometrical manner. (O §§1.1-4, AR §§5.1-4 and 10.1-4)

1.2 Linear transformations and related vector spaces

Linear transformations will be introduced along with their matrix representation. Associated with such matrices are several vector spaces, namely:

- Row and column spaces.
- Ranges and null-spaces.

These will be examined, and relationships between them will be emphasised. In particular, we shall see that:

- The column space and the range of a matrix are identical.
- The dimensions of the range and the null-space of a matrix will be related by the rank-nullity theorem.
- Certain linear transformations can represent a change of basis.
- Solutions to linear simultaneous equations will be analysed in terms of the range and null-space.

A geometrical interpretation of some of these results will also be given where appropriate. (O §§1.5-7 and 3.1-5, AR §§5.5-6, 6.5 and 8.1-5)

1.3 Tests for linear independence

Some tests for linear independence will be introduced. In particular, using determinants to test for linear independence in a real space, and the use of the Wronskian in function spaces. (O §1.8 , AR §5.3)

1.4 An introduction to geometry in $\mathbb{R}^n$

We shall see that vector subspaces correspond to the solution set of a matrix equation $\mathsf{A}\mathbf{x} = \mathbf{0}$. We shall also introduce the idea of an affine set, which is the solution set of a matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$. This will allow us to introduce the notion of a hyperplane which is an $(n-1)$-dimensional geometrical object in $\mathbb{R}^n$. (O §§2.1-2)

1.5 Inner product spaces

Introducing the notion of an inner product, which we can use to generalise our normal idea of the 'distance' between two vectors, we get an inner product space. The defining properties of an inner product will be stated and some examples will be given for the three main vector spaces being considered in this course. The Cauchy-Schwarz Inequality will be derived and the notion of 'angle' will be introduced, the concept of orthogonality will also be revised. Some other results relating to inner products will be derived, including the Generalised Theorem of Pythagoras and the Triangle Inequality. (O §§2.3-7, AR §§6.1-2 and 10.5)

1.6 Orthonormal bases and the Gram-Schmidt procedure

The idea of an orthonormal basis is introduced and the Gram-Schmidt procedure for constructing such bases is described. (O §§2.7-8, AR §6.3)

2 Spectral Theory

In Linear Algebra, spectral theory is mainly concerned with the eigenvalues and eigenvectors of matrices. Spectral theory can be used in a wide variety of applications, mainly through the use of diagonalisation, and some of these will be investigated.

2.1 Eigenvalues and eigenvectors

We start by revising the methods for calculating the eigenvalues and eigenvectors of a matrix which you would have seen in Mathematical Methods (MA100). Some explanation of the vector spaces that are involved in spectral theory due to the presence of eigenvectors will also be given. We shall then introduce the notion of similar matrices and revise how to diagonalise matrices. (O §§5.1-2, AR §§7.1-2)

2.2 Applications of diagonalisation I — systems of differential equations

We shall consider a simple model of population dynamics, and use diagonalisation to solve it. Then, by introducing systems of differential equations, we will construct a slightly more sophisticated model (based around the idea of 'competing species'). This will be used to illustrate the idea of steady state solutions to systems of differential equations, and the stability of these solutions will be discussed. An elementary account of the linearisation of such systems of differential equations will also be given. (AR §9.1)

2.3 Applications of diagonalisation II — [integer] powers of matrices

Multiplying a matrix with itself an arbitrary number of times is a chore, but this task can be massively simplified using diagonalisation. We revise this technique and use it to create a model for age-specific population growth. (AR §11.18)

2.4 Complex matrices

To complement our study of complex vector spaces, we now introduce the idea of a complex matrix — i.e. a matrix that has entries which are complex numbers. We find that these can be divided into several classes, and examine their spectral properties. In particular, we look at

- Hermitian matrices, showing that they have real eigenvalues and that eigenvectors corresponding to distinct eigenvalues are orthogonal
- Unitary and Normal matrices, showing how these lead to the idea of unitary diagonalisation.

We also show how to calculate the spectral decomposition of a matrix, and use this to calculate [arbitrary] powers of matrices. The real matrix analogues of these results will also be discussed. (O §§5.3-4, AR §10.6)

3 Vector spaces II

We now continue our discussion of vector spaces by looking at orthogonal complements, orthogonal projections and some applications.

3.1 Orthogonal complements

We introduce the idea of the orthogonal complement of an arbitrary set of vectors. In the case where this set of vectors forms a vector space, we show that 'the complement of the complement of the set is the set itself'. We also look at the orthogonal complement of the null-space of a matrix and relate it to the range of its transpose. (O §§3.7-8)

3.2 An application of orthogonal complements — bounds on matrix products

Using the results obtained about the range and null-space of a matrix and its transpose, we derive some interesting results concerning the rank of certain matrix products. (O §§3.8-9)

3.3 Direct sums

We introduce the idea of the sum and direct sum of two subspaces. In particular, we show that the sum of two subspaces is itself a subspace. (O §4.1)

3.4 Projections

With reference to the idea of a sum, we introduce the notion of a projection. This is a linear transformation between two subspaces, and the nature of this mapping is considered. Under certain circumstances (i.e. when the projection is symmetric and idempotent), we see that the projection is in fact an orthogonal projection. The properties of this special class of projections are also investigated, including the idea of 'projection onto a column space'. (O §§4.1-6, AR §6.3-4)

3.5 An application of orthogonal projections in real space— least squares fits

We deduce the common statistical method of 'least squares fits' using an orthogonal projection and some of its properties. (O §4.7, AR §9.3)

3.6 An application of orthogonal projections in function space — Fourier series

We consider the possibility of approximating functions by using linear combinations of orthonormal functions. This leads to the derivation of the formulae for Fourier series in the case where the orthonormal functions are sines and cosines. (AR §9.4)

3.7 Generalised inverses

You are all aware that non-singular square matrices have a unique inverse (of course, singular matrices have no inverse). We briefly introduce the idea of a generalised inverse, which is [unsurprisingly] a generalisation of the 'inverse' concept to non-square and singular matrices. Some properties of the generalised inverse are investigated, and we deduce the fact that strong generalised inverses represent orthogonal projections. (O §§8.3-5)

The web-site

More information about the course and copies of the handouts can be obtained from the Department of Mathematics web-site. The URL is `http://www.maths.lse.ac.uk/Courses/ma201.html`.