Further Mathematical Methods (Linear Algebra) 2002

Lectures 9 and 10: Population Dynamics, Steady States and Stability

The second application of diagonalisation which we shall consider is its use in analysing systems of differential equations. This topic will be motivated by looking at how such systems can be used to model 'population dynamics'. In particular, we shall develop a method for finding their steady states and assessing their stability. Further discussion of this topic may be found in the textbook *Mathematics for Economists* (Norton, 1994) by C. Simon and L. Blume.

9.1 Population dynamics: The single species case

Suppose that $y(t)$ is the size of a population of a species of animal at time t. (The time is measured with respect to some reference time, $t = 0$, and the 'size' need not be an integer, since it may, for example, measure the population in thousands.) Modelling the population as a differentiable function over a continuous time parameter, we denote the derivative of the function $y(t)$ with respect to t, i.e. dy/dt, by $\dot{y}(t)$.

The *growth rate* of a population is the rate of change of the size of a population divided by the size itself, i.e. $\dot{y}/y$. If the population has a constant growth rate, say r (which will equal the difference between the birth and death rates), then we have the *Malthus equation*

$$\dot{y}(t) = ry(t),$$

which can easily be solved to yield

$$y(t) = y(0)e^{rt},$$

for some initial population $y(0)$.

This model is over-simplistic, and more realistically, as the population increases, growth-inhibiting factors will come into action. (These might arise, for instance, from limitations on space and scarcity of natural resources.) Consequently, we should modify the simple model used above. We may do this by assuming that the growth rate $\dot{y}/y$ is not constant, but is a decreasing function of y. The simplest type of decreasing function is the linear one, where the growth rate is equal to $a - by$ for positive constants a and b. In this case, we have

$$\dot{y} = (a - by)y,$$

and this is known as the *logistic model* of population growth.[1] This is also easily solved as it is a separable first-order differential equation. As such, we can write

$$\int \frac{dy}{y(a-by)} = \int dt,$$

and using partial fractions to simplify the left-hand side, we find that

$$\frac{1}{a}\int \left[\frac{1}{y} + \frac{b}{a-by}\right] dy = \int dt,$$

which on integrating gives

$$\frac{1}{a}\ln\left[\frac{y}{a-by}\right] = t + c.$$

Simplifying this, we get

$$y(t) = \frac{a}{b + ke^{-at}},$$

[1] Those who study Chaos Theory will encounter such *non*-linear differential equations

where the constant k is related to the initial condition $y(0)$ by the formula

$$y(0) = \frac{a}{b+k}.$$

In turn, this gives rise to two kinds of solution, that is we have

- **either:** $k = 0$ and $y(t) = \frac{a}{b}$ for all times t,
- **or:** $k \neq 0$ and $y(t) = \frac{a}{b + ke^{-at}}$.

Notice that $k = 0$ corresponds to the case where $y(0) = a/b$.

We refer to the first kind of solution (i.e. where $y(t)$ takes the *constant* value a/b at all times) as a *steady state* solution. This is because, if we take the initial value of y in this case, namely a/b, and substitute it into the right-hand-side of the differential equation above we get zero. That is, in this case, the differential equation reduces to $\dot{y} = 0$ and so the value of y does not change with time — it just stays at the initial value a/b. We also observe, for future reference, that in the second kind of solution, the constant k is determined by the initial population $y(0)$ *via* the formula above, but whatever the value of $y(0)$, $y(t) \to a/b$ as $t \to \infty$. The fact that every solution of this differential equation tends towards this value as $t \to \infty$ means that it is *globally asymptotically stable* (more on this later).

9.2 Population dynamics: The competing species case

We now make our model of population dynamics slightly more sophisticated. Suppose that we have two species of animal and that they compete with each other (for food, for instance). Let us denote the corresponding populations at time t by $y_1(t)$ and $y_2(t)$. We assume that, in the absence of the other, the population of either species would exhibit logistic growth, as above. But, given that they compete with each other, we assume that the presence of each has a negative effect on the growth rate of the other. That is, we assume that for some positive numbers a_1, a_2, b_1, b_2, c_1 and c_2, the growth rates of these populations will be given by

$$\begin{aligned}\frac{\dot{y}}{y_1} &= a_1 - b_1 y_1 - c_1 y_2 \\ \frac{\dot{y}}{y_2} &= a_2 - b_2 y_2 - c_2 y_1\end{aligned}$$

We then have the *coupled* system of differential equations given by

$$\begin{aligned}\dot{y}_1 &= a_1 y_1 - b_1 y_1^2 - c_1 y_1 y_2 \\ \dot{y}_2 &= a_2 y_2 - b_2 y_2^2 - c_2 y_1 y_2\end{aligned}$$

and clearly such a model could be extended to more than two species.

9.3 Systems of differential equations

We now take a moment to introduce a new piece of notation which will help us in our analysis of the competing species model of population growth. In general, we say that a [square] *system of differential equations* for the functions $y_1, y_2, \ldots, y_n$ is a set of coupled differential equations of the form

$$\begin{aligned}\dot{y}_1 &= f_1(y_1, y_2, \ldots, y_n) \\ \dot{y}_2 &= f_2(y_1, y_2, \ldots, y_n) \\ &\vdots \\ \dot{y}_n &= f_n(y_1, y_2, \ldots, y_n)\end{aligned}$$

and these can be written as

$$\dot{\mathbf{y}} = F(\mathbf{y}),$$

where F is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

In MA100 you considered the case where the functions $f_1, f_2, \ldots, f_n$ of $y_1(t), y_2(t), \ldots, y_n(t)$ which 'made up' the function F were linear (and we will consider these again in Section 9.5). But, in this course, we shall also be concerned with the case where the functions $f_1, f_2, \ldots, f_n$ of $y_1(t), y_2(t), \ldots,$ $y_n(t)$ are non-linear.

9.4 Steady states and stability

We now introduce some new concepts which will clarify the remarks made at the end of Section 9.1. In particular, we shall define what we mean by a steady state solution and what it is for a solution to be globally asymptotically stable. So, formally, for the former term we say that

Definition 9.1 *The constant vector $\mathbf{y}^* \in \mathbb{R}^n$ is a steady state solution of the [square] system of differential equations*

$$\dot{\mathbf{y}} = F(\mathbf{y}),$$

if it satisfies the equation $F(\mathbf{y}) = \mathbf{0}$, where $\mathbf{0}$ is the null vector in $\mathbb{R}^n$.

As we have seen, if such a system is required to satisfy the initial condition given by $\mathbf{y}(0) = \mathbf{y}^*$, then its solution will be $\mathbf{y}(t) = \mathbf{y}^*$ for all times t. (So, $\mathbf{y}^*$ will be a constant solution of the system.) You may recall that in Section 9.1, when we considered the single species case, we found that $\mathbf{y}^* = a/b$ was a steady state solution of the [1×1 system of] differential equation[s] in question. Let us now look at a slightly harder example.

For example: Consider two species where the populations at a time t are given by the functions y_1 and y_2. These species compete for resources and their populations evolve according to the differential equations

$$\dot{y}_1 = 4y_1 - y_1^2 - y_1 y_2$$
$$\dot{y}_2 = 6y_2 - y_2^2 - 3y_1 y_2$$

This [2×2] system of differential equations is of the form $\dot{\mathbf{y}} = F(\mathbf{y})$ where

$$F(y_1, y_2) = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} = \begin{pmatrix} 4y_1 - y_1^2 - y_1 y_2 \\ 6y_2 - y_2^2 - 3y_1 y_2 \end{pmatrix}.$$

So, to find the steady states of this system we set $F(\mathbf{y}) = \mathbf{0}$ and find the values of $\mathbf{y}$ which satisfy the resulting simultaneous equations, i.e.

$$4y_1 - y_1^2 - y_1 y_2 = 0$$
$$6y_2 - y_2^2 - 3y_1 y_2 = 0$$

to solve these (think about how you found the stationary points of three-dimensional surfaces in MA100!) we re-write these equations as

$$y_1(4 - y_1 - y_2) = 0$$
$$y_2(6 - y_2 - 3y_1) = 0$$

and find that the steady states are $(0, 0), (0, 6), (4, 0)$ and $(1, 3)$. ♣

We now go on to give the formal definition of an asymptotically stable steady state,

Definition 9.2 *A steady state $\mathbf{y}^*$ is an asymptotically stable equilibrium if every solution $\mathbf{y}(t)$ which starts near $\mathbf{y}^*$ converges to $\mathbf{y}^*$ as $t \to \infty$. That is, if there is some $\epsilon > 0$ such that $\dot{\mathbf{y}} = F(\mathbf{y})$ and $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$, then $\mathbf{y}(t) \to \mathbf{y}^*$ as $t \to \infty$.*

Further, we say that

Definition 9.3 *A steady state* $\mathbf{y}^*$ *is a globally asymptotically stable equilibrium if for every* $\mathbf{y}(0)$ *(with the possible exception of a set of* $\mathbf{y}(0)$ *of lower dimension), the solution of* $\dot{\mathbf{y}} = F(\mathbf{y})$ *which satisfies* $\mathbf{y}(0)$ *tends to* $\mathbf{y}^*$ *as* $t \to \infty$*. (That is, 'almost every' solution of the system approaches* $\mathbf{y}^*$ *in the long run.)*

You may recall that in Section 9.1, when we considered the single species case, we found that $\mathbf{y}^* = a/b$ was a globally asymptotically stable equilibrium as *every* solution to the [1×1 system of] differential equation[s] in question tended to the steady state solution a/b as $t \to \infty$.

It is natural to ask: *How* do we determine whether the steady states of a system of differential equations is stable? We are not in a position to answer this question completely, but by examining linear systems, we will be able to find a sufficient condition for steady states to be asymptotically stable equilibria.

9.5 Linear systems of differential equations

A *linear system* of differential equations is one in which the function F is a linear transformation. As we have seen, such a system can be represented by a matrix, i.e.

$$\dot{\mathbf{y}} = \mathsf{A}\mathbf{y},$$

where A is an $n \times n$ matrix whose entries are fixed real numbers. If the matrix A was diagonal, this would be easy to solve. This is because, if

$$\mathsf{A} = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n],$$

then the system is just

$$\dot{y}_1 = \lambda_1 y_1, \;\; \dot{y}_2 = \lambda_2 y_2, \;\; \ldots, \;\; \dot{y}_n = \lambda_n y_n,$$

and so, on solving these differential equations, we find that

$$y_1 = y_1(0)e^{\lambda_1 t}, \;\; y_2 = y_2(0)e^{\lambda_2 t}, \ldots, \;\; y_n = y_n(0)e^{\lambda_n t},$$

where the coefficients $y_1(0), y_2(0), \ldots, y_n(0)$ are the initial conditions that these differential equations must satisfy, i.e. they are the components of the vector $\mathbf{y}(0)$.

However, in general, when we are confronted with a linear system of differential equations they will not be represented by a nice diagonal matrix. So, in an attempt to reduce them to this simple form, we might be able to use *diagonalisation.* Let us assume that the matrix A can indeed be diagonalised. In this case, we can find an invertible matrix P such that

$$\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \mathsf{D},$$

where D is a diagonal matrix containing the eigenvalues of A and P is a matrix whose columns are given by the corresponding eigenvectors, i.e.

$$\mathsf{D} = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \quad \text{and} \quad \mathsf{P} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

We now set $\mathbf{y} = \mathsf{P}\mathbf{z}$ (or, equivalently, $\mathbf{z} = \mathsf{P}^{-1}\mathbf{y}$), and in order to perform this substitution, we need to see how differentiation affects it. But, as the entries of P are just real numbers, it should be clear that

$$\dot{\mathbf{y}} = \frac{d}{dt}(\mathsf{P}\mathbf{z}) = \mathsf{P}\frac{d\mathbf{z}}{dt} = \mathsf{P}\dot{\mathbf{z}},$$

and so, we get

$$\mathsf{P}\dot{\mathbf{z}} = \mathsf{A}\mathbf{y} = \mathsf{A}\mathsf{P}\mathbf{z},$$

or, better still,

$$\dot{\mathbf{z}} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}\mathbf{z} = \mathsf{D}\mathbf{z}.$$

As D is a diagonal matrix, we can now solve this for $\mathbf{z}$ using the method above, and then we can find $\mathbf{y}$ from $\mathbf{y} = \mathsf{P}\mathbf{z}$.

For example: Consider the linear system of differential equations given by the matrix equation

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

To solve this system of differential equations we follow the method given above. We find that the eigenvalues and eigenvectors of A are such that

$$\mathsf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \text{ and } \mathsf{P} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix},$$

where $\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \mathsf{D}$. So, choosing new functions z_1 and z_2 such that

$$\mathbf{y} = \mathsf{P}\mathbf{z},$$

the system becomes $\dot{\mathbf{z}} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}\mathbf{z} = \mathsf{D}\mathbf{z}$, i.e.

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3z_2 \end{bmatrix}.$$

This is now an 'uncoupled' system, and solving these two [easy] differential equations we find that

$$z_1(t) = z_1(0) \quad \text{and} \quad z_2(t) = z_2(0)e^{3t}.$$

However, we want to know about the functions $y_1(t)$ and $y_2(t)$, and so, using $\mathbf{y} = \mathsf{P}\mathbf{z}$, we get

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \implies \begin{aligned} y_1(t) &= z_1(t) + z_2(t) \\ y_2(t) &= -z_1(t) + 2z_2(t) \end{aligned}$$

which, on substituting our expressions for $z_1(t)$ and $z_2(t)$, gives

$$\begin{aligned} y_1(t) &= z_1(0) + z_2(0)e^{3t} \\ y_2(t) &= -z_1(0) + 2z_2(0)e^{3t} \end{aligned}$$

We also want the answer in terms of $y_1(0)$ and $y_2(0)$, and so, using $\mathbf{z} = \mathsf{P}^{-1}\mathbf{y}$, we get

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \implies \begin{aligned} z_1(0) &= \tfrac{1}{3}\{2y_1(0) - y_2(0)\} \\ z_2(0) &= \tfrac{1}{3}\{y_1(0) + y_2(0)\} \end{aligned}$$

Consequently, the final result is given by

$$y_1(t) = \tfrac{1}{3}\{2y_1(0) - y_2(0)\} + \tfrac{1}{3}\{y_1(0) + y_2(0)\}e^{3t},$$

and

$$y_2(t) = -\tfrac{1}{3}\{2y_1(0) - y_2(0)\} + \tfrac{2}{3}\{y_1(0) + y_2(0)\}e^{3t}.$$

Let us now look at the steady states of this system and see whether they are asymptotically stable.

Looking at the differential equations, we find that setting $\dot{\mathbf{y}} = \mathbf{0}$, we get the two simultaneous equations

$$\begin{aligned} y_1 + y_2 &= 0 \\ 2y_1 + 2y_2 &= 0 \end{aligned}$$

But this just gives us one equation relating two variables, that is, setting y_2 (say) equal to the free parameter r, solutions of these simultaneous equations take the form $(y_1, y_2) = (-r, r)$ for all $r \in \mathbb{R}$. Consequently, we have an infinite number of steady states given by $\mathbf{y}^* = (-r, r)$, and so, if the system is required to satisfy one of the initial conditions given by $\mathbf{y}(0) = (-r, r)$, then its solution will be $\mathbf{y}(t) = (-r, r)$ for all times t. This should be clear from the solutions we found, because if we put these initial values in, we find that

$$y_1(t) = \tfrac{1}{3}\{2(-r) - r\} + \tfrac{1}{3}\{(-r) + r\}e^{3t} \implies y_1(t) = -r,$$

and

$$y_2(t) = -\tfrac{1}{3}\{2(-r) - r\} + \tfrac{1}{3}\{(-r) + r\}e^{3t} \implies y_2(t) = r.$$

That is, the time-dependence disappears (as the initial conditions make the coefficient of the e^{3t} terms zero) and we just stay in the state prescribed by the initial conditions for all time.

However, these steady states are not stable. To see this, consider the case where our initial conditions are *not* of the form $\mathbf{y}(0) = (-r, r)$. If any of these *non*-steady states are to give asymptotic stability, we require that in the limit as $t \to \infty$, the solution $\mathbf{y}(t)$ tends to one of the steady states $\mathbf{y}^* = (-r, r)$ for $r \in \mathbb{R}$. But, clearly, if $\mathbf{y}(0) \neq (-r, r)$, the coefficient of the exponential term in the solution is non-zero, and so we find that as $t \to \infty$,

$$y_1(t) \to \infty \quad \text{and} \quad y_2(t) \to \infty,$$

if $y_1(0) + y_2(0) > 0$, and

$$y_1(t) \to -\infty \quad \text{and} \quad y_2(t) \to -\infty,$$

if $y_1(0) + y_2(0) < 0$. Consequently, none of the steady states of this system are stable as $\mathbf{y} \nrightarrow \mathbf{y}^* = (-r, r)$ for any $r \in \mathbb{R}$ if $\mathbf{y}(0) \neq (-r, r)$. ♣

We now return to the general linear system in which A can be diagonalised to see what conclusions we can draw about its steady states and their stability. From the analysis above, we know that for any diagonalisable matrix A, we can write

$$\dot{\mathbf{z}} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}\mathbf{z} = \mathsf{D}\mathbf{z},$$

and so, on solving the n differential equations contained within this system we get

$$\mathbf{z}(t) = \begin{bmatrix} z_1(0)e^{\lambda_1 t} \\ z_2(0)e^{\lambda_2 t} \\ \vdots \\ z_n(0)e^{\lambda_n t} \end{bmatrix}.$$

Now, using $\mathbf{y} = P\mathbf{z}$, we can write

$$\mathbf{y}(t) = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} z_1(0)e^{\lambda_1 t} \\ z_2(0)e^{\lambda_2 t} \\ \vdots \\ z_n(0)e^{\lambda_n t} \end{bmatrix},$$

and multiplying this matrix product out, we find that

$$\mathbf{y}(t) = z_1(0)e^{\lambda_1 t}\mathbf{v}_1 + z_2(0)e^{\lambda_2 t}\mathbf{v}_2 + \cdots + z_n(0)e^{\lambda_n t}\mathbf{v}_n.$$

Indeed, we can use the fact that $\mathbf{z} = \mathsf{P}^{-1}\mathbf{y}$ to find each $z_i(0)$ in terms of the initial conditions, $y_i(0)$. But, all that this will give us is some constant coefficients, say c_i, with which to replace the $z_i(0)$, and so we find that

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n,$$

is the general solution to the system of differential equations $\dot{\mathbf{y}} = \mathsf{A}\mathbf{y}$.

To conclude then, we state two theorems which should be justified by the analysis above. Now, we can show that A can be diagonalised if it has n distinct eigenvalues,[2] and so we claim that

Theorem 9.4 *If an $n \times n$ matrix A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then the system of differential equations given by*

$$\dot{\mathbf{y}} = \mathsf{A}\mathbf{y},$$

has the general solution

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n,$$

where the c_i are constants.

Further, if the eigenvalues are all negative real numbers, then all of the exponential terms tend to zero as $t \to \infty$ and so, $\mathbf{y}(t) \to \mathbf{0}$ in this limit. Indeed, if the matrix has non-zero eigenvalues, then it is non-singular (i.e. it has an inverse),[3] and so the *only* solution to the matrix equation $\mathsf{A}\mathbf{y} = \mathbf{0}$ is $\mathbf{y} = \mathbf{0}$. Consequently, the only steady state in this case is $\mathbf{y}^* = \mathbf{0}$, and since $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$ for all possible sets of initial conditions, this steady state is globally asymptotically stable. So, formally, we have

Theorem 9.5 *If an $n \times n$ matrix A has n distinct negative real eigenvalues, then the only steady state of the system of differential equations*

$$\dot{\mathbf{y}} = \mathsf{A}\mathbf{y},$$

is $\mathbf{y}^* = \mathbf{0}$ and this is globally asymptotically stable.

9.6 Stability in general systems: Linearisation

What about non-linear systems of differential equations? In general, we are unable to solve these as we would in the linear case, and so we can't discover whether the steady states of such systems are stable using the method developed above. But, we can get some information about the stability of non-linear systems of differential equations by using [a version of] Taylor's Theorem to 'relate' it to a linear system. In fact, the version of Taylor's theorem which we shall use is the following:[4]

Theorem 9.6 *[Taylor's Theorem] If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function and $\mathbf{y}^*$ is some constant vector in $\mathbb{R}^n$, then for a vector $\mathbf{h} \in \mathbb{R}^n$,*

$$F(\mathbf{y}^* + \mathbf{h}) = \mathbf{F}(\mathbf{y}^*) + \mathsf{DF}(\mathbf{y}^*)\mathbf{h} + R(\mathbf{h}).$$

Note that if the function $F(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_n(\mathbf{y}))$, then DF is the Jacobian

$$DF = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix},$$

and the matrix $\mathsf{DF}(\mathbf{y}^*)$ is the Jacobian evaluated at $\mathbf{y}^*$. Further, $R(\mathbf{h})$ has the property that

$$\frac{R(\mathbf{h})}{\|\mathbf{h}\|} \longrightarrow 0,$$

as $\mathbf{h} \to \mathbf{0}$.

---

[2] You will prove in Problem Sheet 4 that: If a 2×2 matrix has two distinct eigenvalues, then the eigenvectors corresponding to these eigenvalues are linearly independent. In the Harder Problems on this Sheet, this is generalised to: If an $n \times n$ matrix has n distinct eigenvalues, then the eigenvectors corresponding to these eigenvalues are linearly independent. We also know, from the lectures (or the proof from the Harder Problems of Sheet 4) that: An $n \times n$ matrix is diagonalisable iff it has n linearly independent eigenvectors. The stated result follows from combining these results.

[3] Again, see the Harder Problems on Sheet 4.

[4] Note: a function is *continuously differentiable* if its first-order partial derivatives exist and are continuous.

Loosely speaking, this means that if each entry of $\mathbf{h}$ is small, then

$$F(\mathbf{y}^* + \mathbf{h}) \simeq \mathbf{F}(\mathbf{y}^*) + \mathsf{DF}(\mathbf{y}^*)\mathbf{h},$$

where '$\simeq$' can be interpreted as 'is approximately.'

Now, suppose that $\mathbf{y}^*$ is a steady state solution of the system of differential equations $\dot{\mathbf{y}} = F(\mathbf{y})$, i.e. $\mathbf{F}(\mathbf{y}^*) = \mathbf{0}$, and take $\mathbf{y}(t)$ to be a solution of the system such that $\mathbf{y}(0) - \mathbf{y}^*$ is small. If we now take $\mathbf{h}(t) = \mathbf{y}(t) - \mathbf{y}^*$, then $\mathbf{y}(t) = \mathbf{y}^* + \mathbf{h}(t)$ and our system of differential equations, $\dot{\mathbf{y}} = F(\mathbf{y})$, becomes

$$\frac{d}{dt}\left\{\mathbf{y}^* + \mathbf{h}(t)\right\} = F(\mathbf{y}^* + \mathbf{h}(t)).$$

Consequently, using Taylor's theorem, we have

$$\dot{\mathbf{h}}(t) = \frac{d}{dt}\left\{\mathbf{y}^* + \mathbf{h}(t)\right\} = \mathsf{DF}(\mathbf{y}^*)\mathbf{h}(t) + R(\mathbf{h}(t)),$$

and if $\mathbf{h}(t)$ is small, we can ignore the R term. This means that if the quantity $\mathbf{h}(0) = \mathbf{y}(0) - \mathbf{y}^*$ is small, then the behaviour of the vector $\mathbf{h}(t) = \mathbf{y}(t) - \mathbf{y}^*$ is *qualitatively* the same as the solution to the *linear* system

$$\dot{\mathbf{h}}(t) = \mathsf{DF}(\mathbf{y}^*)\mathbf{h}(t).$$

(Obviously, this argument is not watertight, but it can be made so.) This analysis results in the following theorem:

Theorem 9.7 *Let the constant vector* $\mathbf{y}^*$ *be a steady state solution of the system of differential equations* $\dot{\mathbf{y}} = F(\mathbf{y})$ *where* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuously differentiable function and let the matrix* $\mathsf{DF}(\mathbf{y}^*)$ *denote the Jacobian evaluated at* $\mathbf{y}^*$*. If* $\mathsf{DF}(\mathbf{y}^*)$ *has* n *negative real eigenvalues then* $\mathbf{y}^*$ *is asymptotically stable.*

Let us look at an example to see how this theorem is used.

For example: Returning to the example considered earlier, i.e.

$$\begin{aligned}\dot{y}_1 &= 4y_1 - y_1^2 - y_1y_2\\ \dot{y}_2 &= 6y_2 - y_2^2 - 3y_1y_2\end{aligned}$$

where

$$F(y_1, y_2) = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} = \begin{pmatrix} 4y_1 - y_1^2 - y_1y_2 \\ 6y_2 - y_2^2 - 3y_1y_2 \end{pmatrix},$$

we can see that the Jacobian will be given by

$$DF(\mathbf{y}) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 4 - 2y_1 - y_2 & -y_1 \\ -3y_2 & 6 - 2y_2 - 3y_1 \end{pmatrix}.$$

Evaluating the Jacobian at the steady state $\mathbf{y}^* = (0, 6)$ we get the matrix

$$\mathsf{DF}(\mathbf{y}^*) = \begin{bmatrix} -2 & 0 \\ -18 & -6 \end{bmatrix},$$

whose eigenvalues are -2 and -6 (Verify this!). Consequently, we can deduce that the steady state $\mathbf{y}^* = (0, 6)$ is asymptotically stable as these eigenvalues are both negative real numbers (as required by Theorem 9.7). ♣