Further Mathematical Methods (Linear Algebra) 2002

Problem Sheet 1

(To be discussed in week 2 classes. Please submit answers to the asterisked questions only.)

The discussion of vector spaces and other useful concepts in the notes was quite abstract, and so now we look at some concrete examples. In the first two problems, you should justify your answer by giving either a proof or a counter-example (as appropriate). Further, in the third question you should use Definition 2.6 if you wish to establish that a set of vectors is linearly independent or dependent.

1. * Each of the sets given below are subsets of a vector space, which of them are subspaces? Describe the subsets of $\mathbb{R}^3$ geometrically and indicate what the subsets of $\mathbb{F}^{\mathbb{R}}$ represent.

a. $S_1 = \{[x, y, z]^t \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

b. $S_2 = \{[x, y, x + y]^t \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$.

c. $S_3 = \{\mathbf{f} \in \mathbb{F}^{\mathbb{R}} \mid f(2) = 1\}$.

d. $S_4 = \{\mathbf{f} \in \mathbb{F}^{\mathbb{R}} \mid f(5) = 0\}$.

2. The *union* and *intersection* of two sets, A and B, are the sets given by $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ and $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ respectively. Are either of the sets $S_{1,1,1} \cup S_{1,2,3}$ and $S_{1,1,1} \cap S_{1,2,3}$ subspaces of $\mathbb{R}^3$? Give a geometric description of what these two sets represent and state what this implies for *general* unions and intersections of subspaces.

3. * The plane $x - y + 3z = 0$ represents a subspace of $\mathbb{R}^3$. Find a basis for this subspace and call it B. What is the dimension of this subspace? Further,

- Show that the vector $[4, 7, 1]^t$ is in this subspace. Add this vector to B to form the set B'. Explain why B' is not a basis.
- Show that the vector $[1, 0, 0]^t$ is not in this subspace. Add this vector to B to form the set B''. Explain why B'' is a basis. What vector space is spanned by B''? What is the dimension of this space?

4. Prove the following theorems:

a. * If S is a set of vectors that contains the null vector (i.e. $\mathbf{0}$), then it is linearly dependent. (Hint: Use Definition 2.6.)

b. * Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}\}$ be a set of vectors in a vector space V. If $\mathbf{v}$ can be written as a linear combination of the other vectors in S (i.e. the vectors in the set $S' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$), then S and S' span the same space, i.e. $\mathrm{Lin}(S) = \mathrm{Lin}(S')$. (Hint: Use Definition 2.3 and consider a general vector in $\mathrm{Lin}(S)$. Notice that this result is related to Theorem 2.9.)

c. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a basis of the finite dimensional space V and let $S' \subseteq V$ be any set containing n vectors. If $n < m$, then the vectors in S' cannot span V. (Hint: This result is related to Theorem 2.14 and the proof is similar — except that you want to use the linear dependence that you find to establish a contradiction.)

d. If S is a set of n vectors that spans an n-dimensional vector space V, then S is a basis for V. (Hint: This result is related to Theorem 2.17 and the proof is similar — except that you should use the previous two theorems instead of 2.9 and 2.14.)

Other Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

Here are some more questions on these topics. Everyone *should* try these to further their understanding of the material covered in the lectures. Solutions for these problems will be contained in the Solution Sheet.

5. Show that $S_{1,1,1}$ is a subspace of $\mathbb{R}^3$, and that $S_{1,1,1,1}$ is not. Give a geometric description of what these two sets represent.

6. The sets $S_1 = \{[x, 0, 0]^t \in \mathbb{R}^3 \mid x \in \mathbb{R}\}$ and $S_2 = \{[x, y, z]^t \in \mathbb{R}^3 \mid xz = 0\}$ are subsets of a vector space. Which of them is a subspace? Describe the subsets of $\mathbb{R}^3$ geometrically. Also, explain the relevance of these results in the context of Question 2.

7. Consider a vector, $\mathbf{a}$ of the form $[a^2, b^2, c^2]^t$. With this we can define four subsets as follows: $S_1 = \{\mathbf{a} \in \mathbb{R}^3 \mid a, b, c \in \mathbb{R}\}$, $S_2 = \{\mathbf{a} \in \mathbb{R}^3 \mid a, b, c \neq 0 \text{ and } a, b, c \in \mathbb{R}\}$, $S_3 = \{\mathbf{a} \in \mathbb{C}^3 \mid a, b, c \in \mathbb{C}\}$ and $S_4 = \{\mathbf{a} \in \mathbb{C}^3 \mid a, b, c \neq 0 \text{ and } a, b, c \in \mathbb{C}\}$.

Which of these subsets contain the additive identity required by (A0)? Further, do any of them contain the inverse of the vector $[1, 1, 1]^t$? On the basis of this information, which of these four subsets definitely can't be vector spaces? Establish that any candidate vector spaces are indeed so.

8. Find the sets that represent $\text{Lin}\{\mathbf{1}\}$, $\text{Lin}\{\mathbf{0}\}$ and $\text{Lin}\,\emptyset$. Which of them are vector spaces? For the vector spaces, find a basis or explain why one can't be found. Further, find the dimension of any vector spaces that you find.

Harder Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

For those of you who like the more abstract stuff, here are some more difficult questions for you to think about. Solutions for these problems will be contained in the Solution Sheet. If you want to discuss these solutions (after they have been circulated) you should bother me and not your class teacher.

9. Prove parts (2) and (3) of Theorem 1.2.

10. On pages 10–13 of the hand-out for Lecture 1, we looked at two methods for showing that a subset is not a subspace. To illustrate this, we used the example of the subset $S_{a,b,c,r}$ of $\mathbb{R}^3$ (where $r \neq 0$). Using Method I, we showed that this could not be a subspace for all values of a, b and c in $\mathbb{R}$. (Note that, in the case where $a = b = c = 0$, there are no vectors that satisfy the membership criterion and so $S_{0,0,0,r}$ is the empty set. Thus, it is not a subspace — see Footnote 18.). However, when illustrating Method II, we assumed that $a, b \neq 0$ (for obvious reasons). Devise a proof that works for the case where $a \neq 0$, $b = 0$ and c can take any value. Once this has been done, we have covered all of the possible cases (i.e. we have effectively covered all of the situations where at *most* none, one or two of the constants a, b and c are zero) *except* the one where $a = b = c = 0$, but we can deal with this as above.

11. Further to the remark on page 13 of the hand-out for Lecture 2, justify the claim that the vector space $\{\mathbf{0}\}$ is the *only* vector space that consists of just one vector.

12. Prove the following theorem: If V is a subspace of a finite dimensional vector space W, then V is finite dimensional. Further, $\dim(V) \leq \dim(W)$, and in particular, $\dim(V) = \dim(W)$ iff $V = W$.