# Further Mathematical Methods (Linear Algebra) 2002

# Problem Sheet 10

(To be discussed in week 1 [Lent term] classes. Please submit answers to the asterisked questions only.)

In this sheet, we try our hand at calculating some of the new kinds of inverses that we have introduced in the lectures. We also prove and verify some theorems about them.

**1.** * Find the equations that $u, v, w, x, y$ and $z$ must satisfy so that

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and hence find all right inverses of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Hence, verify that the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ has a solution for every vector $\mathbf{b} = [b_1, b_2]^t \in \mathbb{R}^2$. What does the existence of many right inverses tell us about the number of solutions to the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ for a given $\mathbf{b} \in \mathbb{R}^2$? Explain this relationship in terms of the null-space of $\mathsf{A}$.

**2.** Find the equations that $u, v, w, x, y$ and $z$ must satisfy so that

$$\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and hence find all left inverses of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Using this, find the solutions of the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ for an arbitrary vector $\mathbf{b} = [b_1, b_2, b_3]^t \in \mathbb{R}^3$. Hence, by considering the Cartesian equation of the plane that contains all of the vectors in the range of $\mathsf{A}$, or otherwise, verify that this matrix equation has a unique solution when it has a solution.

**3.** * Show that the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

has a right inverse, but no left inverse. Further, find *one* of these right inverses. [There is no need to find the general form of a right inverse as in Question 1, just find one particular right inverse.]

**4.** * Calculate the strong generalised inverse of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 4 & 5 & 3 \\ 2 & 3 & 5 & 1 \\ 3 & 2 & 5 & -1 \\ 4 & 1 & 5 & -3 \end{bmatrix},$$

by finding an $m \times k$ matrix $\mathsf{B}$ and a $k \times n$ matrix $\mathsf{C}$ such that the ranks of $\mathsf{B}$ and $\mathsf{C}$ are both $k$.

**5.** * Show that: If $\mathsf{A}$ has a right inverse, then $\mathsf{R} = \mathsf{A}^t(\mathsf{A}\mathsf{A}^t)^{-1}$ is the strong generalised inverse of $\mathsf{A}$. Further, show that $\mathbf{x}^* = \mathsf{R}\mathbf{b}$ is the solution of $\mathsf{A}\mathbf{x} = \mathbf{b}$ nearest to the origin. [Hint: Start by showing that $\mathsf{R}$ is a WGI.]

**Other Problems.** (These are *not* compulsory, they are *not* to be handed in, and they will *not* be covered in classes.)

Here are some other questions on generalised inverses which you might like to try.

**6.** Find the singular values decomposition of the matrix

$$\mathsf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and hence calculate its strong generalised inverse. Also, find the orthogonal projections of $\mathbb{R}^3$ onto $R(\mathsf{A})$ and $\mathbb{R}^2$ parallel to $N(\mathsf{A})$.

**7.** Given that the singular values decomposition of an $m \times n$ matrix $\mathsf{A}$ is

$$\mathsf{A} = \sum_{i=1}^{k} \sqrt{\lambda_i}\mathbf{x}_i\mathbf{y}_i^{\dagger},$$

prove that the matrix given by

$$\sum_{i=1}^{k} \frac{1}{\sqrt{\lambda_i}}\mathbf{y}_i\mathbf{x}_i^{\dagger},$$

is the strong generalised inverse of $\mathsf{A}$.

**8.** Suppose that the real matrices $\mathsf{A}$ and $\mathsf{B}$ are such that $\mathsf{A}\mathsf{B}^t\mathsf{B} = \mathsf{0}$. Prove that $\mathsf{A}\mathsf{B}^t = \mathsf{0}$.

**9.** Let $\mathsf{A}$ be an $m \times n$ matrix. Show that the general least squares solution of the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = \mathsf{A}^G\mathbf{b} + (\mathsf{I} - \mathsf{A}^G\mathsf{A})\mathbf{z},$$

where $\mathbf{z}$ is any vector in $\mathbb{R}^n$.

**Harder Problems.** (These are *not* compulsory, they are *not* to be handed in, and they will *not* be covered in classes.)

Here are some results from the lectures on generalised inverses that you might like to try proving.

**10.** Prove that the following statements about an $m \times n$ matrix $\mathsf{A}$ are equivalent:

1. $\mathsf{A}$ has a left inverse, i.e. there is a matrix $\mathsf{L}$ such that $\mathsf{L}\mathsf{A} = \mathsf{I}$. (For example $(\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t$.)
2. $\mathsf{A}\mathbf{x} = \mathbf{b}$ has a *unique* solution when it has a solution.
3. $\mathsf{A}$ has rank $n$.

**11.** Prove that the following statements about an $m \times n$ matrix $\mathsf{A}$ are equivalent:

1. $\mathsf{A}$ has a right inverse, i.e. there is a matrix $\mathsf{R}$ such that $\mathsf{A}\mathsf{R} = \mathsf{I}$. (For example $\mathsf{A}^t(\mathsf{A}\mathsf{A}^t)^{-1}$.)
2. $\mathsf{A}\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.
3. $\mathsf{A}$ has rank $m$.

**12.** Prove that a matrix $\mathsf{A}$ has exactly one strong generalised inverse.

**13.** Consider an $m \times n$ matrix $\mathsf{A}$ that has been decomposed into the product of an $m \times k$ matrix $\mathsf{B}$ and a $k \times n$ matrix $\mathsf{C}$ such that the ranks of $\mathsf{B}$ and $\mathsf{C}$ are both $k$. Show that the matrix

$$\mathsf{A}^G = \mathsf{C}^t(\mathsf{C}\mathsf{C}^t)^{-1}(\mathsf{B}^t\mathsf{B})^{-1}\mathsf{B}^t,$$

is a strong generalised inverse of $\mathsf{A}$.

**14.** Suppose that $\mathsf{A}\mathbf{x} = \mathbf{b}$ is an inconsistent set of equations: show that $\mathbf{x} = \mathsf{A}^G\mathbf{b}$ is the least squares solution that is closest to the origin. Is it necessary that this set of equations is inconsistent?