Further Mathematical Methods (Linear Algebra) 2002

Problem Sheet 2

(To be discussed in week 3 classes. Please submit answers to the asterisked questions only.)

This week we will look at some problems involving linear transformations. As we only covered this material briefly in the lectures, it is vital that you read the hand-out for Lecture 3 before you attempt these exercises. We shall also apply the two tests for linear independence which we devised in the lectures and examine how solutions to sets of simultaneous equations can be analysed in terms of the null space of a matrix.

1. * Prove Theorems 3.3 and 3.7 from the hand-out for Lecture 3.

2. * Consider the transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$T\left([w, x, y, z]^t\right) = \begin{bmatrix} w + x + y \\ x + y + z \end{bmatrix}$$

Show that this transformation is linear. Find the range and null-space of the transformation T, and use these to verify the rank-nullity theorem. Further, find

- the matrix for T with respect to the standard bases in $\mathbb{R}^4$ and $\mathbb{R}^2$, and
- the matrix for T with respect to the bases

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad S' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

  of $\mathbb{R}^4$ and $\mathbb{R}^2$ respectively.

Verify that the sets S and S' are bases of $\mathbb{R}^4$ and $\mathbb{R}^2$.

3. Use the Wronskian to show that the set of functions $\{e^{\alpha x}, e^{\beta x}, e^{\gamma x}\}$ [defined for all values of $x \in \mathbb{R}$] is linearly independent if α, β and γ are all different.

4. * Show that the Wronskian for the functions $f(x)$ and $g(x)$ defined [for all values of $x \in \mathbb{R}$] by

$$f(x) = x^2 \quad \text{and} \quad g(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

is identically zero, but that this pair of functions is linearly independent.

5. Write the set of simultaneous equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4 \\ 2x_1 + x_3 - x_4 &= 2 \\ 2x_2 + x_3 + 3x_4 &= 6 \end{aligned}$$

in the form $\mathsf{A}\mathbf{x} = \mathbf{b}$. Find the general solution to this set of equations writing it in the form

$$\mathbf{x} = \mathbf{v} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_r \mathbf{u}_r$$

where the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ is a basis for the null space of A. Find the Cartesian equation of the affine set given by the general solution and describe it geometrically in terms of hyperplanes.

Other Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

This week, the 'other problems' are here so that you can revise some properties of matrices which you should have encountered in MA100. Everyone *should* try these as we will be using these techniques throughout the course. Solutions for these problems will be contained in the Solution Sheet.

You should recall that the *row* (or *column*) *space* of a matrix is defined to be the space spanned by the row (or column) vectors of a matrix. We shall denote the row and column spaces of a matrix A by $RS(\mathsf{A})$ and $CS(\mathsf{A})$ respectively.

6. Use row operations to find the rank of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & 1 \\ 3 & 0 & 0 & -3 \end{bmatrix}.$$

Hence, write down a smallest spanning set for $RS(\mathsf{A})$ and $CS(\mathsf{A})$.

7. Given the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ where

$$T\left([w, x, y, z]^t\right) = \begin{bmatrix} w + x \\ x + y \\ z + w \end{bmatrix}$$

find the matrix A_T which represents this transformation with respect to the standard basis of $\mathbb{R}^4$. Then, find the range and null-space of this matrix, and use these to verify the rank-nullity theorem holds for matrices. Lastly, determine the set of all vectors $\mathbf{b}$ for which the set of simultaneous equations given by $\mathsf{A}_T\mathbf{x} = \mathbf{b}$ is consistent.

8. Prove that for any real $m \times n$ matrix A, $R(\mathsf{A}) = CS(\mathsf{A})$.

9. Establish that for any real $m \times n$ matrix, the dimension of the column space, the dimension of the row space, and the rank of the matrix are all equal.

Harder Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

For those of you who like the more abstract stuff, here are some more difficult questions for you to think about. Solutions for these problems will be contained in the Solution Sheet. If you want to discuss these solutions (after they have been circulated) you should bother me and not your class teacher.

10. Prove the following two theorems:

- If $S \subseteq V$ is a linearly independent set of vectors that is not already a basis for V, then the set S can be augmented to form a new set S' which is a basis for V by adding appropriate vectors to S.

- If $S \subseteq V$ is a set of vectors which spans V but is not already a basis for V, then the set S can be reduced to form a new set S' which is a basis for V by removing appropriate vectors from S.

11. Prove the rank-nullity theorem (i.e. Theorem 3.9 in the hand-out for Lecture 3) in the cases where $\eta(T) = 0$ and $\eta(T) = n$. Why does the proof given in the hand-out not apply in these cases?

12. Prove Theorems 3.12 and 3.13 from the hand-out for Lecture 3.