# Further Mathematical Methods (Linear Algebra)

# Problem Sheet 3

(To be discussed in week 4 classes. Please submit answers to the asterisked questions only.)

This week, we are going to do some problems on inner product spaces. In particular, we are going to justify the assertion made in the lectures that *many* different inner products can be defined on a given vector space. We shall also use the Gram-Schmidt procedure to generate an orthonormal basis.

**1.** Verify that the Euclidean inner product of two vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]^t$ in $\mathbb{R}^n$, i.e.

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots x_ny_n$$

is indeed an inner product on $\mathbb{R}^n$. Further, given $n$ positive real numbers $w_1, w_2, \ldots, w_n$ and vectors $\mathbf{x}$ and $\mathbf{y}$ as given above, verify that the formula

$$\langle \mathbf{x}, \mathbf{y} \rangle = w_1x_1y_1 + w_2x_2y_2 + \cdots w_nx_ny_n$$

also defines an inner product on $\mathbb{R}^n$.

**2.** Derive the vector equation of a plane in $\mathbb{R}^3$ going through the point with position vector $\mathbf{a}$ and normal $\mathbf{n}$, i.e. $\langle \mathbf{r}, \mathbf{n} \rangle = \langle \mathbf{a}, \mathbf{n} \rangle$. What is the Cartesian equation of this plane? What is the geometrical significance of the quantity $\langle \mathbf{a}, \mathbf{n} \rangle$ if $\mathbf{n}$ is a unit vector? (Note that a unit vector is a vector with a norm of one.)

Use this to calculate the vector and Cartesian equations of the plane which passes through the point with position vector $[1, 2, 1]^t$ and is orthogonal to the vector $[2, 1, 2]^t$. Calculate the quantity mentioned at the end of the previous part.

**3.** * Prove that for all $\mathbf{x}$ and $\mathbf{y}$ in a real inner product space the equalities

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 \quad \text{and} \quad \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 = 4\langle \mathbf{x}, \mathbf{y} \rangle$$

hold. Further, give a geometric interpretation of the significance of the first equality in $\mathbb{R}^3$.

**4.** * Consider the subspace $\mathbb{P}_n^{\mathbb{R}}$ of $\mathbb{F}^{\mathbb{R}}$ and let $x_0$, $x_1$, $\ldots$, $x_n$ be $n+1$ fixed and distinct real numbers. For all vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{P}_n^{\mathbb{R}}$ show that the formula

$$\langle \mathbf{p}, \mathbf{q} \rangle = \sum_{i=0}^{n} p(x_i)q(x_i)$$

defines an inner product on $\mathbb{P}_n^{\mathbb{R}}$.

**5.** * Given two non-zero vectors, prove that if they are orthogonal, then they are linearly independent. Explain why the converse of this result does not hold in general.

**6.** * Show that the set of vectors $S = \{\mathbf{1}, \mathbf{x}, \mathbf{x^2}\} \subseteq \mathbb{P}_2^{[0,1]}$ is linearly independent. Further use these vectors and the Gram-Schmidt procedure to construct an orthonormal basis for $\mathbb{P}_2^{[0,1]}$ where

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)dx$$

is the inner product defined on this vector space. Also, find a matrix $\mathsf{A}$ which will allow you to transform between coordinate vectors that are given relative to these two bases, i.e. find a matrix $\mathsf{A}$ such that for any vector $\mathbf{x} \in \mathbb{P}_2^{[0,1]}$,

$$[\mathbf{x}]_S = \mathsf{A}[\mathbf{x}]_{S'}$$

where $S'$ is the orthonormal basis.

**Other Problems.** (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

Here are some more questions on these topics. Everyone *should* try these to further their understanding of the material covered in the lectures. Solutions for these problems will be contained in the Solution Sheet.

**7.** Consider the vector space of all smooth functions defined on the interval $[0, 1]$, i.e. $\mathbb{S}^{[0,1]}$. Using the inner product given by the formula

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)dx$$

find the inner products of the following pairs of functions:

- $\mathbf{f} : x \to \cos(2\pi x)$ and $\mathbf{g} : x \to \sin(2\pi x)$
- $\mathbf{f} : x \to x$ and $\mathbf{g} : x \to e^x$
- $\mathbf{f} : x \to x$ and $\mathbf{g} : x \to 3x$

Bearing in mind Question 5, comment on the significance of your results in terms of the relationship between orthogonality and linear independence.

**8.** If $\mathbf{p}(x) = a_0 + a_1x + a_2x^2$ and $\mathbf{q}(x) = b_0 + b_1x + b_2x^2$ (for all $x \in \mathbb{R}$) are two general vectors in $\mathbb{P}_2^{\mathbb{R}}$, verify that the formula

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + a_2b_2$$

defines an inner product on $\mathbb{P}_2^{\mathbb{R}}$

**Harder Problems.** (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

Here are some slightly harder questions for those of you who think the stuff above is too easy. Solutions for these problems will be contained in the Solution Sheet. If you want to discuss these solutions (after they have been circulated) you should bother me and not your class teacher.

**9.** Verify that the Euclidean inner product of two vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]^t$ in $\mathbb{C}^n$, i.e.

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1^* + x_2y_2^* + \cdots x_ny_n^*$$

is indeed an inner product on $\mathbb{C}^n$. Further, recall that the norm of a vector $\mathbf{x} \in \mathbb{C}^n$ is defined as

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle},$$

and use this to prove that the following theorems hold in any complex inner product space.

- The *Cauchy-Schwarz Inequality*: If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{C}^n$, then $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$.
- The *Triangle Inequality*: If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{C}^n$, then $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
- *Generalised Theorem of Pythagoras*: If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{C}^n$ and $\mathbf{x} \perp \mathbf{y}$, then $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Recall that two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal*, written $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

**10.** Use the Cauchy-Schwarz inequality to prove that

$$(a \cos\theta + b \sin\theta)^2 \leq a^2 + b^2$$

for all real values of $a$, $b$ and $\theta$.

**11.** Prove that the equality in the Cauchy-Schwarz inequality holds iff the vectors involved are linearly dependent.