Further Mathematical Methods (Linear Algebra) 2002

Problem Sheet 4

(To be discussed in week 5 classes. Please submit answers to the asterisked questions only.)

This week, we practice finding the eigenvalues and eigenvectors of a matrix. The ability to do this will turn out to be very useful, and to demonstrate this, we will use these skills in our study of age-specific population growth.

1. Find the characteristic polynomial of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix},$$

where a is an arbitrary real number. Show that this matrix is diagonalisable when $a = 0$, but not when $a = -1/4$.

2. * Let A be a 2×2 matrix with two distinct eigenvalues λ_1 and λ_2. Show that: If $\mathbf{x}_1$ and $\mathbf{x}_2$ are the eigenvectors corresponding to these eigenvalues, then the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent.

3. Verify that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ b_1/\lambda_1 \\ b_1 b_2/\lambda_1^2 \\ \vdots \\ b_1 b_2 \ldots b_{n-1}/\lambda_1^{n-1} \end{bmatrix}$$

is an eigenvector of the standard $n \times n$ Leslie matrix corresponding to the unique real eigenvalue λ_1. (Hint: Calculate $\mathsf{L}\mathbf{v}_1$ and use the fact that $q(\lambda_1) = 1$, where the function $q(\lambda)$ is defined in the handout for Lecture 8.)

4. * A certain species of animal has a maximum lifespan of ten years. Dividing this into two age classes of 5 years, we find that this species has fertility rates a_1 and a_2 given by 1 and 4 respectively, and a survival rate b_1 of $1/2$. Write down the 2×2 Leslie matrix for this species. This species has an initial population of 2000, with 1000 between 0 and 5 years old, and the other 1000 between 5 and 10 years old. Find an *exact* formula for $\mathbf{x}^{(k)}$. Further, check that

$$\lim_{k \to \infty} \frac{\mathbf{x}^{(k)}}{\lambda_1^k} = c\mathbf{v}_1$$

for some constant c, where λ_1 is the unique positive eigenvalue and $\mathbf{v}_1$ is an eigenvector of the Leslie matrix corresponding to this eigenvalue.

5. * A certain species of animal has a maximum lifespan of thirty years. Dividing this into three age classes of ten years, we find that this species has fertility rates a_1, a_2 and a_3 given by 0, $1/4$ and $1/2$ respectively, and survival rates b_1 and b_2 of $1/2$ and $1/4$ respectively. Write down the 3×3 Leslie matrix for this species. Find the unique real positive eigenvalue of this matrix. Describe the long-term behaviour of the population distribution of this species. (Your answer should address both the growth rate of the population as a whole, and the way in which the population is distributed among the three age classes.)

Other Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

This week, the 'other problems' are here so that you can examine some of the other consequences of our model of age-specific population growth. Everyone *should* try Question 6 as it deals with a possibility that we will not discuss anywhere else.

6. Consider the Leslie matrix

$$\mathsf{L} = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$$

Find the eigenvalues of this matrix. Is the unique positive real eigenvalue λ_1 dominant? Calculate L^3. What does this tell you about the evolution of the population distribution vector $\mathbf{x}^{(k)}$? Explain, *very briefly* why this does *not* contradict the key result in the handout for Lecture 8 (namely, that, in the infinite time limit, there is a fixed proportion of the population in each age class). Note: I am looking for a mathematical reason and not a demography essay!

7. The *net reproduction rate* is defined to be

$$R = a_1 + a_2 b_1 + a_3 b_1 b_2 + \cdots + a_n b_1 b_2 \ldots b_{n-1}.$$

Explain why this is the average number of daughters born to a female during her expected lifetime. Assuming that two consecutive a_i are non-zero, show that the population is eventually increasing iff its net reproduction rate is greater than 1. (Hint: For this, it might be helpful to use the fact that $q(\lambda_1) = 1$.) Why is this result obvious anyway?

Harder Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

For those of you who like the more abstract stuff, here are some more difficult questions for you to think about. Solutions for these problems will be contained in the Solution Sheet. If you want to discuss these solutions (after they have been circulated) you should bother me and not your class teacher.

8. Following on from Question 2, let A be an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that: If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are the eigenvectors corresponding to these eigenvalues, then the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is linearly independent.

9. Let A be an $n \times n$ matrix. Prove that: A is diagonalisable iff A has n linearly independent eigenvectors. (Recall that a matrix A is diagonalisable if there is an invertible matrix P such that the matrix given by $\mathsf{P}^{-1}\mathsf{AP}$ is diagonal.)

10. Prove that: $\lambda = 0$ is an eigenvalue of a matrix A iff A is not invertible.

11. Prove that: If A is an $n \times n$ matrix, then

$$\det(\mathsf{A} - \lambda I) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots c_{n-1}\lambda + c_n,$$

where the c_i $(1 \leq i \leq n)$ are constants. (In particular, note that this is a polynomial of degree n and that the coefficient of the term in λ^n is given by $(-1)^n$.) Further, show that the constant term (i.e. c_n) is given by $\det(\mathsf{A})$.