Further Mathematical Methods (Linear Algebra) 2002

Problem Sheet 5

(To be discussed in week 6 classes. Please submit answers to the asterisked questions only.)

This week we look at some problems involving systems of differential equations. We also start thinking about what happens if we have matrices with complex eigenvalues and how we would calculate functions of matrices other than integer powers and inverses.

1. Given that the vectors $[1,-1,1]^t$, $[-3,0,1]^t$ and $[-1,1,0]^t$ are eigenvectors of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & -2 & -6 \\ 2 & 5 & 6 \\ -2 & -2 & -3 \end{bmatrix}$$

find an invertible matrix P such that $\mathsf{P}^{-1}\mathsf{AP}$ is diagonal. Hence, find the functions $y_1(t)$, $y_2(t)$ and $y_3(t)$ which satisfy the differential equations

$$\begin{aligned} y_1' &= y_1 - 2y_2 - 6y_3 \\ y_2' &= 2y_1 + 5y_2 + 6y_3 \\ y_3' &= -2y_1 - 2y_2 - 3y_3 \end{aligned}$$

with the initial conditions $y_1(0) = y_2(0) = 1$ and $y_3(0) = 0$.

2. * Find the steady states of the following system of differential equations.

$$\begin{aligned} \dot{y}_1 &= y_1(2 - 2y_1 - y_2) \\ \dot{y}_2 &= y_2(2 - 2y_2 - y_1) \end{aligned}$$

One of the steady states has neither y_1 nor y_2 equal to zero. Show that this steady state is asymptotically stable.

3. Following on from the Example on p.9-8 of the handout for Lectures 9 and 10, show that the steady state (4,0) is asymptotically stable.

4. * Let A be a real diagonalisable matrix whose eigenvalues are all non-negative. Prove that there is a matrix B such that $\mathsf{B}^2 = \mathsf{A}$. Is the matrix B unique? (Give a reason for your answer.)

5. * Show that the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 1 \\ -9 & 1 \end{bmatrix}$$

has eigenvalues $1 \pm 3i$. Hence find an invertible matrix P such that $\mathsf{P}^{-1}\mathsf{AP}$ is diagonal. Write down the diagonal matrix D corresponding to your matrix P. Verify that $\mathsf{P}^{-1}\mathsf{AP} = \mathsf{D}$. (Note: Although the eigenvalues are not real numbers, our normal techniques for finding eigenvectors and diagonalising matrices still apply.)

Other Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

This week, the 'other problems' are here so that you can have some more practice using the techniques developed over the last couple of lectures.

6. Find the steady states of the following differential equation.

$$\dot{y} = y^2 + 3y - 10$$

Solve this equation and discuss the stability of these steady states. Investigate the range of initial conditions that will cause y to tend to these steady states in the large time limit.

7. Find the steady states of the following non-linear system of differential equations:

$$\begin{aligned} \dot{x} &= 2xy - 2y^2 \\ \dot{y} &= x - y^2 + 2 \end{aligned}$$

Examine the stability of the steady states that you find.

8. Following on from the Example on p.9-8 of the handout for Lectures 9 and 10, explain what happens if the system in question has a solution whose initial conditions are close to the steady state $(0, 0)$. What happens if a solution has initial conditions close to the steady state $(1, 3)$?

Harder Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

For those of you who like the more abstract stuff, here are some more difficult questions for you to think about. Solutions for these problems will be contained in the Solution Sheet. If you want to discuss these solutions (after they have been circulated) you should bother me and not your class teacher.

9. Consider the general system of linear differential equations given by $\dot{\mathbf{y}} = \mathsf{A}\mathbf{y}$ where A is a 2×2 matrix. Show that $(0, 0)$ is a steady state of this system. Further, show that this steady state is asymptotically stable iff $\det(\mathsf{A}) > 0$ and $\mathrm{Tr}(\mathsf{A}) < 0$. Also show that if either $\det(\mathsf{A}) < 0$ or $\mathrm{Tr}(\mathsf{A}) > 0$, then this steady state is unstable. (Recall that the Trace of a matrix, $\mathrm{Tr}(\mathsf{A})$, is given by the sum of the diagonal elements of A.)

10. Consider our general model of population dynamics where we have two species competing (as given in Section 9.2 of the handout for Lectures 9 and 10). Show that $\mathbf{y}_1^* = (a_1/b_1, 0)$ and $\mathbf{y}_2^* = (0, a_2/b_2)$ will be steady states of this system of non-linear differential equations. Further, show that $\mathbf{y}_2^*$ will be asymptotically stable or unstable depending on the sign of the quantity

$$\Delta = \frac{a_1}{c_1} - \frac{a_2}{b_2}.$$

Explain this condition in terms of the model. What is the corresponding result for $\mathbf{y}_1^*$?