Further Mathematical Methods (Linear Algebra) 2002

Problem Sheet 7

(To be discussed in week 8 classes. Please submit answers to the asterisked questions only.)

This week, we shall start by looking at orthogonal complements and what they tell us about the range and null-space of a matrix. We shall then examine some of the consequences of our results concerning the rank of matrix products. Lastly, we shall look at sums and direct sums of vector spaces.

1. Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $[0, 0, -1]^t$ and $[1, 2, 3]^t$. Find $S^\perp$, the orthogonal complement of S. Interpret your results geometrically.

2. * Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}.$$

Determine the range and null-space of A and its transpose, i.e. find $R(\mathsf{A})$, $R(\mathsf{A}^t)$, $N(\mathsf{A})$ and $N(\mathsf{A}^t)$. Further, verify that $R(\mathsf{A}^t) = N(\mathsf{A})^\perp$ and $R(\mathsf{A})^\perp = N(\mathsf{A}^t)$. Interpret your results geometrically.

3. * Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a finite set of vectors in an inner product space V and let S be the subspace of V spanned by these vectors. Show that $\mathbf{x} \in S^\perp$ iff $\mathbf{x}$ is orthogonal to every vector in the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$.

4. * Prove the following theorems:

- If V and W are subsets of a vector space such that $V \subseteq W$, then $W^\perp \subseteq V^\perp$.
- If A is any $m \times n$ matrix and B is any $n \times m$ matrix where $n < m$, then AB is singular.[1]
- If B is an invertible square matrix and the matrix product AB is defined, then the rank of AB equals the rank of A.

5. * Suppose that Y and Z are the subspaces of $\mathbb{R}^4$ given by

$$Y = \text{Lin}\left\{[1, 0, 1, 0]^t, [0, 0, 0, 1]^t\right\} \quad \text{and} \quad Z = \text{Lin}\left\{[0, 1, 0, 0]^t, [1, 0, 1, -1]^t\right\}.$$

Is the sum $Y + Z$ direct? If so, why, and if not, why not? Find a basis for the subspace $Y + Z$ of $\mathbb{R}^4$.

6. Prove the following theorems:

- If Y and Z are subspaces of a vector space V, then $Y + Z$ is also a subspace of V. Further, $Y + Z$ is the smallest subspace of V containing $Y \cup Z$ (in the sense that every other subspace of V that contains $Y \cup Z$ must contain $Y + Z$).
- If the set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$ is a basis of the vector space V, then

$$V = \text{Lin}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \oplus \text{Lin}\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}.$$

- If Y and Z are subspaces of a vector space V such that $V = Y \oplus Z$, then $\dim(V) = \dim(Y \oplus Z) = \dim(Y) + \dim(Z)$.

[1] Of course, you all know that a *singular* matrix is a matrix that is not invertible.

Other Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

Here are some more questions on these topics. Maybe try them after the class when you have a clearer understanding of what is going on.

7. Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

and repeat Question 2.

8. Find an orthonormal basis for the subspace of $\mathbb{C}^3$ spanned by the vectors $[i, 0, 1]^t$ and $[1, 1, 0]^t$. Further, determine the orthogonal complement of this subspace.

9. Prove that if S is any sub*set* of $\mathbb{R}^n$, then $S \subseteq S^{\perp\perp}$. Further, prove that if S is a sub*space* of $\mathbb{R}^n$, then $S = S^{\perp\perp}$. Consequently, show that if S is a subspace of $\mathbb{R}^n$, then $\dim(S) + \dim(S^\perp) = n$.

10. Suppose that $T : V \to V$ is a linear transformation and that X and Y are subspaces of V such that $T(X) \subseteq X$ and $T(Y) \subseteq Y$. Show that if

$$V = X \oplus Y,$$

then

$$T(V) = T(X) \oplus T(Y).$$

11. Suppose that A is any real $m \times n$ matrix and that $\mathbf{b}$ is an $n \times 1$ column vector. Show that *precisely one* of the following systems has solutions:

- **a.** $\mathsf{A}\mathbf{x} = \mathbf{b}$.
- **b.** $\mathsf{A}^t\mathbf{y} = \mathbf{0}$ and $\mathbf{y}^t\mathbf{b} \neq 0$.

where $\mathbf{0}$ is the null vector in $\mathbb{R}^n$

Harder Problem. (This is *not* compulsory, it is *not* to be handed in, and will *not* be covered in classes.)

This week we have only one harder problem, and you may be surprised to hear that it is an old exam question. (Luckily for you, it is a very old exam question!) See what you make of it.

12. Suppose that A is an $n \times n$ real matrix, prove that

$$\mathbb{R}^n \supseteq R(\mathsf{A}) \supseteq R(\mathsf{A}^2) \supseteq R(\mathsf{A}^3) \supseteq \cdots$$

Further, prove that if $R(\mathsf{A}^s) = R(\mathsf{A}^{s+1})$, then $R(\mathsf{A}^s) = R(\mathsf{A}^q)$ and $N(\mathsf{A}^s) = N(\mathsf{A}^q)$ for all $q \geq s$. Hence, show that if $\rho(A) < n$, then

$$\mathbb{R}^n \supset R(\mathsf{A}) \supset R(\mathsf{A}^2) \supset \cdots \supset R(\mathsf{A}^p) = R(\mathsf{A}^{p+1}) = R(\mathsf{A}^{p+2}) = \cdots$$

for some $p \geq 1$ (where $\mathsf{C} \supset \mathsf{D}$ means that $\mathsf{C} \supseteq \mathsf{D}$ and $\mathsf{C} \neq \mathsf{D}$). Further, prove that $\mathbb{R}^n = N(\mathsf{A}^p) \oplus R(\mathsf{A}^p)$.

(Hint: For the last part of this question it is easiest to use the following theorem:

$$V = Y \oplus Z \quad \text{iff} \quad Y \cap Z = \{\mathbf{0}\} \quad \text{and} \quad \dim(V) = \dim(Y) + \dim(Z).$$

which you should also try to prove.)