# Further Mathematical Methods (Linear Algebra) 2002

# Problem Sheet 9

(To be discussed in week 10 classes. Please submit answers to the asterisked questions only.)

In this sheet, we look at how we can use Linear Algebra to find approximations to functions in terms of a given set of functions. In particular, it should convince you (if you do Questions 6 and 7 as well) that the theory underlying Fourier series is a natural extension of our study of orthogonal projections.

**1.** * Let $\mathbf{x} \in \mathbb{R}^n$ and let $S$ be the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$. Show that the matrix $\mathsf{P}$ representing an orthogonal projection of $\mathbb{R}^n$ onto $S$ is $\mathsf{P} = \mathbf{x}\mathbf{x}^t/\|\mathbf{x}\|^2$.

**2.** (a) Find an orthonormal basis for $\mathbb{P}_3^{[-\pi,\pi]}$, i.e. the vector space spanned by the vectors $\{\mathbf{1}, \mathbf{x}, \mathbf{x^2}, \mathbf{x^3}\}$, using the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx,$$

where $\mathbf{f} : x \to f(x)$, $\mathbf{g} : x \to g(x)$ for all $x \in [-\pi, \pi]$. Hence find a least squares approximation to $\sin x$ in $\mathbb{P}_3^{[-\pi,\pi]}$ and calculate the mean square error.

(b) The first two [non-zero] terms in the Taylor series for $\sin x$ are given by

$$x - \frac{x^3}{3!},$$

for all $x \in [-\pi, \pi]$. Find the mean square error between $\sin x$ and this approximation. Which of these cubics, i.e. the one calculated in (a) or the Taylor series, provides the best approximation to $\sin x$?

**3.** Find a least squares approximation of

- * $x$ by a function of the form $a + be^x$,
- $e^x$ by a function of the form $a + bx$,

over the interval $[0, 1]$ using the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{0}^{1} f(x)g(x)\,dx,$$

where $\mathbf{f} : x \to f(x)$ and $\mathbf{g} : x \to g(x)$ for all $x \in [0, 1]$.

**4.** * Find a vector $\mathbf{q} \in \mathbb{P}_2^{\mathbb{R}}$ such that

$$p(1) = \langle \mathbf{p}, \mathbf{q} \rangle,$$

for every vector $\mathbf{p} : x \to p(x)$ in $\mathbb{P}_2^{\mathbb{R}}$ where the inner product defined on this vector space is given by

$$\langle \mathbf{p}, \mathbf{q} \rangle = \sum_{i=-1}^{1} p(i)q(i).$$

(Notice that this is an instantiation of the inner product given in Question 4 of Problem Sheet 3.)

**5.** * Consider the function $\cos(3x)$ defined in the interval $[-\pi, \pi]$. Find the Fourier series of orders 2, 3 and 4 that represent this function. Discuss what this means in terms of the subspaces of $\mathbb{F}^{[-\pi,\pi]}$ given by $\operatorname{Lin} G_2$, $\operatorname{Lin} G_3$ and $\operatorname{Lin} G_4$. Further, calculate the mean square error in these results and indicate how this corroborates your answer. (If you can't remember what $G_n$ is, see Question 6.)

**Other Problems.** (These are *not* compulsory, they are *not* to be handed in, and they will *not* be covered in classes.)

If you would like to further your understanding of Fourier series, you may like to try these questions too. The first two allow you to test your understanding of the theory given in the lectures, whereas the rest of the questions apply it.

**6.** Consider the subset of $\mathbb{F}^{[-\pi,\pi]}$ given by $G_n = \{\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_n, \mathbf{g}_{n+1}, \ldots, \mathbf{g}_{2n}\}$ where

$$\mathbf{g}_0(x) = \frac{1}{\sqrt{2\pi}},$$

and

$$\mathbf{g}_1(x) = \frac{1}{\sqrt{\pi}}\cos(x),\ \mathbf{g}_2(x) = \frac{1}{\sqrt{\pi}}\cos(2x), \ldots,\ \mathbf{g}_n(x) = \frac{1}{\sqrt{\pi}}\cos(nx),$$

whereas,

$$\mathbf{g}_{n+1}(x) = \frac{1}{\sqrt{\pi}}\sin(x),\ \mathbf{g}_{n+2}(x) = \frac{1}{\sqrt{\pi}}\sin(2x), \ldots,\ \mathbf{g}_{2n}(x) = \frac{1}{\sqrt{\pi}}\sin(nx).$$

Show that $G_n$ is an orthonormal set when using the standard inner product on $\mathbb{F}^{[-\pi,\pi]}$. Further, show that any trigonometric polynomial of order $n$ or less can be represented by a vector in $\operatorname{Lin} G_n$.

**7.** Suppose that $\mathbf{f} \in \mathbb{F}^{[-\pi,\pi]}$ is a function that is not in $\operatorname{Lin} G_n$, using the result of Question 1 show that the orthogonal projection of $\mathbf{f}$ onto $\operatorname{Lin} G_n$ is

$$\sum_{k=0}^{2n} \langle \mathbf{f}, \mathbf{g}_k \rangle \mathbf{g}_k,$$

and explain why this series minimises the mean square error, i.e.

$$\int_{-\pi}^{\pi} [f(x) - g(x)]^2 \, dx,$$

where $\mathbf{f} : x \to f(x)$, $\mathbf{g} : x \to g(x)$ and $\mathbf{g} \in \operatorname{Lin} G_n$. Further, show that this series can be written as

$$\frac{a_0}{2} + \sum_{k=1}^{n} \{a_k \cos(kx) + b_k \sin(kx)\},$$

where the coefficients are given by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\, dx \quad \text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\, dx.$$

This is the Fourier series of order $n$ representing $f(x)$. Indicate how this series can be simplified if $f(x)$ is (a) an odd function, i.e. $f(-x) = -f(x)$, or (b) an even function, i.e. $f(-x) = f(x)$.

**8.** Further to Question 4, find the Fourier series of order $n$ representing the function $\sin(3x)$ over the interval $[-\pi, \pi]$ without integrating. Calculate the mean square error. Indeed, verify that your result is correct by integrating to find the coefficients.

**9.** Following on from Question 5, show that

$$\frac{1}{2} + \cos(x) + \cos(2x) + \cdots + \cos(nx) = \frac{\sin[(n + \frac{1}{2})x]}{2\sin(\frac{1}{2}x)},$$

without integrating to find the coefficients. (Assume that $x \neq 2n\pi$ where $n \in \mathbb{Z}$. What happens if this condition is violated?)

**10.** Find the Fourier series of order $n$ of the following functions which are defined in the interval $[-\pi, \pi]$: (a) $f(x) = \pi - x$, (b) $f(x) = x^2$ and (c) $f(x) = |x|$. Also calculate the mean square error in each case.