Further Mathematical Methods (Linear Algebra) 2002

Solutions For Problem Sheet 1

In this Problem Sheet, we looked at some sets of vectors that were vector spaces and some that were not. We also used some of the 'useful concepts' from the notes and proved some new results about them.

1. We are given four subsets of a vector space and we are asked to find out which of them are subspaces. The reason why we are also asked to give a geometric interpretation of the subsets of $\mathbb{R}^3$ and to describe the subsets of $\mathbb{F}^{\mathbb{R}}$ is so that we can gain some idea of what sort of subsets are (and aren't) subspaces.

(a) The subset $S_1 = \{[x, y, z]^t \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ of $\mathbb{R}^3$ represents a sphere of unit radius centred on the origin. It is not a subspace as:

- Subspaces are themselves vector spaces, and so by Definition 1.1, this means that they *must* contain the additive identity (or null vector, which in this case is $\mathbf{0} = [0, 0, 0]^t$) and as $0^2 + 0^2 + 0^2 = 0 \neq 1$, $\mathbf{0} \notin S_1$.[1] Consequently, S_1 cannot be closed under vector addition.

- By Theorem 1.4, subspaces must be closed under vector addition and to show that S_1 is *not*, we can give a counter-example: The vectors $[1, 0, 0]^t$ and $[0, 1, 0]^t$ are in S_1 (as $1^2 + 0^2 + 0^2 = 1$ and $0^2 + 1^2 + 0^2 = 1$ respectively), however the sum of these two vectors is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

and this new vector is not in S_1 as $1^2 + 1^2 + 0^2 = 2 \neq 1$. Consequently, S_1 cannot be closed under vector addition.

- By Theorem 1.4, subspaces must be closed under scalar multiplication and to show that S_1 is *not*, we can give a counter-example: The vector $[1, 0, 0]^t$ is in S_1 (as $1^2 + 0^2 + 0^2 = 1$), however multiplying this by the scalar 2 (any other real number, except for 1 or -1, would have done!) gives

$$2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix},$$

and this new vector is not in S_1 as $2^2 + 0^2 + 0^2 = 4 \neq 1$. Consequently, S_1 cannot be closed under scalar multiplication.

Of course, any one of these reasons would have been sufficient to establish that the set S_1 is not a subspace of $\mathbb{R}^3$. (Alternatively, you could have shown that S_1 generally fails to satisfy either of the closure conditions required by Theorem 1.4.)

(b) The subset $S_2 = \{[x, y, x + y]^t \in \mathbb{R}^3 \,|\, x, y \in \mathbb{R}\}$ represents a plane through the origin in $\mathbb{R}^3$. We observe that any vector in the set S_2 will have a z-component given by $z = x + y$, and so the Cartesian equation of this plane is $x + y - z = 0$. As this is just the set $S_{1,1,-1}$ in the notation of Chapter 1, we have already shown that this set is a subspace of $\mathbb{R}^3$.

However, if you wish to show that the set S_2 is a subspace of $\mathbb{R}^3$ directly, then we can proceed as follows:

[1] The origin is *obviously* not in S_1 as a sphere of unit radius centred on the origin can be defined to be the set of all points which are exactly one unit *away* from the origin!

- If we take two *general* vectors in S_2, say $\mathbf{x} = [x, y, x+y]^t$ and $\mathbf{x}' = [x', y', x'+y']^t$, where $x, y, x', y' \in \mathbb{R}$, then their vector sum is given by:

$$\mathbf{x}+\mathbf{x}' = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ x'+y' \end{bmatrix} = \begin{bmatrix} x+x' \\ y+y' \\ (x+y)+(x'+y') \end{bmatrix} = \begin{bmatrix} x+x' \\ y+y' \\ (x+x')+(y+y') \end{bmatrix},$$

  where $x+x', y+y', (x+x')+(y+y') \in \mathbb{R}$ too. Clearly, $\mathbf{x}+\mathbf{x}' \in S_2$ and so S_2 is closed under vector addition.

- If we take a *general* vector in S_2, say $\mathbf{x} = [x, y, x+y]^t$, multiplying it by any scalar $\alpha \in \mathbb{R}$ we get

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha(x+y) \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha x+\alpha y \end{bmatrix},$$

  where $\alpha x, \alpha y, \alpha x+\alpha y \in \mathbb{R}$ too. Clearly, $\alpha\mathbf{x} \in S_2$ and so S_2 is closed under scalar multiplication.

Consequently, by Theorem 1.4, S_2 is a subspace of $\mathbb{R}^2$ (as expected).

(c) The subset $S_3 = \{\mathbf{f} \in \mathbb{F}^{\mathbb{R}} \mid f(2) = 1\}$ of $\mathbb{F}^{\mathbb{R}}$ represents the set of all functions [defined over $\mathbb{R}$] which map 2 to 1. Some examples of functions which are represented by vectors in S_3 are $x-1$, x^2-3 and $\sin(\pi x/4)$. It is not a subspace as:

- Subspaces are themselves vector spaces, and so by Definition 1.1, this means that they *must* contain the additive identity (or null vector, which in this case is $\mathbf{0} : x \to 0$ for all $x \in \mathbb{R}$) and as $\mathbf{0}(2) = 0 \neq 1$, $\mathbf{0} \notin S_3$.[2]

- By Theorem 1.4, subspaces must be closed under vector addition and to show that S_3 is *not*, we can give a counter-example: The functions $x-1$ and x^2-3 are in S_3 (as mentioned above), however the sum of these two functions is x^2-x-4 and this new function is not in S_3 as it takes the value -2 at $x=2$. Consequently, S_3 cannot be closed under vector addition.

- By Theorem 1.4, subspaces must be closed under scalar multiplication and to show that S_3 is *not*, we can give a counter-example: The function $x-1$ is in S_3 (as mentioned above), however multiplying this by the scalar 2 (any other real number, except for 1, would have done) gives $2x-2$ and this new function is not in S_3 as it takes the value 2 at $x=2$. Consequently, S_3 cannot be closed under scalar multiplication.

Of course, any one of these reasons would have been sufficient to establish that the set S_3 is not a subspace of $\mathbb{F}^{\mathbb{R}}$. (Alternatively, you could have shown that S_3 generally fails to satisfy either of the closure conditions required by Theorem 1.4.)

(d) The subset $S_4 = \{\mathbf{f} \in \mathbb{F}^{\mathbb{R}} \mid f(5) = 0\}$ of $\mathbb{F}^{\mathbb{R}}$ represents the set of all functions [defined over $\mathbb{R}$] which map 5 to 0. Some examples of functions which are represented by vectors in S_4 are $x-5$, x^2-25 and $\sin(\pi x/5)$.[3] To show that S_4 is a subspace of $\mathbb{F}^{\mathbb{R}}$, we proceed as follows:

- If we take two *general* vectors in S_4, say $\mathbf{f} : x \to f(x)$ and $\mathbf{g} : x \to g(x)$ where $f(5) = 0$ and $g(5) = 0$, then their vector sum is given by

$$[\mathbf{f}+\mathbf{g}](x) = f(x)+g(x) \implies f(5)+g(5) = 0+0 = 0,$$

  and so $\mathbf{f}+\mathbf{g} \in S_4$ too. Consequently, S_4 is closed under vector addition.

[2] The zero-function is obviously not in S_3 as this is the set of all functions which take the value 2 at $x = 1$, whereas the zero-function takes the value zero for all $x \in \mathbb{R}$.

[3] We note in passing that this set also contains the additive identity required by Definition 1.1. Recall that in $\mathbb{F}^{\mathbb{R}}$, this vector is represented by the zero-function $\mathbf{0} : x \to 0$.

- If we take a *general* vector in S_4, say $\mathbf{f} : x \to f(x)$ where $f(5) = 0$, we can multiply it by any scalar $\alpha \in \mathbb{R}$ to get
$$[\alpha\mathbf{f}](x) = \alpha f(x) \implies \alpha f(5) = \alpha \cdot 0 = 0,$$
and so $\mathbf{f} \in S_4$ too. Consequently, S_4 is closed under scalar multiplication.

So, by Theorem 1.4, S_2 is a subspace of $\mathbb{R}^2$.

2. The sets $S_{1,1,1}$ and $S_{1,2,3}$ are both subspaces of $\mathbb{R}^3$ as established in the Example of Section 1.4.2. Indeed, they both represent sets of vectors [or points] that lie in planes passing through the origin, specifically the planes $x + y + z = 0$ and $x + 2y + 3z = 0$. In this question we ask, in this specific setting, whether the union or the intersection of two subspaces is also a subspace. This may seem unimportant now, but later in the course we shall define other set operations[4] and we will want to know if performing these set operations on subspaces give rise to other subspaces.

We start by looking at the *union* of these two sets, namely
$$S_{1,1,1} \cup S_{1,2,3} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, \mathbf{x} \in S_{1,1,1} \text{ or } \mathbf{x} \in S_{1,2,3}\},$$
and this represents the set of all vectors [or points] that lie on either one of the two planes. At first sight, it *appears* that this new set is also a subspace since:

- The null vector, i.e. $\mathbf{0}$, is in both of the sets under consideration and so it is in their union too.
- If we take any vector in $S_{1,1,1}$ (or $S_{1,2,3}$) and multiply it by a scalar we get another vector in $S_{1,1,1}$ (or $S_{1,2,3}$) since both of these sets are closed under scalar multiplication (as individually, they are both subspaces).
- If we take any two vectors in $S_{1,1,1}$ (or $S_{1,2,3}$) and add them together we get another vector in $S_{1,1,1}$ (or $S_{1,2,3}$) since both of these sets are closed under vector addition (as individually, they are both subspaces).

However, this is not the case since we can easily find a counter-example to the claim that it is closed under vector addition. To do this, we take a vector from $S_{1,1,1}$ (which is not in $S_{1,2,3}$) and a vector from $S_{1,2,3}$ (which is not in $S_{1,1,1}$), for instance the vectors
$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in S_{1,1,1} \text{ and } \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \in S_{1,2,3},$$
and notice that their sum is $[3, -2, 0]^t$. But, this vector does not lie in the union since it is *neither* in $S_{1,1,1}$ (since $3 - 2 + 0 = 1 \neq 0$) *nor* $S_{1,1,1}$ (since $3 - 4 + 0 = -1 \neq 0$). Thus, the union of the sets $S_{1,1,1}$ and $S_{1,2,3}$ is not closed under vector addition and hence it is not a subspace of $\mathbb{R}^3$.

Next, looking at the *intersection* of these two sets, namely
$$S_{1,1,1} \cap S_{1,2,3} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, \mathbf{x} \in S_{1,1,1} \text{ and } \mathbf{x} \in S_{1,2,3}\},$$
this represents the set of all vectors [or points] that lie on both of the two planes. In particular, if the vector $[x, y, z]^t$ lies in this set its components must satisfy both $x+y+z = 0$ and $x+2y+3z = 0$, that is, they must be solutions to the simultaneous equations:
$$\begin{aligned} x + y + z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$$
But, in this case, we have two equations in three variables and so we can set one of the variables, say z, equal to the free parameter $r \in \mathbb{R}$. Thus, solving the equations we find that $x = r$ and $y = -2r$ which means that only vectors of the form
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

[4] A set operation is something that you 'do' to a set (or pair of sets) to yield another set. (For example, both the union, and the intersection, of two sets yield another set.)

will lie in the intersection, i.e. the vectors [or points] in this set lie on a line through the origin which 'points' in the direction $[1, -2, 1]^t$. Indeed, this analysis provides us with a particularly nice way of proving that $S_{1,1,1} \cap S_{1,2,3}$ does give us a subspace, as taking any two vectors in this set, say

$$r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ and } s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

where $r, s \in \mathbb{R}$, and any scalar $\alpha \in \mathbb{R}$ we can establish closure under:

- **Vector Addition:** since

$$r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = (r+s) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

  where $r + s \in \mathbb{R}$, and

- **Scalar Multiplication:** since

$$\alpha \cdot r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = (\alpha r) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

  where $\alpha r \in \mathbb{R}$.

Thus, by Theorem 1.4, $S_{1,1,1} \cap S_{1,2,3}$ is a subspace.

What does this question imply for *general* unions and intersections of subspaces? Well, it should lead you to *suspect* that the *union* of two [different] subspaces will *not* yield a subspace, whereas the *intersection* will. We have not proved this, but it provides us with an example of a set operation that does, and a set operation that does not, yield a subspace when it is applied to two specific subspaces.

3. We are given the plane $x - y + 3z = 0$ which represents a subspace of $\mathbb{R}^3$.[5] To find a basis, we notice that we have one equation in three variables, and so two of them must be free. So, taking y and z to be the free variables, we set them equal to the free parameters r and s respectively, and find that $x = r - 3s$. Consequently, vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r - 3s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix},$$

satisfy the equation of the plane, and so the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\},$$

will span this subspace.[6] However, to be a basis, this set must be linearly independent too. But by choosing the free parameters to correspond to different variables we automatically get linear independence.[7] Consequently, this set is a basis and so we let

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and as it contains two vectors we can see that the subspace of $\mathbb{R}^3$ we are considering has a dimension of two.

[5] This is obviously a subspace of $\mathbb{R}^3$ as the set of all points in this plane is just $S_{1,-1,3}$ — see Section 1.4.2.

[6] This is the method used in the Example following Definition 2.2.

[7] But you may verify that they are linearly independent by using the method of the Example following Theorem 2.7.

Further, the vector $[4,7,1]^t$ is in this subspace because its components satisfy the equation of the plane (i.e. $4 - 7 + 3 = 0$). However, if we add this vector to B to form the set

$$B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \right\},$$

we no longer have a basis as this new set is linearly dependent. To see this, observe that the vector $[4,7,1]^t$ can be written as a linear combination of the other vectors in B', i.e.

$$\begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Or, alternatively, just note that as $[4,7,1]^t$ is in the subspace it must be in the linear span of B, i.e it must be a linear combination of the other vectors in B'.

Further still, the vector $[1,0,0]^t$ is not in this subspace because its components do not satisfy the equation of the plane (i.e. $1 - 0 + 0 = 1 \neq 0$). So, adding this vector to B we form the set

$$B'' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now, this set is linearly independent as if we look at the vector equation

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0},$$

we get the simultaneous equations

$$\begin{aligned} 1\alpha_1 - 3\alpha_2 + 1\alpha_3 &= 0 \\ 1\alpha_1 + 0\alpha_2 + 0\alpha_3 &= 0 \Longrightarrow \alpha_1 = 0 \\ 0\alpha_1 + 1\alpha_2 + 0\alpha_3 &= 0 \Longrightarrow \alpha_2 = 0 \end{aligned}$$

and as these values give $\alpha_3 = 0$ in the top equation, we *only* get a trivial solution to the vector equation given above. Consequently, as the set B'' is linearly independent it is a basis. The space spanned by B'' is $\mathbb{R}^3$ as a set of three linearly independent vectors in the three-dimensional space $\mathbb{R}^3$ will be a basis for (and will hence span) $\mathbb{R}^3$ by Theorem 2.17. Obviously, the dimension of this space is three.

4. We are asked to prove the following four theorems:

(a) If S is a set of vectors that contains the null vector (i.e. $\mathbf{0}$), then it is linearly dependent.

Proof: Let us take the set S which contains the null vector to be $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{0}\}$. It is suggested that we use Definition 2.6, which tells us that the set S is linearly dependent *if* the vector equation

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n + \alpha_{n+1} \mathbf{0} = \mathbf{0},$$

has non-trivial solutions. Clearly, this is the case here as, regardless of the values of $\alpha_1, \alpha_2, \ldots, \alpha_n$, the coefficient of $\mathbf{0}$ can take *any* real value (i.e. a_{n+1} need not be zero). Consequently, the set S is linearly dependent (as required).

(b) Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}\}$ be a set of vectors in a vector space V. If $\mathbf{v}$ can be written as a linear combination of the other vectors in S (i.e. the vectors in the set $S' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$), then S and S' span the same space, i.e. $\text{Lin}(S) = \text{Lin}(S')$.

Proof: We take the sets S and S' defined in the question and note that as $\mathbf{v}$ is a linear combination of the vectors in the latter set we can write

$$\mathbf{v} = \sum_{i=1}^{m} \alpha_i \mathbf{u}_i,$$

where the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m$ are scalars. Then, using Definition 2.3 as suggested, we note that

$$\begin{aligned}
\text{Lin}(S) &= \left\{ \sum_{i=1}^{m} \gamma_i \mathbf{u}_i + \gamma_{m+1}\mathbf{v} \;\middle|\; \text{for scalars } \gamma_1, \gamma_2, \ldots, \gamma_m, \gamma_{m+1} \right\} \\
&= \left\{ \sum_{i=1}^{m} \gamma_i \mathbf{u}_i + \gamma_{m+1} \sum_{i=1}^{m} \alpha_i \mathbf{u}_i \;\middle|\; \text{for scalars } \alpha_1, \alpha_2, \ldots, \alpha_m, \gamma_1, \gamma_2, \ldots, \gamma_m, \gamma_{m+1} \right\} \\
&= \left\{ \sum_{i=1}^{m} (\gamma_i + \gamma_{m+1}\alpha_i) \mathbf{u}_i \;\middle|\; \text{for scalars } \alpha_1, \alpha_2, \ldots, \alpha_m, \gamma_1, \gamma_2, \ldots, \gamma_m, \gamma_{m+1} \right\} \\
&= \left\{ \sum_{i=1}^{m} \beta_i \mathbf{u}_i \;\middle|\; \text{for scalars } \beta_1, \beta_2, \ldots, \beta_m \right\} \\
\therefore\ \text{Lin}(S) &= \text{Lin}(S')
\end{aligned}$$

where we have let $\beta_i = \gamma_i + \gamma_{m+1}\alpha_i$ for $1 \leq i \leq m$ (as required).[8]

(c) Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a basis of the finite dimensional space V and let $S' \subseteq V$ be any set containing n vectors. If $n < m$, then the vectors in S' cannot span V.

Proof: We take the set S defined in the question to be a basis of the finite dimensional space V[9] Also, let S' be *any* subset of V containing $n < m$ vectors, say $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$. We want to show that S' can *not* span V.

To do this we assume, for contradiction, that S' *does* span V. If this is the case, then by Definition 2.2, every vector in V can be written as a linear combination of the vectors in S', and in particular, so can the vectors in S. Thus, we can write

$$\begin{array}{ccccccccc}
\mathbf{u}_1 & = & a_{11}\mathbf{v}_1 & + & a_{12}\mathbf{v}_2 & + & \cdots & + & a_{1n}\mathbf{v}_n \\
\mathbf{u}_2 & = & a_{21}\mathbf{v}_1 & + & a_{22}\mathbf{v}_2 & + & \cdots & + & a_{2n}\mathbf{v}_n \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\mathbf{u}_m & = & a_{m1}\mathbf{v}_1 & + & a_{m2}\mathbf{v}_2 & + & \cdots & + & a_{mn}\mathbf{v}_n
\end{array}$$

Now, consider the vector equation

$$\gamma_1 \mathbf{u}_1 + \gamma_2 \mathbf{u}_2 + \cdots + \gamma_m \mathbf{u}_m = \mathbf{0}, \tag{1}$$

which, on substituting in the expressions for $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ above, gives

$$\gamma_1(a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + \cdots + a_{1n}\mathbf{v}_n) + \cdots + \gamma_m(a_{m1}\mathbf{v}_1 + a_{m2}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_n) = \mathbf{0},$$

and re-arranging we get

$$(\gamma_1 a_{11} + \gamma_2 a_{21} + \cdots + \gamma_m a_{m1})\mathbf{v}_1 + \cdots + (\gamma_1 a_{1n} + \gamma_2 a_{2n} + \cdots + \gamma_m a_{mn})\mathbf{v}_n = \mathbf{0}.$$

But, whatever the coefficients of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in this equation may actually be, we notice that if they are all equal to zero, then they satisfy the equation. That is, the set of simultaneous

[8] Notice that this result is related to Theorem 2.9: We have just shown that if we *remove* a linearly dependent vector from a set of vectors, the set will still span the same space. However, Theorem 2.9 tells us that if we *add* a linearly independent vector to a linearly independent set of vectors, the set will still be linearly independent.

[9] Notice that, by Definition 2.13, this means that V is an m-dimensional vector space.

equations

$$\begin{array}{ccccccccc} a_{11}\gamma_1 & + & a_{21}\gamma_2 & + & \cdots & + & a_{m1}\gamma_m & = & 0 \\ a_{12}\gamma_1 & + & a_{22}\gamma_2 & + & \cdots & + & a_{m2}\gamma_m & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{1n}\gamma_1 & + & a_{2n}\gamma_2 & + & \cdots & + & a_{mn}\gamma_m & = & 0 \end{array}$$

will *always* be a solution to the vector equation in question. However, since $n < m$, we have more unknowns than equations[10] and so there will be $m - n > 0$ free variables which, in general, will be non-zero. Consequently, there will be non-trivial solutions to Equation 1 and hence the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly dependent. But, this is contrary to the fact that these vectors form a basis and must therefore be linearly independent. Thus, by contradiction, S can not span V (as required).[11]

(d) If S is a set of n vectors that spans an n-dimensional vector space V, then S is a basis for V.

Proof: As S is a set of n vectors that spans an n-dimensional vector space V, to show that S is a basis for V, we only need to establish that it is a linearly independent set.

> To do this, we assume, for contradiction, that S is linearly dependent. That is, there is a vector in S that can be written as a linear combination of the other vectors in S. So, by **(b)**, we can remove this vector from S to get a set S' which contains $n-1$ vectors but still spans V. But, this is contrary to **(c)** which tells us that a set containing $n - 1 < n$ vectors cannot span an n-dimensional space such as V. Thus, by contradiction, S must be linearly independent.

Consequently, as S also spans V, it is a basis for V (as required).[12]

Other problems

The Other Problems on this sheet were intended to give you some further insight into what kinds of sets form vector spaces.

5. The set $S_{1,1,1} = \{[x, y, z]^t \in \mathbb{R}^3 \mid x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$. To see why, refer to the first Example of Section 1.4.2 where I proved that all sets of the form $S_{a,b,c} = \{[x, y, z]^t \in \mathbb{R}^3 \mid ax + by + cz = 0\}$ are subspaces of $\mathbb{R}^3$. Geometrically, $S_{1,1,1}$ is the set of all points on the plane with Cartesian equation $x + y + z = 0$. (Note that this plane clearly goes through the origin.)

The set $S_{1,1,1,1} = \{[x, y, z]^t \in \mathbb{R}^3 \mid x + y + z = 1\}$ is not a subspace of $\mathbb{R}^3$. To see why, refer to either the second or third Examples of Section 1.4.2 where I proved that all sets of the form $S_{a,b,c,r} = \{[x, y, z]^t \in \mathbb{R}^3 \mid ax + by + cz = r \neq 0\}$ are not subspaces of $\mathbb{R}^3$. Geometrically, $S_{1,1,1,1}$ is the set of all points on the plane with Cartesian equation $x + y + z = 1$. (Note that this plane clearly does not go through the origin, and so the set in question does not contain the null vector. Consequently, it should be obvious that this subset of $\mathbb{R}^3$ is not a subspace.)

6. The set $S_1 = \{[x, 0, 0]^t \in \mathbb{R}^3 \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. To see this, we note that for two general vectors $\mathbf{x} = [x, 0, 0]^t$ and $\mathbf{y} = [y, 0, 0]^t$ and a general scalar α (where, obviously, $x, y, \alpha \in \mathbb{R}$) we have:

- It is closed under vector addition as

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x + y \\ 0 \\ 0 \end{bmatrix},$$

[10] Notice that we have n simultaneous equations relating the m variables $\gamma_1, \gamma_2, \ldots, \gamma_m$.

[11] Notice that this result is related to Theorem 2.14: We have just shown that if we have a set containing *less* vectors than a basis, then the set cannot span the space. However, Theorem 2.14 tells us that if we have a set containing *more* vectors than a basis, then the set cannot be linearly independent.

[12] Notice that this result is related to Theorem 2.17: We have just shown that a set of n vectors that spans an n-dimensional space is a basis for that space, whereas Theorem 2.17 tells us that a set of n linearly independent vectors in an n-dimensional space is a basis for that space. Consequently, a set of n vectors in an n-dimensional space V only needs to span V *or* be linearly independent to be a basis for V.

and so $\mathbf{x}+\mathbf{y} \in S_1$ as $x+y \in \mathbb{R}$.

- It is closed under scalar multiplication as

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x \\ 0 \\ 0 \end{bmatrix},$$

  and so $\alpha\mathbf{x} \in S_1$ as $\alpha\mathbf{x} \in \mathbb{R}$.

and so it is a subspace of $\mathbb{R}^3$ by Theorem 1.4. Geometrically, this set corresponds to the line which is the *intersection* of the *planes* $y=0$ *and* $z=0$, i.e. the x-axis. (Of course, we can represent this line as the intersection of many different pairs of planes,[13] but I have just chosen the 'simplest' pair.)

The set $S_2 = \{[x,y,z]^t \in \mathbb{R}^3 \mid xz = 0\}$ is not a subspace of $\mathbb{R}^3$. To see this note that the vectors $[1,0,0]^t$ and $[0,0,1]^t$ are in S_2, but their vector sum, namely

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

is not in S_2 (as $1 \times 1 = 1 \neq 0$). Consequently, this set is not closed under vector addition and is therefore not a subspace. Geometrically, as the condition $xz=0$ means that $x=0$ *or* $z=0$, this set corresponds to the *union* of the yz and xy-planes. (Notice that, as in Problem 2, when trying to find a counter-example in the 'union of two subspaces' case, you have to take a vector from each of the subspaces.)

7. We consider four sets which are defined in terms of a vector $\mathbf{a}$ of the form $[a^2,b^2,c^2]^t$. In the vector spaces $\mathbb{R}^3$ and $\mathbb{C}^3$, the role of the additive identity is played by the null vector $[0,0,0]^t$. So, for the subsets in question to be vector spaces, Definition 1.1(A0) requires that they contain this vector, which rules out S_2 and S_4 straightaway as they require that $a,b,c \neq 0$.

Also, the vector $[1,1,1]^t$ is contained in all four sets, and Definition 1.1(AI) requires that a vector space contains the additive inverse of such vectors, namely $[-1,-1,-1]^t$. So this condition rules out S_1 and S_2 as $a,b,c \in \mathbb{R}$ means $a^2,b^2,c^2 \geq 0$.

Consequently, S_1, S_2 and S_4 cannot be vector spaces. Let us now establish that the remaining subset, namely S_3, is indeed a vector space. To do this, as it will be a subspace of $\mathbb{C}^3$, we only need to show that it satisfies Theorem 1.4. So, for two general vectors $\mathbf{x} = [a_1^2, b_1^2, c_1^2]^t$ and $\mathbf{y} = [a_2^2, b_2^2, c_2^2]^t$ and a general scalar γ (where, obviously, $a_1,b_1,c_1,a_2,b_2,c_2,\gamma \in \mathbb{C}$) we have:

- It is closed under vector addition as

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} a_1^2 \\ b_1^2 \\ c_1^2 \end{bmatrix} + \begin{bmatrix} a_2^2 \\ b_2^2 \\ c_2^2 \end{bmatrix} = \begin{bmatrix} a_1^2+a_2^2 \\ b_1^2+b_2^2 \\ c_1^2+c_2^2 \end{bmatrix},$$

  and so $\mathbf{x}+\mathbf{y} \in S_3$ as $a_1^2+a_2^2, b_1^2+b_2^2, c_1^2+c_2^2 \in \mathbb{C}$.

- It is closed under scalar multiplication as

$$\gamma\mathbf{x} = \gamma \begin{bmatrix} a_1^2 \\ b_1^2 \\ c_1^2 \end{bmatrix} = \begin{bmatrix} \gamma a_1^2 \\ \gamma b_1^2 \\ \gamma c_1^2 \end{bmatrix},$$

  and so $\gamma\mathbf{x} \in S_3$ as $\gamma a_1^2, \gamma b_1^2, \gamma c_1^2 \in \mathbb{C}$.

Thus, S_3 is a subspace of $\mathbb{C}^3$.

[13] For example, the x-axis is also the intersection of the planes $y+z=0$ and $y-z=0$.

8. Using Definition 2.3, it should be clear that if the three given linear spans are subsets of some vector space V,

$$\text{Lin}\{\mathbf{1}\} = \{\alpha \cdot \mathbf{1} \mid \text{for all scalars } \alpha\},\ \text{Lin}\{\mathbf{0}\} = \{\mathbf{0}\} \quad \text{and} \quad \text{Lin}\,\varnothing = \varnothing,$$

Now, $\text{Lin}\{\mathbf{1}\}$ is a vector space by Theorem 2.4, and looking at Section 1.4.1 we can see that $\text{Lin}\{\mathbf{0}\} = \{\mathbf{0}\}$ is the [trivial] vector space, whereas $\text{Lin}\,\varnothing = \varnothing$ is not a vector space.

Only vector spaces can have bases, and a basis for $\text{Lin}\{\mathbf{1}\}$ would be $\{\mathbf{1}\}$, which means that it is a one-dimensional vector space. However, as $\text{Lin}\{\mathbf{0}\} = \{\mathbf{0}\}$, we cannot find a basis for this vector space, although by Definition 2.16, we can *stipulate* that it is a zero-dimensional vector space.[14]

Harder problems

The Harder Problems on this sheet were intended to give you some further practice in proving results about vector spaces.

9. We are asked to prove parts (2) and (3) of Theorem 1.2. We shall do them in turn, firstly:

> **Theorem 1.2 (2):** The additive inverse of a vector $\mathbf{u} \in V$, namely $-\mathbf{u}$, is such that $(-1) \cdot \mathbf{u} = -\mathbf{u}$.
>
> **Proof:** To show that $(-1) \cdot \mathbf{u} = -\mathbf{u}$ we demonstrate that $\mathbf{u} + (-1) \cdot \mathbf{u} = \mathbf{0}$ (i.e. (AI) is satisfied). This can be done by noting that:
>
> $$\begin{aligned} \mathbf{u} + (-1) \cdot \mathbf{u} &= 1 \cdot \mathbf{u} + (-1) \cdot \mathbf{u} && \text{:by (M1)} \\ &= (1 + (-1)) \cdot \mathbf{u} && \text{:by (MD2)} \\ &= 0 \cdot \mathbf{u} && \text{:as } 1 + (-1) = 0 \\ \therefore\ \mathbf{u} + (-1) \cdot \mathbf{u} &= \mathbf{0} && \text{:by Theorem 1.2 (1)} \end{aligned}$$
>
> as required.

and secondly:

> **Theorem 1.2 (3):** If $\alpha \cdot \mathbf{u} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.
>
> **Proof:** We are given that $\alpha \cdot \mathbf{u} = \mathbf{0}$ and we need to establish that this implies that $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$. To do this, we note that there are two possible cases, namely:
>
> - If $\alpha = 0$, then we have $\alpha \cdot \mathbf{u} = 0 \cdot \mathbf{u} = \mathbf{0}$ by Theorem 1.2 (1).
> - If $\alpha \neq 0$, then we can write
>
> $$\begin{aligned} \alpha^{-1} \cdot (\alpha \cdot \mathbf{u}) &= (\alpha^{-1}\alpha) \cdot \mathbf{u} && \text{:by (MA)} \\ &= 1 \cdot \mathbf{u} && \text{:as } \alpha^{-1}\alpha = 1 \\ \therefore\ \alpha^{-1} \cdot (\alpha \cdot \mathbf{u}) &= \mathbf{u} && \text{:by (M1)} \end{aligned}$$
>
>   However, we also have
>
> $$\begin{aligned} \alpha^{-1} \cdot (\alpha \cdot \mathbf{u}) &= \alpha^{-1} \cdot \mathbf{0} && \text{:as } \alpha \cdot \mathbf{u} = \mathbf{0} \\ \therefore\ \alpha^{-1} \cdot (\alpha \cdot \mathbf{u}) &= \mathbf{0} && \text{:by Theorem 1.2 (1)} \end{aligned}$$
>
>   Thus, equating these two expressions, we get $\mathbf{u} = \mathbf{0}$.
>
> Consequently, if $\alpha \cdot \mathbf{u} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$ (as required).

[14] To see why this is so, look at the discussion which follows Definition 2.16.

where these results hold for any vector $\mathbf{u}$ in the vector space V and all scalars, α.

10. We are asked to show that the set $S_{a,b,c,r} = \{\mathbf{x} = [x_1, x_2, x_3]^t \in \mathbb{R}^3 \mid ax_1 + bx_2 + cx_3 = r\}$ where $a, r \neq 0$, $b = 0$ and c can take any value, is *not* a subspace of $\mathbb{R}^3$ by finding a counter-example. This can be done in one of two ways:[15]

- We can add the vector $[r/a, 0, 0]^t \in S_{a,b,c,r}$ to itself to get the vector $[2r/a, 0, 0]^t$ which is not in $S_{a,b,c,r}$ (since $2r + 0 + 0 = 2r \neq r$) and as such $S_{a,b,c,r}$ is not closed under vector addition.
- We can multiply the vector $[r/a, 0, 0]^t$ by any scalar α (where $\alpha \neq 1$) to get the vector $[\alpha r/a, 0, 0]^t$ which is not in $S_{a,b,c,r}$ (since $\alpha r + 0 + 0 = \alpha r \neq r$) and as such $S_{a,b,c,r}$ is not closed under scalar multiplication

Thus, by [the contrapositive of] Theorem 1.4, $S_{a,b,c,r}$ is not a subspace of $\mathbb{R}^3$.

11. We are asked to show that the vector space $\{\mathbf{0}\}$ is the only vector space containing just one vector. Let us consider the case where $\{\mathbf{0}\}$ is a subspace of a larger vector space V. Clearly any *other* set containing just one vector, say $\mathbf{x} \neq \mathbf{0}$, will be $\{\mathbf{x}\} \subseteq V$, and this cannot be a vector space as it doesn't contain an additive identity as required by Definition 1.1(A0). Further, the vector space $\{\mathbf{0}\}$ is itself unique as, by Theorem 1.2(1), the additive identity $\mathbf{0} \in V$ is unique.

12. We are asked to prove the following theorem: If V is a subspace of a finite dimensional vector space W, then V is finite dimensional. Further, $\dim(V) \leq \dim(W)$, and in particular, $\dim(V) = \dim(W)$ iff $V = W$.

Proof: Let V be a subspace of the finite dimensional vector space W. To prove that V must be finite dimensional too we perform the following construction (note that $j \geq 1$):

- **Step** 0**:** If $V = \{\mathbf{0}\}$, then V is finite dimensional and we have finished. If $V \neq \{\mathbf{0}\}$, then we choose a vector $\mathbf{v}_1 \in V$.
- **Step** j**:** If $V = \text{Lin}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$, then V is finite dimensional and we are finished. If $V \neq \text{Lin}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$, then choose a vector $\mathbf{v}_{j+1} \in V$ such that $\mathbf{v}_{j+1} \notin \text{Lin}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$.

After each step, as long as the process continues, we have constructed a set of vectors where no vector in the set is in the linear span of the other vectors. Thus, after each step we have constructed a linearly independent set of vectors by Theorem 2.9. Indeed, this linearly independent set of vectors is contained within W (as $V \subseteq W$) and so, by [the contrapositive of] Theorem 2.14, it cannot contain more vectors than a basis of W. Thus, as W is finite dimensional the number of vectors that this set contains must be finite too (i.e. the construction above must terminate) and consequently, V is finite dimensional (as required).

Now, if we take a basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq V \subseteq W$ of V, then $\dim(V) = m$ and either S is also a basis for W or it is not. That is, either

- S is a basis for W and $\dim(V) = \dim(W)$, or
- S is not a basis for W but, by Theorem 2.9, we can add vectors to the linearly independent set S to make it a basis for W, i.e. $\dim(V) < \dim(W)$.

Consequently, it must be the case that $\dim(V) \leq \dim(W)$. In particular, to establish that $\dim(V) = \dim(W)$ iff $V = W$, we have:

- **LTR:** If $\dim(V) = \dim(W)$, then S (a basis for V) is a set of m linearly independent vectors in an m-dimensional space W, and so, by Theorem 2.17, S is a basis for W too. Consequently, as the vectors in S span both V and W, we have $V = \text{Lin}(S) = W$.
- **RTL:** If $V = W$, then obviously $\dim(V) = \dim(W)$.

as required.

[15] There is, of course, no need to do both.