# Further Mathematical Methods (Linear Algebra) 2002

# Solutions For Problem Sheet 5

In this Problem Sheet we looked at some problems involving systems of differential equations. We also started thinking about how matrices with complex eigenvalues could be manipulated and how we could calculate functions of matrices other than integer powers and inverses.

**1.** As we are told that the vectors $[1,-1,1]^t$, $[-3,0,1]^t$ and $[-1,1,0]^t$ are eigenvectors of the matrix $\mathsf{A}$, to show that there is an invertible matrix $\mathsf{P}$ such that $\mathsf{P}^{-1}\mathsf{AP}$ is diagonal, it is sufficient to establish that the matrix

$$\mathsf{P} = \begin{bmatrix} 1 & -3 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

which has column vectors given by the eigenvectors of $\mathsf{A}$, is invertible. So, evaluating the determinant of this matrix by performing a co-factor expansion along the second row we find that

$$\det(\mathsf{P}) = 1(0+1) - 1(1+3) = -3,$$

and so the matrix $\mathsf{P}$ is invertible (as required).

Indeed, in the next part of the question we will need to make use of the diagonal matrix $\mathsf{D}$ which is given by $\mathsf{P}^{-1}\mathsf{AP}$. The easiest way to find this matrix is to evaluate $\mathsf{AP}$ and use this to infer what $\mathsf{D}$ is from the fact that $\mathsf{AP} = \mathsf{PD}$. That is, calculating

$$\mathsf{AP} = \begin{bmatrix} 1 & -2 & -6 \\ 2 & 5 & 6 \\ -2 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -3 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -9 & -3 \\ 3 & 0 & 3 \\ -3 & 3 & 0 \end{bmatrix},$$

we see that this new matrix is just the matrix $\mathsf{P}$ with its first, second and third columns multiplied by the numbers $-3$, $3$ and $3$ respectively. Therefore, as $\mathsf{AP} = \mathsf{PD}$, the required diagonal matrix is

$$\mathsf{D} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

where, of course, the diagonal entries are the eigenvalues to which the given eigenvectors of $\mathsf{A}$ correspond.

We are now asked to look at a system of coupled linear differential equations relating the functions $y_1(t)$, $y_2(t)$ and $y_3(t)$. As the system in question is linear, we follow the method given in the handout for Lectures 9 and 10 and write it as a matrix equation, i.e.

$$\dot{\mathbf{y}} = \mathsf{A}\mathbf{y},$$

where $\mathbf{y}(t) = [y_1(t), y_2(t), y_3(t)]^t$ and the matrix $\mathsf{A}$ from the previous part [conveniently] corresponds to the matrix of coefficients for this system. So, using the earlier analysis there is an invertible matrix $\mathsf{P}$ such that $\mathsf{P}^{-1}\mathsf{AP} = \mathsf{D}$ and so we can write

$$\dot{\mathbf{y}} = \mathsf{PDP}^{-1}\mathbf{y} \implies (\mathsf{P}^{-1}\dot{\mathbf{y}}) = \mathsf{D}(\mathsf{P}^{-1}\mathbf{y}),$$

which on making the substitution $\mathbf{z} = \mathsf{P}^{-1}\mathbf{y}$ yields the uncoupled system of linear differential equations

$$\dot{\mathbf{z}} = \mathsf{D}\mathbf{z},$$

and this can easily be solved (see the handout for Lectures 9 and 10) to yield

$$\mathbf{z}(t) = \begin{bmatrix} Ae^{-3t} \\ Be^{3t} \\ Ce^{3t} \end{bmatrix},$$

where $A$, $B$ and $C$ are arbitrary constants. Consequently, using the fact that $\mathbf{y} = \mathsf{P}\mathbf{z}$, we find that

$$\mathbf{y}(t) = \mathsf{P}\mathbf{z}(t) = \begin{bmatrix} Ae^{-3t} - (3B+C)e^{3t} \\ -Ae^{-3t} + Ce^{3t} \\ Ae^{-3t} + Be^{3t} \end{bmatrix},$$

is the *general* solution to this system of coupled linear differential equations.

Lastly, to find the *particular* solution associated with the initial conditions $y_1(0) = y_2(0) = 1$ and $y_3(0) = 0$, we have to find the appropriate values of $A$, $B$ and $C$. So, noting that in this case,

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

and setting $t = 0$ in the general solution, we get a set of three simultaneous equations, i.e.

$$\begin{aligned} A - 3B - C &= 1 \\ -A + C &= 1 \\ A + B &= 0 \end{aligned}$$

Solving these we find that $A = \frac{2}{3}$, $B = -\frac{2}{3}$ and $C = \frac{5}{3}$, which on substitution into our expression for $\mathbf{y}(t)$ and simplifying yields

$$\mathbf{y}(t) = \frac{1}{3}\begin{bmatrix} 2e^{-3t} + e^{3t} \\ -2e^{-3t} + 5e^{3t} \\ 2e^{-3t} - 2e^{3t} \end{bmatrix},$$

which is the sought after particular solution.

**2.** The steady states of the system of differential equations represented by $\dot{\mathbf{y}} = F(\mathbf{y})$ are the solutions to the equation $\dot{\mathbf{y}} = \mathbf{0}$. Thus, solving the pair of quadratic simultaneous equations given by $F(\mathbf{y}) = \mathbf{0}$, i.e.

$$\begin{aligned} y_1(2 - 2y_1 - y_2) &= 0 \\ y_2(2 - 2y_2 - y_1) &= 0 \end{aligned}$$

you should have found *four* steady states $\mathbf{y}^* = [y_1^*, y_2^*]^t$, namely $[0,0]^t$, $[0,1]^t$, $[1,0]^t$ and $\left[\frac{2}{3}, \frac{2}{3}\right]^t$.

The question then asks you to show that the steady state $\left[\frac{2}{3}, \frac{2}{3}\right]^t$ is asymptotically stable. To do this, we use Theorem 9.7 from the handout for Lectures 9 and 10 and calculate the Jacobian of $F(\mathbf{y})$ so, bearing in mind that $F(\mathbf{y}) = [f_1, f_2]^t$, this is given by

$$DF(\mathbf{y}) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 2 - 4y_1 - y_2 & -y_1 \\ -y_2 & 2 - 4y_2 - y_1 \end{bmatrix},$$

which at the steady state $\mathbf{y}^* = \left[\frac{2}{3}, \frac{2}{3}\right]^t$, is

$$\mathsf{DF}\left(\left[\tfrac{2}{3}, \tfrac{2}{3}\right]^t\right) = \begin{bmatrix} -\frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3} \end{bmatrix}.$$

It should now be straightforward to show that the eigenvalues of this matrix are $-2$ and $-2/3$, and because these are both real and negative, Theorem 9.7 tells us that the steady state $\mathbf{y}^* = \left[\frac{2}{3}, \frac{2}{3}\right]^t$ is asymptotically stable.

**3.** In the handout for Lectures 9 and 10, we found that $\mathbf{y}^* = [4, 0]^t$ is a steady state of the system of non-linear differential equations given by

$$\begin{aligned} \dot{y}_1 &= 4y_1 - y_1^2 - y_1 y_2 \\ \dot{y}_2 &= 6y_2 - y_2^2 - 3y_1 y_2 \end{aligned}$$

and we also found that the Jacobian of this system was given by

$$DF(\mathbf{y}) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 4 - 2y_1 - y_2 & -y_1 \\ -3y_2 & 6 - 2y_2 - 3y_1 \end{bmatrix}.$$

Evaluating this Jacobian at the given steady state we get

$$\mathsf{DF}(\mathbf{y}^*) = \begin{bmatrix} -4 & -4 \\ 0 & -6 \end{bmatrix},$$

and you should be able to show that the eigenvalues of this matrix are $-4$ and $-6$. Consequently, by Theorem 9.7, we can deduce that the steady state $\mathbf{y}^* = [4, 0]^t$ is asymptotically stable as these eigenvalues are both negative real numbers.

**4.** Suppose that $\mathsf{A}$ is a real diagonalisable matrix whose eigenvalues are all non-negative, we need to prove that there is a matrix $\mathsf{B}$ such that $\mathsf{B}^2 = \mathsf{A}$. To do this, we use the fact that $\mathsf{A}$ is diagonalisable, i.e. there exists a matrix $\mathsf{P}$ such that $\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \mathsf{D}$ is diagonal, and on re-arranging, this gives $\mathsf{A} = \mathsf{P}\mathsf{D}\mathsf{P}^{-1}$. Now, the matrix $\mathsf{D}$ has entries given by the eigenvalues of $\mathsf{A}$, i.e. $\mathsf{D} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and so we consider the matrix

$$\mathsf{B} = \mathsf{P}\,\text{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\right)\mathsf{P}^{-1},$$

where the square roots are unproblematic because the $\lambda_i \geq 0$ for $i = 1, 2, \ldots, n$. Thus, as

$$\begin{aligned}
\mathsf{B}^2 &= \mathsf{P}\,\text{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\right)\mathsf{P}^{-1}\mathsf{P}\,\text{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\right)\mathsf{P}^{-1} \\
&= \mathsf{P}\,\text{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\right)\text{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\right)\mathsf{P}^{-1} \\
&= \mathsf{P}\,\text{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_n\right)\mathsf{P}^{-1} \\
&= \mathsf{P}\mathsf{D}\mathsf{P}^{-1} \\
\implies \mathsf{B}^2 &= \mathsf{A}
\end{aligned}$$

we can see that our matrix $\mathsf{B}$ is such that $\mathsf{B}^2 = \mathsf{A}$ (as required).[1]

We are now asked whether the matrix $\mathsf{B}$ is unique, i.e. is the matrix $\mathsf{B}$ given above the *only* matrix such that $\mathsf{B}^2 = \mathsf{A}$? The answer is, of course, no it's not. To see why, note that each of the [non-zero] eigenvalues can give rise to two square roots, and so the diagonal matrix which we use to construct $\mathsf{B}$ could be any one of the diagonal matrices which we can find using

$$\text{diag}\left(\pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}, \ldots, \pm\sqrt{\lambda_n}\right),$$

where we choose either the '+' or the '−' for each entry. That is, for an $n \times n$ matrix $\mathsf{A}$ we could construct $2^n$ different matrices $\mathsf{B}$ such that $\mathsf{B}^2 = \mathsf{A}$.[2]

**5.** We are asked to show that the eigenvalues of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 1 \\ -9 & 1 \end{bmatrix},$$

[1] Some of you may have claimed that a matrix $\mathsf{B}$ such that $\mathsf{B}^2 = \mathsf{A}$ can be formed from the spectral decomposition of $\mathsf{A}$, i.e.

$$\mathsf{B} = \sqrt{\lambda_1}\mathsf{E}_1 + \sqrt{\lambda_2}\mathsf{E}_2 + \cdots + \sqrt{\lambda_n}\mathsf{E}_n,$$

where $\mathsf{E}_i = \mathbf{x}_i\mathbf{x}_i^t$ and $\mathbf{x}_i$ is an eigenvector of $\mathsf{A}$ corresponding to the eigenvalue $\lambda_i$ for $i = 1, 2, \ldots, n$. (You should check that this expression for $\mathsf{B}$ will give $\mathsf{A}$ when squared.) However, recall that the spectral decomposition only exists when the matrix $\mathsf{A}$ is unitarily diagonalisable (i.e. when $\mathsf{A}$ is a normal matrix) and not all diagonalisable matrices are *unitarily* diagonalisable. Consequently, this line of reasoning is not general enough to answer the question.

[2] Provided, of course, all of the eigenvalues are non-zero. If $m$ of the $n$ eigenvalues are zero, then we can construct $2^{n-m}$ different matrices $\mathsf{B}$ such that $\mathsf{B}^2 = \mathsf{A}$. (In particular, this means that if $\mathsf{A}$ was an $n \times n$ matrix with $n$ eigenvalues which were all zero, then we could find only one (i.e. a unique) $\mathsf{B}$ such that $\mathsf{B}^2 = \mathsf{A}$. But, there is only one diagonalisable matrix $\mathsf{A}$ which can have $n$ eigenvalues which were all zero, i.e. $\mathsf{A} = \mathsf{0}$, and in this case the unique $\mathsf{B}$ would be $\mathsf{0}$ too!)

are $1 \pm 3i$ and hence find an invertible matrix $\mathsf{P}$ such that the matrix $\mathsf{P}^{-1}\mathsf{AP} = \mathsf{D}$ is diagonal, i.e. we need to find the eigenvectors corresponding to these eigenvalues as well. There are two ways of doing this: the first is probably the most obvious, whereas the second is probably the easiest. Let us look at them both in action:

**Method 1:** The most obvious way of doing this is to solve the determinant equation

$$\det(\mathsf{A} - \lambda \mathsf{I}) = 0,$$

as we know that the solutions will be the eigenvalues. In this case, we get

$$\begin{vmatrix} 1-\lambda & 1 \\ -9 & 1-\lambda \end{vmatrix} = 0,$$

and expanding this out, we get

$$(1-\lambda)^2 + 9 = 0 \quad \Longrightarrow \quad 1 - \lambda = \pm 3i \quad \Longrightarrow \quad \lambda = 1 \pm 3i,$$

which are, as expected, the required eigenvalues. To find the eigenvectors, we note that they are the *non-zero* vectors $\mathbf{x}$ which are solutions to the matrix equation

$$(\mathsf{A} - \lambda \mathsf{I})\mathbf{x} = \mathbf{0}.$$

So, if $\lambda = 1 + 3i$, we want a non-zero vector $\mathbf{x} = [x, y]^t$ that satisfies

$$\begin{bmatrix} -3i & 1 \\ -9 & -3i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0},$$

and so we require $x$ and $y$ such that $3ix = y$, which means that (letting $x$ be a free parameter) the eigenvectors have the form $x[1, 3i]^t$. Similarly, if $\lambda = 1 - 3i$, we want a non-zero vector $\mathbf{x} = [x, y]^t$ that satisfies

$$\begin{bmatrix} 3i & 1 \\ -9 & 3i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0},$$

and so we require $x$ and $y$ such that $-3ix = y$, which means that (again, letting $x$ be a free parameter) the eigenvectors have the form $x[1, -3i]^t$. Consequently, setting $x = 1$ in both of these expressions, we find that $[1, 3i]^t$ and $[1, -3i]^t$ are eigenvectors corresponding to the eigenvalues $1 + 3i$ and $1 - 3i$ respectively.

**Method 2:** However, the easiest way to do this is to note that the equation $\det(\mathsf{A} - \lambda \mathsf{I}) = 0$ gives us the eigenvalues as they are the numbers which make the columns of the matrix $\mathsf{A} - \lambda \mathsf{I}$ linearly dependent.[3] So, to verify that $1 \pm 3i$ are eigenvalues of the matrix $\mathsf{A}$ we just note that the matrix

$$\mathsf{A} - (1 \pm 3i)\mathsf{I} = \begin{bmatrix} 1 - (1 \pm 3i) & 1 \\ -9 & 1 - (1 \pm 3i) \end{bmatrix} = \begin{bmatrix} \mp 3i & 1 \\ -9 & \mp 3i \end{bmatrix},$$

has linearly dependent columns.[4] Further, this method also gives us an easy way of calculating the eigenvectors because an eigenvector corresponding to a given eigenvalue $\lambda$ is just any vector of coefficients which allows us to express the null vector in terms of

[3] Recall that this was how we motivated the use of the determinant equation in the lectures!

[4] That is, if we take $1 + 3i$, the column vectors of the matrix $\mathsf{A} - (1 + 3i)\mathsf{I}$ are $[-3i, -9]^t$ and $[1, -3i]^t$ which are clearly linearly dependent, whereas if we take $1 - 3i$, the column vectors of the matrix $\mathsf{A} - (1 - 3i)\mathsf{I}$ are $[3i, -9]^t$ and $[1, 3i]^t$ which are also clearly linearly dependent.

the column vectors of $\mathsf{A} - \lambda\mathsf{I}$.[5] So, noting that the columns of $\mathsf{A} - (1+3i)\mathsf{I}$ are linearly dependent as the second is $-3i$ times the first, we have

$$\begin{bmatrix} -3i \\ -9 \end{bmatrix} + 3i \begin{bmatrix} 1 \\ -3i \end{bmatrix} = \mathbf{0},$$

and so $[1, 3i]^t$ is an eigenvector of $\mathsf{A}$ corresponding to the eigenvalue $1+3i$ (as before). Similarly, noting that the columns of $\mathsf{A} - (1-3i)\mathsf{I}$ are linearly dependent as the second is $3i$ times the first, we have

$$\begin{bmatrix} 3i \\ -9 \end{bmatrix} - 3i \begin{bmatrix} 1 \\ 3i \end{bmatrix} = \mathbf{0},$$

and so $[1, -3i]^t$ is an eigenvector of $\mathsf{A}$ corresponding to the eigenvalue $1-3i$ (as before).

Thus, with this information, we can find an invertible matrix $\mathsf{P}$ such that the matrix product $\mathsf{P}^{-1}\mathsf{AP}$ gives a diagonal matrix $\mathsf{D}$. For instance, the matrices

$$\mathsf{P} = \begin{bmatrix} 1 & 1 \\ 3i & -3i \end{bmatrix} \quad \text{and} \quad \mathsf{D} = \begin{bmatrix} 1+3i & 0 \\ 0 & 1-3i \end{bmatrix},$$

will do the job. To verify that $\mathsf{P}^{-1}\mathsf{AP} = \mathsf{D}$, you should check that $\mathsf{AP} = \mathsf{PD}$ and observe that $\mathsf{P}$ *is* invertible as it has linearly independent column vectors (i.e. $\det(\mathsf{P}) \neq 0$). If you really must calculate the inverse of $\mathsf{P}$, you should find that

$$\mathsf{P}^{-1} = \frac{1}{6i} \begin{bmatrix} 3i & 1 \\ 3i & -1 \end{bmatrix},$$

in this case.

### Other problems

Here are the solutions for the other problems. As these were not covered in class the solutions will be a bit more detailed. (Note that I use $\epsilon$'s in these solutions to help you understand the definitions given in the handout for Lectures 9 and 10 and as such, people who have done Real Analysis should not interpret them too literally.)

**6.** We are asked to consider the differential equation,

$$\dot{y} = y^2 + 3y - 10.$$

By Definition 9.1, the steady states occur when $\dot{y} = 0$, and so we must solve the quadratic equation

$$y^2 + 3y - 10 = 0,$$

to find them. But this is just

$$(y+5)(y-2) = 0,$$

and so the steady states are $y^* = -5$ and $y^* = 2$.

To solve this differential equation we use the 'separation of variables' technique, that is, we write

$$\int \frac{dy}{(y+5)(y-2)} = \int dt,$$

[5] Recall from the lectures that solving the matrix equation,

$$(\mathsf{A} - \lambda\mathsf{I})\mathbf{x} = \mathbf{0},$$

for the eigenvectors of $\mathsf{A}$ corresponding to the eigenvalue $\lambda$ is equivalent to writing $\mathbf{x}$ as $[x_1, x_2, \ldots, x_n]^t$ and finding the coefficients $x_1, x_2, \ldots, x_n$ which are solutions to the vector equation,

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{0},$$

where the vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ are the column vectors of the matrix $\mathsf{A} - \lambda\mathsf{I}$.

and note that the left-hand-side can be expressed in terms of partial fractions, i.e.

$$-\frac{1}{7}\int\left[\frac{1}{y+5}-\frac{1}{y-2}\right]dy=\int dt,$$

which in turn yields the solution

$$-\frac{1}{7}\left[\ln(y+5)-\ln(y-2)\right]=t+c,$$

where $c$ is an arbitrary constant. So, rearranging this gives

$$\frac{y+5}{y-2}=Ae^{-7t} \quad \text{or,} \quad y(t)=\frac{2Ae^{-7t}+5}{Ae^{-7t}-1},$$

where $A=e^{-7c}$.

Now, to gain our first insight into the stability of the steady states, we notice that for any [finite] value of the arbitrary constant $A$, $y\to-5$ as $t\to\infty$. This seems to suggest that the steady state $y^*=-5$ is asymptotically stable. Further, we notice that there are no [finite] values of $A$ for which the steady state $y^*=2$ is realised. This is because in the limit $t\to\infty$, in order to have $y\to2$, we require an $A$ such that

$$2=\frac{2A+5}{A-1},$$

which on rearranging yields the inconsistency $-2=5$! Thus, the steady state $y^*=2$ does not seem to be asymptotically stable.[6]

Now, to discuss the asymptotic stability of the steady states properly, we need to consider how these solutions depend on the initial condition $y(0)\equiv y_0$. Clearly, this is related to the arbitrary constant $A$ by the equation

$$y_0=\frac{2A+5}{A-1} \quad \text{or, on rearranging,} \quad A=\frac{y_0+5}{y_0-2}.$$

Thus, substituting for $A$ and simplifying, our solutions now take the form

$$y(t)=\frac{y_0(2e^{-7t}+5)+10(e^{-7t}-1)}{y_0(e^{-7t}-1)+(5e^{-7t}+2)}.$$

Firstly, we note that if we start in the steady states, then we stay in them for all time, i.e.

- If $y_0=2$, then $y(t)=2$.
- If $y_0=-5$, then $y(t)=-5$.

Secondly, we notice that as $t\to\infty$, these solutions take the form

$$y(t)\to\begin{cases}-5 & \text{if } y_0\neq2\\ 2 & \text{if } y_0=2\end{cases}$$

Thus, the steady state $y^*=-5$ is asymptotically stable as there is an $\epsilon>0$ such that if $|y(0)-y^*|<\epsilon$, then $y(t)\to y^*$ as $t\to\infty$. For instance, if we take any $\epsilon$ in the range $0<\epsilon\leq7$, say $\epsilon=7$, then $y(0)\equiv y_0$ will lie in the range $|y_0-y^*|<7$, or $-7+y^*<y_0<7+y^*$. So, as we are considering the case where $y^*=-5$, we have $-12<y_0<2$ and we know that for these values of $y_0$, $y(t)\to-5$ as

[6] However, notice that if we remove the restriction that $A$ is finite, this equation can be written as

$$2=\frac{2+(5/A)}{1-(1/A)},$$

which gives us a consistent equation if $A$ is 'infinite.' Thus, there are conditions under which the steady state $y^*=2$ is realised. Of course, this must be the case, as systems which are started in a steady state remain in them for all time, i.e. if we pick the initial condition to be $y(0)=2$, then this will be a situation where the steady state $y^*=2$ is realised. (Further, notice that $y(0)=2$ implies that $A$ will be 'infinite' as expected!) This is discussed in more detail in the next paragraph.

$t \to \infty$. But, the steady state $y^* = 2$ is not asymptotically stable as there is no such $\epsilon > 0$ (i.e. we only tend towards this steady state when we have $y_0 = 2$ and this entails that $\epsilon = 0$!).[7]

Thirdly, if you are very keen, you can notice that the steady state $y^* = -5$ is *globally* asymptotically stable. This is because every[8] value of $y_0$ gives us a $y(t)$ that tends to $y^* = -5$ as $t \to \infty$.

**7.** We are asked to find the steady states of the system of non-linear differential equations given by

$$\dot{x} = 2xy - 2y^2$$
$$\dot{y} = x - y^2 + 2$$

As always, the steady states occur when $\dot{x}$ and $\dot{y}$ are simultaneously equal to zero, i.e. when

$$2xy - 2y^2 = 0$$
$$x - y^2 + 2 = 0$$

So, to satisfy the first equation we require that $y = 0$ or $x = y$. Substituting the former into the second equation gives $x + 2 = 0$ and so $[x^*, y^*]^t = [-2, 0]^t$ is a steady state. Similarly, substituting the latter into the second equation gives

$$y^2 - y - 2 = 0 \implies (y-2)(y+1) = 0 \implies y = 2 \text{ or } y = -1,$$

which as $x = y$, gives the steady states $[2, 2]^t$ and $[-1, -1]^t$. Consequently, this system has three steady states, namely $[-2, 0]^t$, $[2, 2]^t$ and $[-1, -1]^t$.

To examine the stability of these steady states, we need to look at the Jacobian of this system, i.e.

$$DF([x, y]^t) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2y & 2x - 4y \\ 1 & -2y \end{pmatrix},$$

where we have used the fact that

$$F([x, y]^t) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 2xy - 2y^2 \\ x - y^2 + 2 \end{pmatrix}.$$

So, taking the steady states in turn, we have:

For $[-2, 0]^t$: In this case, the Jacobian matrix is

$$\mathsf{DF}([-2, 0]^t) = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix},$$

and this has eigenvalues given by the determinant equation

$$\begin{vmatrix} -\lambda & -4 \\ 1 & -\lambda \end{vmatrix} = 0 \implies \lambda^2 + 4 = 0.$$

So, the eigenvalues are $\pm 2i$ and as these are not both real and negative, we cannot use Theorem 9.7 from the handout for Lectures 9 and 10 to assess their stability. However, we know from the lectures that these eigenvalues describe the evolution of the vector

[7] Incidentally, you may be wondering what Theorem 9.7 says about this case. Clearly, the Jacobian of the 'system' $\dot{y} = f(t) = y^2 + 3y - 10$ is given by

$$DF(y) = (\partial f / \partial y) = (2y + 3).$$

Now, when $y^* = -5$, we get the $1 \times 1$ matrix $\mathsf{DF}(-5) = [-7]$ and this has a negative real eigenvalue of $-7$ (Obviously!) which implies that this steady state is asymptotically stable. But, when $y^* = 2$, we get $\mathsf{DF}(2) = [7]$ and this matrix has a non-negative real eigenvalue of 7 (Again, this is obvious.) which implies that this steady state is not asymptotically stable.

[8] Except for a set of $y_0$ of lower dimension, i.e. the set containing $y_0 = 2$! This is clearly a set of lower dimension as it represents a point (dimension zero) and all the other values of $y_0$ form a set that represents a line (dimension one).

$\mathbf{h}(t) = \mathbf{y}(t) - \mathbf{y}^*$ and *qualitatively,*[9] this behaves in the same way as solutions to the *linear* system of differential equations given by

$$\dot{\mathbf{h}}(t) = \mathsf{DF}(\mathbf{y}^*)\mathbf{h}(t),$$

and we also know from the handout for Lectures 9 and 10 that *linear* systems of differential equations have solutions of the form

$$\mathbf{h}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2,$$

where, in this case, $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$ of $\mathsf{DF}(\mathbf{y}^*)$ and the coefficients $c_1$ and $c_2$ are constants. Thus, using this information, we can see that

$$\mathbf{h}(t) = \mathbf{y}(t) - \mathbf{y}^* = c_1 e^{2it}\mathbf{v}_1 + c_2 e^{-2it}\mathbf{v}_2,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues $2i$ and $-2i$ respectively. But, what does this mean? To answer this question, we calculate the eigenvectors of $\mathsf{DF}(\mathbf{y}^*)$ corresponding to the two eigenvalues and find that $\mathbf{v}_1 = [2i, 1]^t$ and $\mathbf{v}_2 = [-2i, 1]^t$, that is, we have

$$\mathbf{h}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} = c_1 e^{2it}\begin{bmatrix} 2i \\ 1 \end{bmatrix} + c_2 e^{-2it}\begin{bmatrix} -2i \\ 1 \end{bmatrix}.$$

However, the presence of complex quantities is distracting and it would be nice if we could somehow 'eliminate' them to get some idea of how the system is behaving. To do this, we use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$ to write our expression for $\mathbf{h}(t)$ as

$$\mathbf{h}(t) = \begin{bmatrix} 2i(c_1 - c_2)\cos(2t) - 2(c_1 + c_2)\sin(2t) \\ (c_1 + c_2)\cos(2t) + i(c_1 - c_2)\sin(2t) \end{bmatrix},$$

and then note that any set of initial conditions $\mathbf{y}(0)$ (sufficiently close to $\mathbf{y}^*$) that we might wish to consider can be translated into initial conditions for $\mathbf{h}(t)$ (where $\|\mathbf{h}(0)\|$ is correspondingly close to zero). So, taking an appropriate set of initial conditions for $\mathbf{h}(t)$, say $\mathbf{h}(0) = [h_1(0), h_2(0)]^t$, we have

$$\mathbf{h}(0) = \begin{bmatrix} h_1(0) \\ h_2(0) \end{bmatrix} = \begin{bmatrix} 2i(c_1 - c_2) \\ c_1 + c_2 \end{bmatrix},$$

and substituting these into our expression for $\mathbf{h}(t)$ we get

$$\mathbf{h}(t) = \begin{bmatrix} h_1(0)\cos(2t) - 2h_2(0)\sin(2t) \\ h_2(0)\cos(2t) + \frac{1}{2}h_1(0)\sin(2t) \end{bmatrix},$$

where all of the quantities are now real. Thus, re-writing our result gives

$$\mathbf{h}(t) = \mathbf{y}(t) - \mathbf{y}^* = \cos(2t)\begin{bmatrix} h_1(0) \\ h_2(0) \end{bmatrix} - \frac{\sin(2t)}{2}\begin{bmatrix} 4h_2(0) \\ -h_2(0) \end{bmatrix},$$

which is, believe it or not, the equation of an elliptical helix centred on the vertical line which passes through the point $\mathbf{h} = \mathbf{0}$ (or $\mathbf{y} = \mathbf{y}^*$ in '$\mathbf{y}$'-coordinates) if we choose our axes appropriately — see Figure 1. Now, *intuitively,* a steady state $\mathbf{y}^*$ is

- asymptotically stable if $\mathbf{y}(t) \to \mathbf{y}^*$ (i.e. $\mathbf{h}(t) \to 0$) for initial conditions suitably close to $\mathbf{y}^*$, and
- asymptotically unstable if $\mathbf{y}(t) \not\to \mathbf{y}^*$ (i.e. $\mathbf{h}(t) \not\to 0$) for initial conditions suitably close to $\mathbf{y}^*$,

[9] This means that for initial conditions that are suitably close to $\mathbf{y}^*$ (i.e. initial conditions such that $\|\mathbf{h}(0)\|$ is suitably close to zero) the solution that we get for $\mathbf{h}(t)$ should give us a good idea of how $\mathbf{y}(t)$ is behaving.

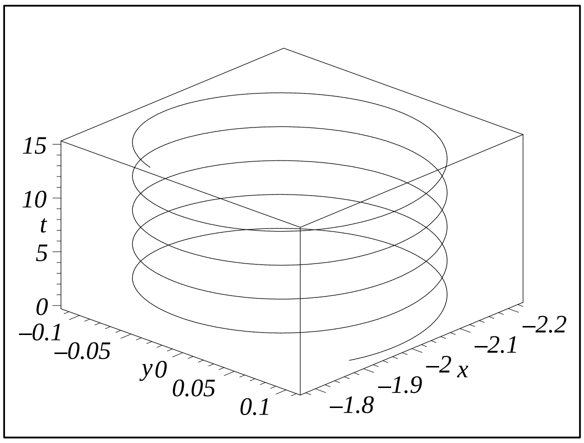



Figure 1: *An [exact] solution to the system of non-linear differential equations considered in Question 7. In this case, the initial conditions are close to the steady state* $\mathbf{y}^* = [x^*, y^*]^t = [-2, 0]^t$ *— specifically, the initial conditions for the solution indicated are* $\mathbf{y}(0) = [x(0), y(0)]^t = [-1.9, 0.1]^t$*. Notice that the curve is an elliptical helix which is centred on the steady state under consideration as expected from our qualitative analysis.*

where the latter usually indicates that we are going to get $\mathbf{y}(t)$ tending either to infinity, or some *other* steady state, as $t \to \infty$. In this case though, we have a solution that 'circles' around the steady state and this seems to fall somewhere between the two forms of asymptotic behaviour noted above. However, *formally*, a steady state $\mathbf{y}^*$ is asymptotically stable if there is some $\epsilon > 0$ such that $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$ implies $\mathbf{y}(t) \to \mathbf{y}^*$ as $t \to \infty$. So, formally, this steady state is *not* asymptotically stable as there is *no* $\epsilon > 0$ such that $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$ implies $\mathbf{y}(t) \to \mathbf{y}^*$ as $t \to \infty$.

<u>For $[2, 2]^t$</u>: In this case, the Jacobian matrix is

$$\mathsf{DF}([2,2]^t) = \begin{bmatrix} 4 & -4 \\ 1 & -4 \end{bmatrix},$$

and this has eigenvalues given by the determinant equation

$$\begin{vmatrix} 4-\lambda & -4 \\ 1 & -4-\lambda \end{vmatrix} = 0 \implies (4-\lambda)(-4-\lambda) + 4 = 0 \implies \lambda^2 - 12 = 0.$$

So, the eigenvalues are $\pm\sqrt{12}$ and as these are not both real and negative, we cannot use Theorem 9.7 from the handout for Lectures 9 and 10 to assess their stability. But, using a similar *qualitative*[10] analysis to the one above, we can see that in this situation we have

$$\mathbf{h}(t) = \mathbf{y}(t) - \mathbf{y}^* = c_1 e^{\sqrt{12}t}\mathbf{v}_1 + c_2 e^{-\sqrt{12}t}\mathbf{v}_2,$$

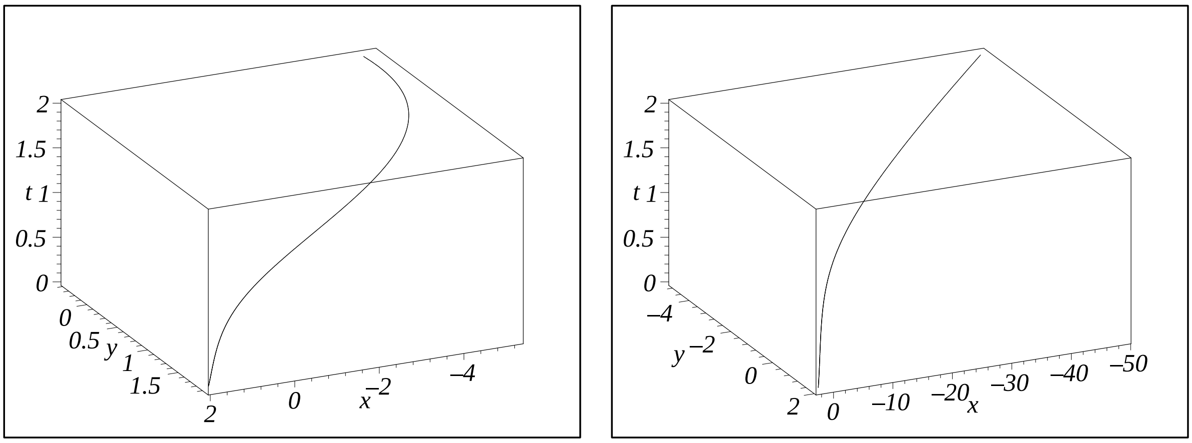
where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues $\sqrt{12}$ and $-\sqrt{12}$ of $\mathsf{DF}(\mathbf{y}^*)$ and the coefficients $c_1$ and $c_2$ are constants. Clearly, due to the increasing exponential in the first term, solutions which start close to this steady state will [almost always[11]] move away from it and so this steady state is asymptotically unstable — see Figure 2.

<u>For $[-1, -1]^t$</u>: In this case, the Jacobian matrix is

$$\mathsf{DF}([-1,-1]^t) = \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix},$$

[10]See Footnote 9.

[11]This caveat is due to the fact that some initial conditions will make $c_1 = 0$, and under such circumstances $\mathbf{y}(t)$ will tend to $\mathbf{y}^*$ (i.e. $\mathbf{h}(t)$ will tend to zero). However, this steady state is *not* asymptotically stable, since for any $\epsilon > 0$, there are initial conditions where $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$, but $\mathbf{y}(t) \not\to \mathbf{y}^*$ (due to the presence of the increasing exponential).




Exact expression     Approximate expression

Figure 2: *Solutions to the system of non-linear differential equations considered in Question 7 with initial conditions close to the steady state* $\mathbf{y}^* = [x^*, y^*]^t = [2, 2]^t$ *— specifically, the initial conditions for the solutions indicated are* $\mathbf{y}(0) = [x(0), y(0)]^t = [1.9, 1.9]^t$*. The exact solution was obtained by solving the non-linear equations (using Maple) and the approximate solution was obtained by solving the 'qualitatively similar' linear equations. Notice that the approximate solution becomes increasingly unreliable as* $t$ *increases.*

and this has eigenvalues given by the determinant equation

$$\begin{vmatrix} -2-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0 \implies (2-\lambda)(-2-\lambda) - 2 = 0 \implies \lambda^2 - 6 = 0.$$

So, the eigenvalues are $\pm\sqrt{6}$ and as these are not both real and negative, we cannot use Theorem 9.7 from the handout for Lectures 9 and 10 to assess their stability. But, again, using a similar *qualitative*[12] analysis to the one above, we can see that in this situation we have

$$\mathbf{h}(t) = \mathbf{y}(t) - \mathbf{y}^* = c_1 e^{\sqrt{6}t}\mathbf{v}_1 + c_2 e^{-\sqrt{6}t}\mathbf{v}_2,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues $\sqrt{6}$ and $-\sqrt{6}$ of $\mathsf{DF}(\mathbf{y}^*)$ and the coefficients $c_1$ and $c_2$ are constants. Clearly, due to the increasing exponential in the first term, solutions which start close to this steady state will [almost always[13]] move away from it and so this steady state is also asymptotically unstable — see Figure 3.

**8.** In Question 3, we considered the system of non-linear differential equations given in the handout for Lectures 9 and 10 and of the four steady states that we found for this system, we have analysed two. We now turn to the two steady states that are left over, namely $(0, 0)$ and $(1, 3)$.

Evaluating the Jacobian at the former of these steady states gives

$$\mathsf{DF}(\mathbf{y}^*) = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix},$$

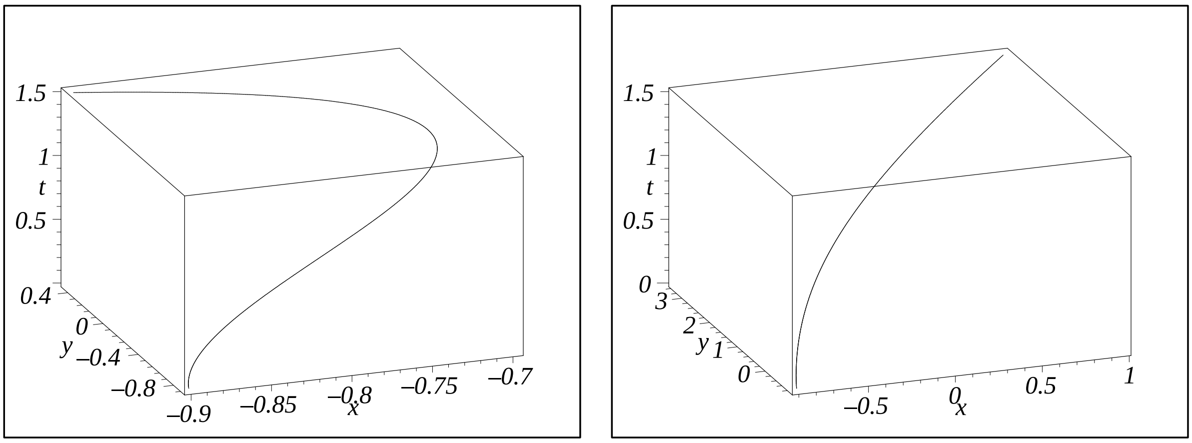
and so it should be obvious that the eigenvalues of this matrix are 4 and 6. So, as per the analysis of Question 7, it should be clear that the presence of eigenvalues which are positive real numbers indicates that the steady state $(0, 0)$ is asymptotically unstable.

Evaluating the Jacobian at the latter of these steady states gives

$$\mathsf{DF}(\mathbf{y}^*) = \begin{bmatrix} -1 & -1 \\ -9 & -3 \end{bmatrix},$$

[12] See Footnote 9.

[13] See Footnote 11.




Exact expression — Approximate expression

Figure 3: *Solutions to the system of non-linear differential equations considered in Question 7 with initial conditions close to the steady state* $\mathbf{y}^* = [x^*, y^*]^t = [-1, -1]^t$ *— specifically, the initial conditions for the solutions indicated are* $\mathbf{y}(0) = [x(0), y(0)]^t = [-0.9, -0.9]^t$*. The exact solution was obtained by solving the non-linear equations (using Maple) and the approximate solution was obtained by solving the 'qualitatively similar' linear equations. Notice that the approximate solution becomes increasingly unreliable as* $t$ *increases.*

and you should be able to show that the eigenvalues of this matrix are $-2 + \sqrt{10}$ and $-2 - \sqrt{10}$. So, as per the analysis of Question 7, it should be clear that as the first of these eigenvalues is a positive real number (note that $\sqrt{10} > 2$), the steady state $(1, 3)$ is also asymptotically unstable.

**Harder problems**

Here are the solutions for the harder problems. Again, as these were not covered in class the solutions will be a bit more detailed. (Note that, as in the Other Problems, I use $\epsilon$'s in these solutions — particularly in the 'proofs'. People who have done Real Analysis will probably find the proofs a bit unsatisfactory, but as this is a Methods course they cannot be made more precise here. As such, the 'proofs' should be regarded as a way of seeing what you would have to do without actually doing it in any detail.)

**9.** We are asked to show that $(0, 0)$ is a steady state of the general system of linear differential equations given by $\dot{\mathbf{y}} = \mathsf{A}\mathbf{y}$ where $\mathsf{A}$ is a $2 \times 2$ matrix. To do this, we take $\mathbf{y} = [0, 0]^t = \mathbf{0}$ and note that this gives $\dot{\mathbf{y}} = \mathsf{A}\mathbf{0} = \mathbf{0}$, i.e. it is a steady state (as required). Further, we are asked to show that this steady state is asymptotically stable iff $\det(A) > 0$ and $\mathrm{Tr}(A) < 0$. But, to do this, we need to note a Lemma:

> **Lemma:** If $\mathsf{A}$ is a $2 \times 2$ matrix with eigenvalues $\lambda_1$ and $\lambda_2$, then $\det(\mathsf{A}) = \lambda_1 \lambda_2$ and $\mathrm{Tr}(\mathsf{A}) = \lambda_1 + \lambda_2$.
>
> **Proof:** Let $\mathsf{A}$ be the $2 \times 2$ matrix given by
>
> $$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
>
> i.e. $\det(\mathsf{A}) = ad - bc$ and $\mathrm{Tr}(\mathsf{A}) = a + d$. The eigenvalues of this matrix can be calculated by solving the determinant equation $\det(\mathsf{A} - \lambda \mathsf{I}) = 0$, i.e.
>
> $$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \implies (a - \lambda)(d - \lambda) - bc = 0 \implies \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

and we can solve this quadratic equation by using the 'formula'. That is, the eigenvalues are

$$\lambda_1 = \frac{1}{2}\left\{a + d + \sqrt{(a+d)^2 - 4(ad - bc)}\right\},$$

and,

$$\lambda_2 = \frac{1}{2}\left\{a + d - \sqrt{(a+d)^2 - 4(ad - bc)}\right\},$$

which implies that

$$\lambda_1 + \lambda_2 = a + d = \mathrm{Tr}(\mathsf{A}),$$

and,

$$\lambda_1\lambda_2 = \frac{1}{4}\Big\{(a+d)^2 - \big[(a+d)^2 - 4(ad-bc)\big]\Big\} = ad - bc = \det(\mathsf{A}),$$

(as required).

and make two observations:

**Observation 1:** We take the matrix $\mathsf{A}$ to be real and so the equation that is used to find the eigenvalues has real coefficients. As such, the solutions must be a pair of [not necessarily distinct] real numbers or a pair of complex numbers that are complex conjugates.

**Observation 2:** The solution of the system of linear differential equations in question will be of the form

$$\mathbf{y}(t) = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$ respectively and the coefficients $A$ and $B$ are constants.

With this information available, we are now in a position to establish the result:

**Theorem:** Let $\mathsf{A}$ be a real $2 \times 2$ matrix and let $\dot{\mathbf{y}} = \mathsf{A}\mathbf{y}$. The steady state $\mathbf{y}^* = \mathbf{0}$ is asymptotically stable iff $\det(\mathsf{A}) > 0$ and $\mathrm{Tr}(\mathsf{A}) < 0$.

**Proof:** This is an 'if and only if' statement and so we have to prove it both ways:

**LTR:** As the steady state $\mathbf{y}^* = \mathbf{0}$ is asymptotically stable, there must be an $\epsilon > 0$ such that if $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$, then $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$. But, this means that the solution

$$\mathbf{y}(t) = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2,$$

of the system of linear differential equations given by $\dot{\mathbf{y}} = \mathsf{A}\mathbf{y}$ is such that $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$ for appropriate sets of initial conditions, i.e.

- **Either:** $\lambda_1$ and $\lambda_2$ are both real and negative. Thus, $\mathrm{Tr}(\mathsf{A}) = \lambda_1 + \lambda_2 < 0$ and $\det(\mathsf{A}) = \lambda_1\lambda_2 > 0$.
- **Or:** $\lambda_1$ and $\lambda_2$ are both complex and form a complex conjugate pair, i.e. $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < 0$.[14] Thus, $\mathrm{Tr}(\mathsf{A}) = \lambda_1 + \lambda_2 = 2\alpha < 0$ and $\det(\mathsf{A}) = \lambda_1\lambda_2 = \alpha^2 + \beta^2 > 0$.

Consequently, the two possible cases lead to the desired conclusion.

**RTL:** Given that $\mathrm{Tr}(\mathsf{A}) < 0$ and $\det(\mathsf{A}) > 0$ we can see that $\lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 > 0$. But, for these two inequalities to hold, it must be the case that:

---

[14] This is because in the complex case, the solution to the system of linear differential equations being considered is

$$\mathbf{y}(t) = e^{\alpha t}\left(Ae^{i\beta t}\mathbf{v}_1 + Be^{-i\beta t}\mathbf{v}_2\right),$$

and this will only tend to $\mathbf{0}$ as $t \to \infty$ if $\alpha < 0$.

- **Either:** $\lambda_1$ and $\lambda_2$ are both real negative numbers, in which case there will be an $\epsilon > 0$ such that $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$ and $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$. Thus, the steady state is asymptotically stable.
- **Or:** $\lambda_1$ and $\lambda_2$ are complex and form a complex conjugate pair, i.e. $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < 0$. Thus, as the solution to the linear system of differential equations is given by

$$\mathbf{y}(t) = e^{\alpha t}\left(Ae^{i\beta t}\mathbf{v}_1 + Be^{-i\beta t}\mathbf{v}_2\right),$$

  in this case, there will be an $\epsilon > 0$ such that $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$ and $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$. Thus, the steady state is asymptotically stable.

Consequently, the two possible cases lead to the desired conclusion.

Lastly, we are asked to show that if either $\det(\mathsf{A}) < 0$ or $\mathrm{Tr}(\mathsf{A}) > 0$, then this steady state is unstable, that is,

> **Theorem:** Let $\mathsf{A}$ be a real $2 \times 2$ matrix and let $\dot{\mathbf{y}} = \mathsf{A}\mathbf{y}$. If either $\det(\mathsf{A}) < 0$ or $\mathrm{Tr}(\mathsf{A}) > 0$, then the steady state $\mathbf{y}^* = \mathbf{0}$ is unstable.

> **Proof:** We have two cases to consider, i.e. the case where $\det(\mathsf{A}) < 0$ and the case where $\mathrm{Tr}(\mathsf{A}) > 0$, and in both of these cases we have to establish that the steady state $\mathbf{y}^* = \mathbf{0}$ is unstable.

- If $\det(\mathsf{A}) < 0$, then $\lambda_1\lambda_2 < 0$. That is, $\lambda_1 > 0$ and $\lambda_2 < 0$, or $\lambda_1 < 0$ and $\lambda_2 > 0$.[15] Clearly, whichever of these two possibilities obtains, we are going to have one positive eigenvalue and consequently, an increasing exponential in the [general] solution. As such, there will be *no* $\epsilon > 0$ such that $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$ and $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$.[16] Thus, the steady state is not asymptotically stable.
- If $\mathrm{Tr}(\mathsf{A}) > 0$, then $\lambda_1 + \lambda_2 > 0$ and so there are two possibilities:
  - $\lambda_1$ and $\lambda_2$ are both real, in which case, at least one of them must be positive. Thus, there will be [at least one] increasing exponential in the [general] solution and as such, there will be *no* $\epsilon > 0$ such that $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$ and $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$.[17] Consequently, the steady state is not asymptotically stable.
  - $\lambda_1$ and $\lambda_2$ are both complex and form a complex conjugate pair, say $\alpha \pm i\beta$ where $\alpha, \beta \in \mathbb{R}$. In this case, as $\lambda_1 + \lambda_2 = 2\alpha > 0$, we have $\alpha > 0$ and so the solution to the linear system of differential equations, i.e.

$$\mathbf{y}(t) = e^{\alpha t}\left(Ae^{i\beta t}\mathbf{v}_1 + Be^{-i\beta t}\mathbf{v}_2\right),$$

    contains an increasing exponential. As such, there will be *no* $\epsilon > 0$ such that $\|\mathbf{y}(0) - \mathbf{y}^*\| < \epsilon$ and $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$.[18] Thus, the steady state is not asymptotically stable.

  Thus, if $\mathrm{Tr}(\mathsf{A}) > 0$, the steady state is not asymptotically stable.

(As required.)

**10.** Our general model of population dynamics in the presence of two competing species was given (in Section 2 of the handout for Lectures 9 and 10) as

$$\dot{y}_1 = a_1y_1 - b_1y_1^2 - c_1y_2y_1$$
$$\dot{y}_2 = a_2y_2 - b_2y_2^2 - c_2y_1y_2$$

[15] The assumption that $\det(\mathsf{A}) < 0$ precludes the possibility of two complex eigenvalues which form a complex conjugate pair, say $\alpha \pm i\beta$, as this can only occur if $\det(\mathsf{A}) = \lambda_1\lambda_2 = \alpha^2 + \beta^2 \geq 0$.

[16] Since, for *any* $\epsilon > 0$, there will be some initial conditions where $\|\mathbf{y}(0) - \mathbf{0}\| < \epsilon$, but $\mathbf{y}(t) \not\to \mathbf{0}$ (due to the presence of the increasing exponential).

[17] See Footnote 16.

[18] See Footnote 16.

where $a_1, a_2, b_1, b_2, c_1$and $c_2$ are all positive numbers. So, to show that $\mathbf{y}_1^* = (a_1/b_1, 0)$ and $\mathbf{y}_2^* = (0, a_2/b_2)$ are steady states of this system of non-linear differential equations, we just have to verify that they satisfy $\dot{y}_1 = \dot{y}_2 = 0$. That is, $\mathbf{y}_1^*$ and $\mathbf{y}_2^*$ should make the right-hand-sides of the equations

$$\begin{aligned}\dot{y}_1 &= y_1(a_1 - b_1 y_1 - c_1 y_2)\\ \dot{y}_2 &= y_2(a_2 - b_2 y_2 - c_2 y_1)\end{aligned}$$

simultaneously zero, which they obviously do.[19] Further, to show that $\mathbf{y}_2^*$ will be asymptotically stable or unstable depending on the sign of the quantity

$$\Delta = \frac{a_1}{c_1} - \frac{a_2}{b_2},$$

we need to examine the eigenvalues of the Jacobian at this steady state. So, to calculate the Jacobian in this general setting, we use the fact that

$$F([y_1, y_2]^t) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a_1 y_1 - b_1 y_1^2 - c_1 y_2 y_1 \\ a_2 y_2 - b_2 y_2^2 - c_2 y_1 y_2 \end{pmatrix},$$

to discover that

$$DF([y_1, y_2]^t) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} a_1 - 2b_1 y_1 - c_1 y_2 & -c_1 y_1 \\ -c_2 y_2 & a_2 - 2b_2 y_2 - c_2 y_1 \end{pmatrix}.$$

Thus, at the steady state $\mathbf{y}_2^* = [0, a_2/b_2]^t$, the Jacobian matrix is given by

$$\mathsf{DF}([0, a_2/b_2]^t) = \frac{1}{b_2}\begin{bmatrix} a_1 b_2 - a_2 c_1 & 0 \\ -a_2 c_2 & -a_2 b_2 \end{bmatrix},$$

and so we can calculate the eigenvalues by solving the determinant equation given by $\det(\mathsf{A} - \lambda \mathsf{I}) = 0$, i.e.

$$\begin{vmatrix} a_1 b_2 - a_2 c_1 - b_2 \lambda & 0 \\ -a_2 c_2 & -a_2 b_2 - b_2 \lambda \end{vmatrix} = 0 \implies (a_1 b_2 - a_2 c_1 - b_2 \lambda)(a_2 b_2 - b_2 \lambda) = 0,$$

which means that the eigenvalues are $\lambda = -a_2$ which is negative (as $a_2$ is positive) and

$$\lambda = c_1 \left( \frac{a_1}{c_1} - \frac{a_2}{b_2} \right) = c_1 \Delta,$$

using the definition of $\Delta$ given above. Now, as $c_1$ is positive and $\Delta$ is real it should be clear that:

- If $\Delta < 0$, then we have two negative real eigenvalues and the steady state $\mathbf{y}_2^*$ will be asymptotically stable.
- If $\Delta > 0$, then we have a positive real eigenvalue and so the steady state $\mathbf{y}_2^*$ will be asymptotically unstable.[20]

which is the desired result.

Further, in the special case where $\Delta = 0$, we can examine the behaviour of solutions with initial conditions close to the steady state $\mathbf{y}_2^*$ by considering the linear system of differential equations which

[19] Alternatively, you could set the right-hand-sides equal to zero and solve the resulting simultaneous equations for the steady states. If you did this, you would find that the steady states were $(0, 0)$, $(a_1/b_1, 0)$, $(0, a_2/b_2)$ and

$$\left( \frac{a_2 c_1 - a_1 b_2}{c_1 c_2 - b_1 b_2}, \frac{a_1 c_2 - a_2 b_1}{c_1 c_2 - b_1 b_2} \right).$$

Clearly, the second and third of these steady states are $\mathbf{y}_1^*$ and $\mathbf{y}_2^*$, as desired.

[20] Since, for *any* $\epsilon > 0$, there will be some initial conditions where $\|\mathbf{y}(0) - \mathbf{y}_2^*\| < \epsilon$, but $\mathbf{y}(t) \not\to \mathbf{y}_2^*$ (due to the presence of the increasing exponential).

*qualitatively* describe the evolution of this system. In this case, we find that these solutions take the form

$$\mathbf{y}(t) - \mathbf{y}_2^* = Ae^{-a_2 t}\mathbf{v}_1 + B\mathbf{v}_2,$$

where $A$ and $B$ are arbitrary constants and $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues $\lambda_1 = -a_2$ and $\lambda_2 = c_1\Delta = 0$. Thus, as $t \to \infty$, the right-hand-side of this expression doesn't tend to zero and so we can see that the system is [formally] asymptotically unstable in this case too.

To explain this condition in terms of the model, we write our system of coupled non-linear differential equations as

$$\frac{\dot{y}_1}{y_1} = a_1 - b_1 y_1 - c_1 y_2 \quad \text{and} \quad \frac{\dot{y}_2}{y_2} = a_2 - b_2 y_2 - c_2 y_1,$$

and recall that $a_1$ and $a_2$ are the growth rates of the populations $y_1$ and $y_2$ respectively in the absence of any growth-inhibiting factors (such as limitations on space, scarcity of natural resources or one species preying on another.) The growth inhibiting factors that we consider in this model then fall into two categories, namely:

- $b_1$ and $b_2$ which measure the effect of 'external' factors on the populations $y_1$ and $y_2$ respectively (such as limitations on space and scarcity of natural resources). That is, as these populations grow, the environment will cause them to die out.
- $c_1$ and $c_2$ which measure the effect of 'internal' factors on the populations $y_1$ and $y_2$ respectively (such as how effectively one of the species can prey on the other). For example, if the population of species 1 grows, there will be more of them for species 2 to prey upon causing the population of species 1 to be reduced at a greater rate. Further, if species 2 grows, there will be more of them to prey upon species 1 and this also leads to a reduction in the population of species 1. (And, of course, vice versa.)

Obviously, the evolution of both of these populations as they interact will be very sensitive to both the initial populations and the relative values of the six positive constants in the model. Indeed, this part of the question sets out to explore this sensitivity in a limited range of situations. The first of these involves situations where we start close to the steady state $\mathbf{y}_2^*$, and we discuss this in the next two paragraphs. The second involves situations where we start close to the steady state $\mathbf{y}_1^*$, and we shall discuss this at the end of the question.

If we start close to the steady state $\mathbf{y}_2^* = [0, a_2/b_2]^t$, then the initial population of species 1 and 2 will be close to 0 and $a_2/b_2$ respectively. As such, the population of species 1 is unlikely to be affected by 'external' factors and the population of species 2 is unlikely to be affected by 'internal' factors,[21] which means that our equations can be written as

$$\frac{\dot{y}_1}{y_1} \simeq a_1 - c_1 y_2 \quad \text{and} \quad \frac{\dot{y}_2}{y_2} \simeq a_2 - b_2 y_2,$$

or indeed, as

$$\frac{\dot{y}_1}{y_1} \simeq c_1\left[\frac{a_1}{c_1} - y_2\right] \quad \text{and} \quad \frac{\dot{y}_2}{y_2} \simeq b_2\left[\frac{a_2}{b_2} - y_2\right].$$

Further, the quantity $y_2$ is close to $a_2/b_2$, and so this would imply that if we are sufficiently close to the steady state $y_2^*$ we could actually write these equations as

$$\frac{\dot{y}_1}{y_1} \simeq c_1\left[\frac{a_1}{c_1} - \frac{a_2}{b_2}\right] \quad \text{and} \quad \frac{\dot{y}_2}{y_2} \simeq 0.$$

Now, under these circumstances, $\dot{y}_2 \simeq 0$ and so the value of $y_2$ will change very slowly with time, however the population of species 1 will evolve according to

$$\dot{y}_1 \simeq c_1\Delta y_1 \quad \Longrightarrow \quad y_1 \simeq Ae^{c_1\Delta t},$$

where $A$ is an arbitrary constant and $\Delta$ is defined as above. Clearly, this leads us to two conclusions:

[21] As, in both cases, these effects depend on $y_1$, and this quantity is small as we are starting near $y_1 = 0$.

- If $\Delta > 0$, then $y_1$ will increase until the approximations used in this calculation cease to be valid.
- If $\Delta < 0$, then $y_1$ will decrease to zero (with the approximations used in this calculation remaining valid).

Thus, although this analysis is not as 'rigorous' as the one above, it does give us some idea of how to interpret the conditions that we derived earlier in terms of the model.

Now, note that the quantity $a_1$ measures the growth rate of species 1 and the quantity $a_2/b_2$ gives us a measure of the population of species 2 which is present. Also, $c_1$ tells us how effective species 2 is at preying on species 1, so the quantity $c_1a_2/b_2$ measures the rate at which species 1 is losing members due to the presence of species 2. Now, this means that the quantity

$$a_1 - c_1\frac{a_2}{b_2} = c_1\left[\frac{a_1}{c_1} - \frac{a_2}{b_2}\right] = c_1\Delta,$$

measures the net growth rate of species 1 bearing in mind that this population is being preyed on by species 2. Further, this means that

- If $\Delta > 0$, then species 1 reproduces at a faster rate than the rate at which its population is decreasing due to the presence of species 2 and so in this case, the population of species 1 cannot be kept in check by species 2, i.e. its population will start to increase.
- If $\Delta < 0$, then species 1 reproduces at a slower rate than the rate at which its population is decreasing due to the presence of species 2 and so in this case, species 1 will eventually die out.

So, in terms of the model, the sign of $\Delta$ indicates whether the growth rate of species 1 is greater (or less) than the rate at which species 2 is preying upon it.

To find the corresponding result for $\mathbf{y}_1^* = [a_1/b_1, 0]^t$, we note that at this steady state, the Jacobian matrix is given by

$$\mathsf{DF}([a_1/b_1, 0]^t) = \frac{1}{b_1}\begin{bmatrix} -a_1b_1 & -a_1c_1 \\ 0 & a_2b_1 - a_1c_2 \end{bmatrix},$$

and so we can calculate the eigenvalues by solving the determinant equation given by $\det(\mathsf{A} - \lambda\mathsf{I}) = 0$, i.e.

$$\begin{vmatrix} -a_1b_1 - b_1\lambda & -a_1c_1 \\ 0 & a_2b_1 - a_1c_2 - b_1\lambda \end{vmatrix} = 0 \implies -b_1(a_1 + \lambda)(a_2b_1 - a_1c_2 - b_1\lambda) = 0,$$

which means that the eigenvalues are $\lambda = -a_1$ which is negative (as $a_1$ is positive) and

$$\lambda = c_2\left(\frac{a_2}{c_2} - \frac{a_1}{b_1}\right) = c_2\Delta',$$

where $\Delta'$ is the natural analogue of $\Delta$. Now, as $c_2$ is positive and $\Delta'$ is real it should be clear that:

- If $\Delta' < 0$, then we have two negative real eigenvalues and the steady state $\mathbf{y}_1^*$ will be asymptotically stable.
- If $\Delta' > 0$, then we have a positive real eigenvalue and so the steady state $\mathbf{y}_1^*$ will be asymptotically unstable.[22]

which is the desired result.

Further, in the special case where $\Delta' = 0$, we can examine the behaviour of solutions with initial conditions close to the steady state $\mathbf{y}_1^*$ by considering the linear system of differential equations which *qualitatively* describe the evolution of this system. In this case, we find that these solutions take the form

$$\mathbf{y}(t) - \mathbf{y}_1^* = Ae^{-a_1t}\mathbf{v}_1 + B\mathbf{v}_2,$$

[22] Since, for *any* $\epsilon > 0$, there will be some initial conditions where $\|\mathbf{y}(0) - \mathbf{y}_1^*\| < \epsilon$, but $\mathbf{y}(t) \not\to \mathbf{y}_1^*$ (due to the presence of the increasing exponential).

where $A$ and $B$ are arbitrary constants and $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues $\lambda_1 = -a_1$ and $\lambda_2 = c_2\Delta' = 0$. Thus, as $t \to \infty$, the right-hand-side of this expression doesn't tend to zero and so we can see that the system is [formally] asymptotically unstable in this case too.

To see what this means in terms of the model, we assume an analysis similar to the one above and note that the quantity $a_2$ measures the growth rate of species 2 and the quantity $a_1/b_1$ gives us a measure of the population of species 1 which is present. Also, $c_2$ tells us how effective species 1 is at preying on species 2, so the quantity $c_2a_1/b_1$ measures the rate at which species 2 is losing members due to the presence of species 1. Now, this means that the quantity

$$a_2 - c_2\frac{a_1}{b_1} = c_2\left[\frac{a_2}{c_2} - \frac{a_1}{b_1}\right] = c_2\Delta',$$

measures the net growth rate of species 2 bearing in mind that this population is being preyed on by species 1. Further, this means that

- If $\Delta' > 0$, then species 2 reproduces at a faster rate than the rate at which its population is decreasing due to the presence of species 1 and so in this case, the population of species 2 cannot be kept in check by species 1, i.e. its population will start to increase.
- If $\Delta' < 0$, then species 2 reproduces at a slower rate than the rate at which its population is decreasing due to the presence of species 1 and so in this case, species 2 will eventually die out.

So, in terms of the model, the sign of $\Delta'$ indicates whether the growth rate of species 2 is greater (or less) than the rate at which species 1 is preying upon it.