Further Mathematical Methods (Linear Algebra)

Solutions For Problem Sheet 8

In this sheet, we looked at orthogonal projections and how to analyse sets of data using least squares fits. These latter questions were all fairly easy as they just involved 'number crunching'.

1. Let X be the subspace of $\mathbb{R}^3$ spanned by the vectors $[1,2,3]^t$ and $[1,1,-1]^t$. We are asked to find a matrix P such that $\mathsf{P}\mathbf{x}$ is the orthogonal projection of $\mathbf{x} \in \mathbb{R}^3$ onto X. To do this, we recall from the lectures that:

> If A is an $m \times n$ matrix of rank n, then the orthogonal projection onto $R(\mathsf{A})$ is given by $\mathsf{P} = \mathsf{A}(\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t$.

and so if we can find a matrix A such that $R(\mathsf{A}) = X$, then the matrix which we seek will be

$$\mathsf{P} = \mathsf{A}(\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t.$$

So, since $R(\mathsf{A}) = CS(\mathsf{A})$, a suitable matrix would be

$$\mathsf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & -1 \end{bmatrix},$$

as this has rank two (since the vectors $[1,2,3]^t$ and $[1,1,-1]^t$ are linearly independent) and $R(\mathsf{A}) = X$. Thus, the desired orthogonal projection is given by

$$\mathsf{A}^t\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 3 \end{bmatrix} \implies (\mathsf{A}^t\mathsf{A})^{-1} = \frac{1}{42}\begin{bmatrix} 3 & 0 \\ 0 & 14 \end{bmatrix},$$

and so,

$$(\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t = \frac{1}{42}\begin{bmatrix} 3 & 0 \\ 0 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{42}\begin{bmatrix} 3 & 6 & 9 \\ 14 & 14 & -14 \end{bmatrix}.$$

Thus, the matrix

$$\mathsf{P} = \mathsf{A}(\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t = \frac{1}{42}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 & 6 & 9 \\ 14 & 14 & -14 \end{bmatrix} = \frac{1}{42}\begin{bmatrix} 17 & 20 & -5 \\ 20 & 26 & 4 \\ -5 & 4 & 41 \end{bmatrix},$$

will give us the sought after orthogonal projection onto X.

Note: We can check that this matrix does represent *an* orthogonal projection since it is clearly symmetric (i.e. $\mathsf{P}^t = \mathsf{P}$) and it is also idempotent as

$$\mathsf{P}^2 = \mathsf{P}\mathsf{P} = \frac{1}{42^2}\begin{bmatrix} 17 & 20 & -5 \\ 20 & 26 & 4 \\ -5 & 4 & 41 \end{bmatrix}\begin{bmatrix} 17 & 20 & -5 \\ 20 & 26 & 4 \\ -5 & 4 & 41 \end{bmatrix} = \frac{1}{42}\begin{bmatrix} 17 & 20 & -5 \\ 20 & 26 & 4 \\ -5 & 4 & 41 \end{bmatrix} = \mathsf{P}.$$

Indeed, if you are really keen, you can check that this matrix represents an orthogonal projection *onto* X by noting that for any vector $\mathbf{x} \in \mathbb{R}^3$ we have

$$42\mathsf{P}\mathbf{x} = \begin{bmatrix} 17 & 20 & -5 \\ 20 & 26 & 4 \\ -5 & 4 & 41 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 17 \\ 20 \\ -5 \end{bmatrix} + x_2\begin{bmatrix} 20 \\ 26 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 4 \\ 41 \end{bmatrix},$$

and showing that the set of vectors $\text{Lin}\{[17,20,-5]^t, [20,26,4]^t, [-5,4,41]^t\} = X$, as this will imply that the vector $\mathsf{P}\mathbf{x}$ is in X. (This can be done, for instance, by using these vectors to form the rows of a matrix and then performing row operations on this matrix until its rows give you a set of vectors that clearly spans X.)[1]

2. We are given a real $n \times n$ matrix A which is orthogonally diagonalisable, i.e. there exists an orthogonal matrix P such that

$$\mathsf{P}^t\mathsf{A}\mathsf{P} = \mathsf{D},$$

where D is a diagonal matrix made up from the eigenvalues of A and we are told that all of the eigenvalues and eigenvectors of this matrix are real. The orthogonal matrix P has column vectors given by the orthonormal set of vectors $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ where $\mathbf{x}_i$ is the eigenvector corresponding to the eigenvalue λ_i, i.e. $\mathsf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i$. As such, we have $\mathsf{P}\mathsf{P}^t = \mathsf{I}$ and so, writing this out in full we have:

$$\mathsf{I} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \text{---} & \mathbf{x}_1^t & \text{---} \\ \text{---} & \mathbf{x}_2^t & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{x}_n^t & \text{---} \end{bmatrix},$$

which can be multiplied out to give (see the aside at the end of this question),

$$\mathsf{I} = \mathbf{x}_1\mathbf{x}_1^t + \mathbf{x}_2\mathbf{x}_2^t + \cdots + \mathbf{x}_n\mathbf{x}_n^t.$$

(Notice, that this is a spectral decomposition of the identity matrix.)[2] Then, multiplying both sides of this expression by A, we get

$$\mathsf{A} = \mathsf{A}\mathbf{x}_1\mathbf{x}_1^t + \mathsf{A}\mathbf{x}_2\mathbf{x}_2^t + \cdots + \mathsf{A}\mathbf{x}_n\mathbf{x}_n^t,$$

which gives

$$\mathsf{A} = \lambda_1\mathbf{x}_1\mathbf{x}_1^t + \lambda_2\mathbf{x}_2\mathbf{x}_2^t + \cdots + \lambda_n\mathbf{x}_n\mathbf{x}_n^t.$$

Thus, defining the $n \times n$ matrix E_i to be such that $\mathsf{E}_i = \mathbf{x}_i\mathbf{x}_i^t$ we get

$$\mathsf{A} = \lambda_1\mathsf{E}_1 + \lambda_2\mathsf{E}_2 + \cdots + \lambda_n\mathsf{E}_n,$$

which is a spectral decomposition of A.

To prove that the matrices E_i are such that

$$\mathsf{E}_i\mathsf{E}_j = \begin{cases} \mathsf{E}_i & \text{if } i = j \\ \mathsf{0} & \text{if } i \neq j \end{cases}$$

we note that, by definition,

$$\mathsf{E}_i\mathsf{E}_j = \mathbf{x}_i\mathbf{x}_i^t\mathbf{x}_j\mathbf{x}_j^t = \langle \mathbf{x}_i, \mathbf{x}_j \rangle \mathbf{x}_i\mathbf{x}_j^t,$$

where we have used our convention. Thus, as S is an orthonormal set, we have

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and so,

$$\mathsf{E}_i\mathsf{E}_j = \begin{cases} 1 \cdot \mathbf{x}_i\mathbf{x}_i^t & \text{if } i = j \\ 0 \cdot \mathbf{x}_i\mathbf{x}_j^t & \text{if } i \neq j \end{cases}$$

which means that,

$$\mathsf{E}_i\mathsf{E}_j = \begin{cases} \mathsf{E}_i & \text{if } i = j \\ \mathsf{0} & \text{if } i \neq j \end{cases}$$

where $\mathsf{0}$ is the $n \times n$ zero matrix (as required).

To show that the matrix E_i represents an orthogonal projection we note that since:

[1] But, life is too short.

[2] The identity matrix has $\lambda = 1$ (multiplicity n) as its eigenvalues. Also, each of the $\mathbf{x}_i$ will be an eigenvector of the identity matrix corresponding to this eigenvalue since $\mathsf{I}\mathbf{x}_i = 1 \cdot \mathbf{x}_i$.

- $\mathsf{E}_i^2 = \mathsf{E}_i\mathsf{E}_i = \mathsf{E}_i$ (from above), the matrix E_i is idempotent and as such it represents a projection.

- $\mathsf{E}_i^t = (\mathbf{x}_i\mathbf{x}_i^t)^t = \mathbf{x}_i\mathbf{x}_i^t = \mathsf{E}_i$, the matrix E_i is symmetric and so it represents an orthogonal projection.

Thus, E_i represents an orthogonal projection (as required). Now, we know from the lectures that S is a basis for $\mathbb{R}^n$ and so, any vector $\mathbf{x} \in \mathbb{R}^n$ can be written as

$$\mathbf{x} = \sum_{j=1}^{n} \alpha_j \mathbf{x}_j,$$

which means that as $\mathsf{E}_i = \mathbf{x}_i\mathbf{x}_i^t$, and the vectors in S are orthonormal, we have

$$\mathsf{E}_i\mathbf{x} = \sum_{j=1}^{n} \alpha_j \mathsf{E}_i \mathbf{x}_j = \sum_{j=1}^{n} \alpha_j (\mathbf{x}_i\mathbf{x}_i^t)\mathbf{x}_j = \sum_{j=1}^{n} \alpha_j \mathbf{x}_i(\mathbf{x}_i^t\mathbf{x}_j) = \sum_{j=1}^{n} \alpha_j \langle \mathbf{x}_i, \mathbf{x}_j \rangle \mathbf{x}_i = \alpha_i \mathbf{x}_i,$$

i.e. E_i orthogonally projects any vector in $\mathbb{R}^n$ onto $\text{Lin}\{\mathbf{x}_i\}$ (as required). Consequently, we can see that

$$\mathsf{A}\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathsf{E}_i \mathbf{x} = \sum_{i=1}^{n} \lambda_i \alpha_i \mathbf{x}_i,$$

and so the linear transformation represented by A takes the 'component' of $\mathbf{x}$ in the direction of each eigenvector (i.e. $\alpha_i\mathbf{x}_i$ for each eigenvector $\mathbf{x}_i$) and multiplies it by a factor given by the corresponding eigenvalue. (For example, if $\lambda_i > 1$, then the vector $\alpha_i\mathbf{x}_i$ is 'stretched' [or 'dilated'] by a factor of λ_i.) So, basically, the spectral decomposition allows us to describe linear transformations in terms of the sum of the 'components' of a vector in the direction of each eigenvector *scaled* by the appropriate eigenvalue. A simple illustration of this is given in Figure 1.

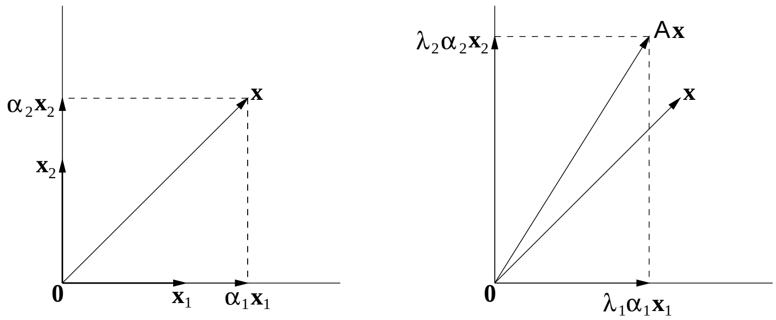

Figure 1: *This figure illustrates the results of Question 2 in* $\mathbb{R}^2$*. Notice that the eigenvectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are orthogonal (i.e. perpendicular) and have the same [i.e. unit] length. In the left-hand diagram we see that the vector* $\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$ *can be decomposed into 'components'* $\alpha_1\mathbf{x}_1$ *and* $\alpha_2\mathbf{x}_2$ *in the directions of the eigenvectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *respectively. Indeed, these two 'components' are given by* $\mathsf{E}_1\mathbf{x}$ *and* $\mathsf{E}_2\mathbf{x}$ *[respectively]. In the right-hand diagram, the two 'components' have been multiplied by the relevant eigenvalue (notice that* $0 \leq \lambda_1 \leq 1$ *and* $\lambda_2 \geq 1$*) and the sum of these new vectors gives us the vector* $\mathsf{A}\mathbf{x}$ *as expected from the theory above.*

Aside: You may be surprised that the matrix product PP^t can be multiplied out to give:

$$\mathsf{PP}^t = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \text{---} & \mathbf{x}_1^t & \text{---} \\ \text{---} & \mathbf{x}_2^t & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{x}_n^t & \text{---} \end{bmatrix} = \mathbf{x}_1\mathbf{x}_1^t + \mathbf{x}_2\mathbf{x}_2^t + \cdots + \mathbf{x}_n\mathbf{x}_n^t,$$

a fact which we have just used in both the lectures and the previous question. I will not justify this in any general way, but I will show why it holds in the case where P is a 3×3 matrix. To see this, suppose that the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ which constitute the columns of the matrix P are given by,

$$\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}, \text{ and } \mathbf{x}_3 = \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix}.$$

That is, the matrix product PP^t is given by,

$$\mathsf{PP}^t = \begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix},$$

and multiplying these two matrices together yields,

$$\mathsf{PP}^t = \begin{bmatrix} \sum x_{i1}x_{i1} & \sum x_{i1}x_{i2} & \sum x_{i1}x_{i3} \\ \sum x_{i2}x_{i1} & \sum x_{i2}x_{i2} & \sum x_{i2}x_{i3} \\ \sum x_{i3}x_{i1} & \sum x_{i3}x_{i2} & \sum x_{i3}x_{i3} \end{bmatrix},$$

where all of these summations run from $i = 1$ to 3. However, as we can add matrices by adding their corresponding elements, this is the same as

$$\mathsf{PP}^t = \begin{bmatrix} x_{11}x_{11} & x_{11}x_{12} & x_{11}x_{13} \\ x_{12}x_{11} & x_{12}x_{12} & x_{12}x_{13} \\ x_{13}x_{11} & x_{13}x_{12} & x_{13}x_{13} \end{bmatrix} + \begin{bmatrix} x_{21}x_{21} & x_{21}x_{22} & x_{21}x_{23} \\ x_{22}x_{21} & x_{22}x_{22} & x_{22}x_{23} \\ x_{23}x_{21} & x_{23}x_{22} & x_{23}x_{23} \end{bmatrix} + \begin{bmatrix} x_{31}x_{31} & x_{31}x_{32} & x_{31}x_{33} \\ x_{32}x_{31} & x_{32}x_{32} & x_{32}x_{33} \\ x_{33}x_{31} & x_{33}x_{32} & x_{33}x_{33} \end{bmatrix},$$

which, you will notice, is just

$$\mathsf{PP}^t = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \end{bmatrix} + \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} \begin{bmatrix} x_{21} & x_{22} & x_{23} \end{bmatrix} + \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} \begin{bmatrix} x_{31} & x_{32} & x_{33} \end{bmatrix},$$

i.e. we have,

$$\mathsf{PP}^t = \mathbf{x}_1\mathbf{x}_1^t + \mathbf{x}_2\mathbf{x}_2^t + \mathbf{x}_3\mathbf{x}_3^t,$$

if P is a 3×3 matrix, as desired.

Note: In the solutions to Questions 3 to 7 we start by considering the system of equations which the data would 'ideally' satisfy. That is, if there were *no* errors in the data, then given a rule with certain parameters, we would expect to get values for these parameters which were solutions to this system of equations. However, as there *are* errors in the data, this system of equations will be inconsistent, and hence we will be unable to solve them for the parameters. This is why we look for a least squares fit! The values of the parameters that we find using this analysis are the ones that minimise the least squares error between the rule and the data. (Notice that, in general, these parameters will not satisfy any of the equations that the data would 'ideally' satisfy.)[3]

3. The quantities x and y are related by a rule of the form $y = ax + b$ for some constants a and b. We are given some data, i.e.

x	1	2	3	4
y	5	3	2	1

[3] Indeed, this raises an interesting question: Do we need to check that each system of equations is actually inconsistent before we apply this method? The answer is, of course, no. In the unlikely event ('unlikely' because surely no-one would set a 'least squares' question that can be solved without using a least squares analysis) that the equations are consistent (that is, in the case where the error terms are all zero), the least squares solution will still give the correct answer. This is because, in such a situation, the least squares solution $\mathbf{x}^* = (\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t\mathbf{b}$ will give the unique solution to the set of equations that the data satisfy. To see why, look at Question 8. (For a further illustration of this idea, see the remarks following Questions 5 and 7.

and asked to find the least squares estimate of the parameters a and b in the rule above. To do this, we note that ideally[4] a and b would satisfy the system of equations

$$\begin{aligned} a + b &= 5 \\ 2a + b &= 3 \\ 3a + b &= 2 \\ 4a + b &= 1 \end{aligned}$$

and writing them in matrix form, i.e. setting

$$\mathsf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

we let $\mathsf{A}\mathbf{x} = \mathbf{b}$, so that we can use the fact that $\mathbf{x}^* = (\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t\mathbf{b}$ gives a least squares solution to this system. Then, as

$$\mathsf{A}^t\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \implies (\mathsf{A}^t\mathsf{A})^{-1} = \frac{1}{20} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix},$$

and

$$\mathsf{A}^t\mathbf{b} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 11 \end{bmatrix} \implies \mathbf{x}^* = \frac{1}{20} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix} \begin{bmatrix} 21 \\ 11 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -13 \\ 60 \end{bmatrix},$$

the required least squares estimates are $a^* = -1.3$ and $b^* = 6$.[5]

4. The quantities x and y are known to be related by a rule of the form

$$y = \frac{m}{x} + c,$$

for some constants m and c. We are given some data, and as we need to fit this data to a curve containing a $1/x$ term, it is convenient to supplement the table of data with an extra row, i.e.

x	$1/5$	$1/4$	$1/3$	$1/2$	1
$1/x$	5	4	3	2	1
y	4	3	2	2	1

Now, to find the least squares estimate of m and c in the rule above, we note that ideally[6] these parameters would satisfy the system of equations

$$\begin{aligned} 5m + c &= 4 \\ 4m + c &= 3 \\ 3m + c &= 2 \\ 2m + c &= 2 \\ m + c &= 1 \end{aligned}$$

[4] As noted above, I say ideally as these equations are inconsistent. (Again, this is why we are looking for a least squares solution!) We can see that they are inconsistent as subtracting the first two equations gives $a = -2$, and hence $b = 7$. But this solution satisfies neither the third equation (as $-6 + 7 = 1 \neq 2$), nor the fourth equation (as $-8 + 7 = -1 \neq 1$).

[5] Thus, putting these values into the rule, we see that

$$y = -1.3x + 6,$$

is the curve which minimises the least square error between the rule and the data. Notice that this line has a negative gradient as one might expect from the data (i.e. as x increases, y decreases!).

[6] As noted above, I say ideally as these equations are inconsistent. We can see that this is the case because the third and fourth equations imply that $m = 0$, and hence $c = 2$, but this 'solution' fails to satisfy any of the three remaining equations.

So, writing them in matrix form, i.e. setting

$$\mathsf{A} = \begin{bmatrix} 5 & 1 \\ 4 & 1 \\ 3 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} m \\ c \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \\ 1 \end{bmatrix},$$

and letting $\mathsf{A}\mathbf{x} = \mathbf{b}$, we can use the fact that $\mathbf{x}^* = (\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t\mathbf{b}$ gives a least squares solution to this system. Then, as

$$\mathsf{A}^t\mathsf{A} = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 4 & 1 \\ 3 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \implies (\mathsf{A}^t\mathsf{A})^{-1} = \frac{1}{50} \begin{bmatrix} 5 & -15 \\ -15 & 55 \end{bmatrix},$$

and

$$\mathsf{A}^t\mathbf{b} = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 43 \\ 12 \end{bmatrix} \implies \mathbf{x}^* = \frac{1}{50} \begin{bmatrix} 5 & -15 \\ -15 & 55 \end{bmatrix} \begin{bmatrix} 43 \\ 12 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 35 \\ 15 \end{bmatrix},$$

the required least squares estimates are $m^* = 0.7$ and $c^* = 0.3$.[7]

We are also asked why it would be wrong to suppose that this was equivalent to the problem of fitting a curve of the form

$$z = xy = cx + m,$$

through the data points (xy, x). The reason why this supposition would be wrong is that in the problem which we solved, we effectively introduced an error term $\mathbf{r}$ which made the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ consistent. That is, the system of equations represented by the matrix equation

$$\mathsf{A}\mathbf{x} = \mathbf{b} - \mathbf{r},$$

would have solutions that corresponded to *the* values of the parameters for the data in question. However, as we are ignorant of the values which the components of $\mathbf{r}$ take, we have to content ourselves with finding the values of the parameters which minimise the least squares error given by $\|\mathbf{r}\|$. Due to the way that this is set up, each component of the vector $\mathbf{r}$ is the 'vertical distance' between a data point and a curve given by the rule. Thus, when we find the curve that minimises the least squares error, we are therefore just minimising the sum of the squares of these 'vertical distances.' Now, when we adopt the rule

$$z = xy = cx + m,$$

through the data points (xy, x), although this is algebraically the same, the errors that we introduce will not be the vertical distances between the data points and the curve $y = \frac{m}{x} + c$, but those between the data points (z, x) and the curve $z = cx + m$. So, as this transformation of the data points alters the 'distances' that we are trying to minimise, it will alter the least squares fit that we find.

[7] Thus, putting these values into the rule, we see that

$$y = \frac{7}{10}\frac{1}{x} + \frac{3}{10},$$

is the curve which minimises the least square error between the rule and the data.

5. We are asked to find the least squares fit of the form $y = m^*x + c^*$ through the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. To do this, we use the matrix approach developed in the lectures to show that the parameters m^* and c^* are given by

$$m^* = \frac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2} \quad \text{and} \quad c^* = \frac{(\sum y_i)(\sum x_i^2) - (\sum x_i)(\sum x_i y_i)}{n\sum x_i^2 - (\sum x_i)^2},$$

where all summations in these formulae run from $i = 1$ to n. In this general case, the parameters would ideally[8] satisfy the system of equations

$$\begin{aligned} c + x_1 m &= y_1 \\ c + x_2 m &= y_2 \\ &\vdots \\ c + x_n m &= y_n \end{aligned}$$

and writing them in matrix form, i.e. setting

$$\mathsf{A} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} c \\ m \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

and letting $\mathsf{A}\mathbf{x} = \mathbf{b}$, we can use the fact that $\mathbf{x}^* = (\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t\mathbf{b}$ gives a least squares solution to this system. So, noting that all of the summations below will run from $i = 1$ to n, we get

$$\mathsf{A}^t\mathsf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$\implies (\mathsf{A}^t\mathsf{A})^{-1} = \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix},$$

and

$$\mathsf{A}^t\mathbf{b} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\implies \mathbf{x}^* = \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\therefore \quad \mathbf{x}^* = \begin{bmatrix} c^* \\ m^* \end{bmatrix} = \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} (\sum x_i^2)(\sum y_i) - (\sum x_i)(\sum x_i y_i) \\ -(\sum x_i)(\sum y_i) + n\sum x_i y_i \end{bmatrix},$$

which gives the desired result.

Remark: As we have mentioned above, if the data was free from error, and as such, the data points were all on the curve in question, then the above system of equations would be consistent. Consequently, it would be easy to solve them for the parameters m and c. But, just for the sake of completeness, we will now show that the above result will also give this answer. To do this, let us

[8] As this is a least squares fit question we assume that the data points are such that these equations are inconsistent. (But, also see the remark below.)

assume that we have solved the [now] consistent set of equations and found the parameters to be m and c. This means that each data point (x_i, y_i) would satisfy the equation $y_i = mx_i + c$, and as such, our result becomes

$$\mathbf{x}^* = \frac{1}{n\sum x_i^2 - (\sum x_i)^2}\begin{bmatrix} \left(\sum x_i^2\right)\left(\sum\{mx_i+c\}\right) - \left(\sum x_i\right)\left(\sum x_i\{mx_i+c\}\right) \\ -\left(\sum x_i\right)\left(\sum\{mx_i+c\}\right) + n\sum x_i\{mx_i+c\} \end{bmatrix}$$

$$= \frac{1}{n\sum x_i^2 - (\sum x_i)^2}\begin{bmatrix} m\left(\sum x_i^2\right)\left(\sum x_i\right) + c\left(\sum x_i^2\right)\left(\sum 1\right) - m\left(\sum x_i\right)\left(\sum x_i^2\right) - c\left(\sum x_i\right)^2 \\ -m\left(\sum x_i\right)^2 - c\left(\sum x_i\right)\left(\sum 1\right) + nm\left(\sum x_i^2\right) + nc\sum x_i \end{bmatrix}$$

$$= \frac{1}{n\sum x_i^2 - (\sum x_i)^2}\begin{bmatrix} cn\sum x_i^2 - c(\sum x_i)^2 \\ -m(\sum x_i)^2 + nm\sum x_i^2 \end{bmatrix} \quad \text{:Using the fact that } \textstyle\sum 1 = n$$

$$\implies \mathbf{x}^* = \begin{bmatrix} c \\ m \end{bmatrix},$$

as expected.

Other Problems.

Here are the solutions to the other questions on the analysis of data sets using least squares fits that you might have tried.

6. An input variable θ and a response variable y are related by a law of the form

$$y = a + b\cos^2\theta,$$

where a and b are constants. The observation of y is subject to error, and we are asked to use the following data to estimate a and b using the method of least squares. As we need to fit this data to a curve containing a $\cos^2\theta$ term, it is convenient to supplement the table of data with a couple of rows, i.e.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\cos^2\theta$	1.00	0.75	0.50	0.25	0.00
y	4.1	3.4	2.7	2.1	1.6

We now note that ideally,[9] the parameters a and b would satisfy the system of equations

$$\begin{aligned} a + 1.00b &= 4.1 \\ a + 0.75b &= 3.4 \\ a + 0.50b &= 2.7 \\ a + 0.25b &= 2.1 \\ a + 0.00b &= 1.6 \end{aligned}$$

and writing them in matrix form, i.e. setting

$$\mathsf{A} = \begin{bmatrix} 1 & 1.00 \\ 1 & 0.75 \\ 1 & 0.50 \\ 1 & 0.25 \\ 1 & 0.00 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4.1 \\ 3.4 \\ 2.7 \\ 2.1 \\ 1.6 \end{bmatrix},$$

[9] As noted above, I say ideally as these equations are inconsistent. We can see that this is the case because the last equation implies that $a = 1.6$, and so the first equation gives $b = 2.5$, but this 'solution' fails to satisfy any of the three remaining equations.

and letting $\mathsf{A}\mathbf{x} = \mathbf{b}$, we can use the fact that $\mathbf{x}^* = (\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t\mathbf{b}$ gives a least squares solution to this system. So, as

$$\mathsf{A}^t\mathsf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.00 & 0.75 & 0.50 & 0.25 & 0.00 \end{bmatrix} \begin{bmatrix} 1 & 1.00 \\ 1 & 0.75 \\ 1 & 0.50 \\ 1 & 0.25 \\ 1 & 0.00 \end{bmatrix} = \begin{bmatrix} 5 & 2.50 \\ 2.50 & 1.875 \end{bmatrix}$$

$$\Longrightarrow \quad (\mathsf{A}^t\mathsf{A})^{-1} = \frac{1}{3.125} \begin{bmatrix} 1.875 & -2.50 \\ -2.50 & 5 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 \\ -0.8 & 1.6 \end{bmatrix},$$

and

$$\mathsf{A}^t\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.00 & 0.75 & 0.50 & 0.25 & 0.00 \end{bmatrix} \begin{bmatrix} 4.1 \\ 3.4 \\ 2.7 \\ 2.1 \\ 1.6 \end{bmatrix} = \begin{bmatrix} 13.9 \\ 8.525 \end{bmatrix}$$

$$\Longrightarrow \quad \mathbf{x}^* = \begin{bmatrix} 0.6 & -0.8 \\ -0.8 & 1.6 \end{bmatrix} \begin{bmatrix} 13.9 \\ 8.525 \end{bmatrix} = \begin{bmatrix} 1.52 \\ 2.52 \end{bmatrix},$$

the required least squares estimates are $a^* = 1.52$ and $b^* = 2.52$.[10]

7. We are asked to find the least squares solution to the system given by $x_1 = a_1, a_2, \ldots, a_n$. The notation in this question is a bit misleading (and often leads to confusion), but essentially all we have is a single variable x_1 which is found to be equal to n constants $a_1, a_2, \ldots, a_n$. So, ideally,[11] we would be able to find a value of x_1 such that

$$\begin{aligned} x_1 &= a_1 \\ x_1 &= a_2 \\ &\vdots \\ x_1 &= a_n \end{aligned}$$

and writing these in matrix form, i.e. setting

$$\mathsf{A} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \mathbf{x} = [x_1] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

and letting $\mathsf{A}\mathbf{x} = \mathbf{b}$, we can use the fact that $\mathbf{x}^* = (\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t\mathbf{b}$ gives a least squares solution to this system. So, as

$$\mathsf{A}^t\mathsf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = [n] \quad \Longrightarrow \quad (\mathsf{A}^t\mathsf{A})^{-1} = \left[\frac{1}{n}\right],$$

[10] Thus, putting these values into the rule, we see that

$$y = 1.52 + 2.52\cos^2\theta,$$

is the curve which minimises the least square error between the rule and the data.

[11] As this is a least squares fit question we assume that the data points are such that these equations are inconsistent, i.e. at least two of the a_i are distinct. (But, also see the remark below.)

and

$$\mathsf{A}^t\mathbf{b} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \left[\sum_{i=1}^n a_i \right] \implies \mathbf{x}^* = \left[\frac{1}{n} \sum_{i=1}^n a_i \right],$$

the least squares solution is

$$x_1^* = \frac{1}{n} \sum_{i=1}^n a_i,$$

and this is just the average of the a_i for $1 \leq i \leq n$.[12]

Remark: If the a_i were all equal, say $a_1 = a_2 = \cdots = a_n = a$, then this system of equations would be consistent, and the solution would be [unsurprisingly] $x_1 = a$. As before, our least squares result can also be used to get this answer, because in this case

$$x_1^* = \frac{1}{n} \sum_{i=1}^n a = \frac{1}{n} na = a,$$

as expected.

8. We are asked to show that

> If $\mathsf{A}\mathbf{x} = \mathbf{b}$ is consistent, then every solution of $\mathsf{A}^t\mathsf{A}\mathbf{x} = \mathsf{A}^t\mathbf{b}$ also solves the original matrix equation.

So, we are given that the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ is consistent, i.e. there are vectors $\mathbf{x}$ which satisfy it. Let us take $\mathbf{y}$ to be such a vector, i.e. $\mathsf{A}\mathbf{y} = \mathbf{b}$. To show that this result is true, we have to show that if the vector $\mathbf{x}$ is a solution of the matrix equation

$$\mathsf{A}^t\mathsf{A}\mathbf{x} = \mathsf{A}^t\mathbf{b},$$

then $\mathbf{x}$ is a solution of the original matrix equation, i.e. $\mathsf{A}\mathbf{x} = \mathbf{b}$ too. This can be done by noting that we can write

$$\mathsf{A}^t\mathsf{A}\mathbf{x} = \mathsf{A}^t\mathbf{b} \text{ as } \mathsf{A}^t(\mathsf{A}\mathbf{x} - \mathbf{b}) = \mathbf{0},$$

and as we have $\mathsf{A}\mathbf{y} = \mathbf{b}$ for some vector $\mathbf{y}$ (since this matrix equation is assumed to be consistent) we have

$$\mathsf{A}^t\mathsf{A}(\mathbf{x} - \mathbf{y}) = \mathbf{0},$$

i.e. the vector $\mathbf{x} - \mathbf{y} \in N(\mathsf{A}^t\mathsf{A})$. However, we can see that $N(\mathsf{A}^t\mathsf{A}) = N(\mathsf{A})$ since[13]

- For any $\mathbf{u} \in N(\mathsf{A}^t\mathsf{A})$, we have $\mathsf{A}^t\mathsf{A}\mathbf{u} = \mathbf{0}$ which means that

$$\mathbf{u}^t\mathsf{A}^t\mathsf{A}\mathbf{u} = \mathbf{0} \implies (\mathsf{A}\mathbf{u})^t\mathsf{A}\mathbf{u} = \mathbf{0} \implies \langle \mathsf{A}\mathbf{u}, \mathsf{A}\mathbf{u} \rangle = 0,$$

  using our convention. Thus, $\|\mathsf{A}\mathbf{u}\|^2 = 0$ and so we have $\mathsf{A}\mathbf{u} = \mathbf{0}$, i.e. $\mathbf{u} \in N(\mathsf{A})$. Consequently, $N(\mathsf{A}^t\mathsf{A}) \subseteq N(\mathsf{A})$.

- For any $\mathbf{u} \in N(\mathsf{A})$, we have $\mathsf{A}\mathbf{u} = \mathbf{0}$ and so, $\mathsf{A}^t\mathsf{A}\mathbf{u} = \mathbf{0}$ too. Consequently, $N(\mathsf{A}) \subseteq N(\mathsf{A}^t\mathsf{A})$.

[12] Incidentally, if we are trying to fit a *constant* function to the data, this is the one which will minimise the least square error. (Notice that the right-hand-side of this expression *is* a constant for any given set of data!)

[13] Compare this with the proof that $N(\mathsf{A}^t) = N(\mathsf{A}\mathsf{A}^t)$ given in the lectures. (This is in the proof of $\rho(\mathsf{A}) = \rho(\mathsf{A}^t\mathsf{A}) = \rho(\mathsf{A}\mathsf{A}^t)$ for any real matrix A.) Notice that substituting A^t for A in this result would also have given the result needed for this question!

and so we have $\mathbf{x} - \mathbf{y} \in N(\mathsf{A})$. Thus, for some vector $\mathbf{z} \in N(\mathsf{A})$ we have $\mathbf{x} - \mathbf{y} = \mathbf{z}$, and so our solution to the matrix equation

$$\mathsf{A}^t\mathsf{A}\mathbf{x} = \mathsf{A}^t\mathbf{b},$$

is $\mathbf{x} = \mathbf{y} + \mathbf{z}$ for some $\mathbf{y}$ and $\mathbf{z}$ such that $\mathsf{A}\mathbf{y} = \mathbf{b}$ and $\mathsf{A}\mathbf{z} = \mathbf{0}$ respectively. Consequently, we now note that this vector $\mathbf{x}$ is also a solution to the original matrix equation since

$$\mathsf{A}\mathbf{x} = \mathsf{A}(\mathbf{y} + \mathbf{z}) = \mathsf{A}\mathbf{y} + \mathsf{A}\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b},$$

as required.[14]

This result is important in the context of the least squares analyses considered in this problem sheet since it guarantees that: If the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ is consistent (i.e. there are no errors in the data), then any solution to the matrix equation

$$\mathsf{A}^t\mathsf{A}\mathbf{x} = \mathsf{A}^t\mathbf{b},$$

will have to be the *unique* solution given by

$$\mathbf{x} = (\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t\mathbf{b},$$

(as the matrix $\mathsf{A}^t\mathsf{A}$ is assumed to be invertible in such least squares analyses), and the result that we have just proved guarantees that this will also be a solution of $\mathsf{A}\mathbf{x} = \mathbf{b}$. That is, if there are no errors, then such a least squares analysis will still give the right solution as suggested in Footnote 3.

Note: Once we have established that $\mathbf{x} - \mathbf{y} \in N(\mathsf{A})$ in the proof of this result, the following steps should be obvious since we know that the solution set of the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ is just the affine set given by

$$\{\mathbf{x} \,|\, \mathsf{A}\mathbf{x} = \mathbf{b}\} = \{\mathbf{y} + \mathbf{z} \,|\, \mathsf{A}\mathbf{y} = \mathbf{b} \text{ and } \mathbf{z} \in N(A)\},$$

i.e. the solution set is just the translate of $N(\mathsf{A})$ by the vector $\mathbf{y}$. (See Figure 2.)

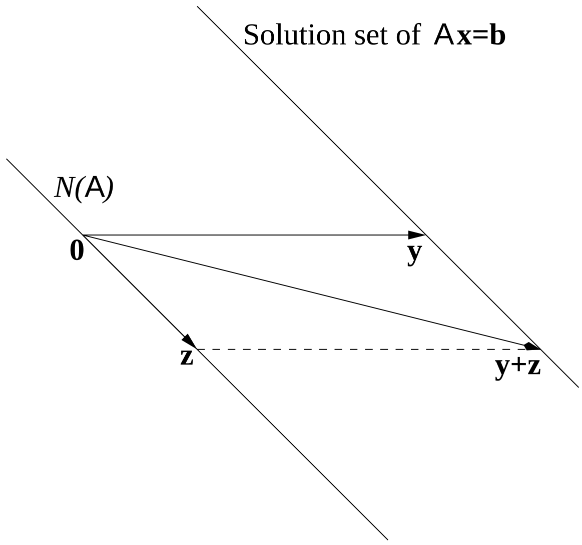

Figure 2: *Any vector that differs from a solution to the matrix equation* $\mathsf{A}\mathbf{x} = \mathbf{b}$*, say* $\mathbf{y}$*, by a vector* $\mathbf{z} \in N(\mathsf{A})$*, is also a solution to this matrix equation. (Notice that the solution set of the matrix equation* $\mathsf{A}\mathbf{x} = \mathbf{b}$ *is just the affine set given by the translate of* $N(\mathsf{A})$ *by* $\mathbf{y}$*.)*

[14]Notice that arguing that

$$\mathsf{A}^t\mathsf{A}\mathbf{x} = \mathsf{A}^t\mathbf{b} \implies \mathsf{A}\mathbf{x} = \mathbf{b},$$

is not sufficient to establish this result since it assumes that the matrix A^t is invertible and this may not be the case. Indeed, the matrix A^t may not even be square!

Remark: A simpler proof of the result in this question can be obtained by noting that since

$$\mathsf{A}^t(\mathsf{A}\mathbf{x} - \mathbf{b}) = \mathbf{0},$$

the vector given by $\mathsf{A}\mathbf{x} - \mathbf{b}$ is in the null space of A^t. However, we know that:

- $N(\mathsf{A}^t) = R(\mathsf{A})^\perp$ and so, we have $\mathsf{A}\mathbf{x} - \mathbf{b} \in R(\mathsf{A})^\perp$.
- The matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ is assumed to be consistent and so $\mathbf{b} \in R(\mathsf{A})$. But, the vector given by $\mathsf{A}\mathbf{x}$ is in the range of A as well. So, as $R(\mathsf{A})$ is closed under vector addition (since it is a subspace), we have $\mathsf{A}\mathbf{x} - \mathbf{b} \in R(\mathsf{A})$ too.

So, as the vector $\mathsf{A}\mathbf{x} - \mathbf{b}$ is in both $R(\mathsf{A})$ and $R(\mathsf{A})^\perp$, we have $\mathsf{A}\mathbf{x} - \mathbf{b} \in R(\mathsf{A}) \cap R(\mathsf{A})^\perp$. Consequently, as we know that a subspace and its orthogonal complement can be used to form a direct sum, we have

$$R(\mathsf{A}) \oplus R(\mathsf{A})^\perp \text{ and hence, } R(\mathsf{A}) \cap R(\mathsf{A})^\perp = \{\mathbf{0}\},$$

i.e. it must be the case that $\mathsf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$. Hence, any vector $\mathbf{x}$ satisfying the matrix equation

$$\mathsf{A}^t(\mathsf{A}\mathbf{x} - \mathbf{b}) = \mathbf{0},$$

will also satisfy the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ (as required).

Harder problems

Here are the solutions for the Harder Problems. As these were not covered in class the solutions will be a bit more detailed.

9. Let X be a subspace of the vector space V and let P denote the orthogonal projection onto X. We are asked to show that:

> If $\mathsf{Q} = \mathsf{I} - \mathsf{P}$, then for any subspace Y of V,
>
> $$\text{Lin}(X \cup Y) = \text{Lin}(X \cup Q(Y)),$$
>
> where $Q(Y)$ is given by
>
> $$Q(Y) = \{\mathsf{Q}\mathbf{y} \mid \mathbf{y} \in Y\},$$
>
> (i.e. it is the set of vectors which is found by multiplying each of the vectors in Y by Q.)[15]

So, to establish this result, we need to use the information given above to show that

$$\text{Lin}(X \cup Y) = \text{Lin}(X \cup Q(Y)),$$

and we can do this by noting that:

- Taking any vector $\mathbf{z} \in \text{Lin}(X \cup Y)$, we can write $\mathbf{z} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. But, since $\mathsf{Q} = \mathsf{I} - \mathsf{P}$ we can write

  $$\mathsf{Q}\mathbf{y} = (\mathsf{I} - \mathsf{P})\mathbf{y} = \mathbf{y} - \mathsf{P}\mathbf{y} \implies \mathbf{y} = \mathsf{Q}\mathbf{y} + \mathsf{P}\mathbf{y},$$

  where $\mathsf{P}\mathbf{y} \in X$ as P the [orthogonal] projection onto X and $\mathsf{Q}\mathbf{y} \in Q(Y)$. Thus, we have

  $$\mathbf{z} = \underbrace{\mathbf{x} + \mathsf{P}\mathbf{y}}_{\text{in } X} + \mathsf{Q}\mathbf{y},$$

  and so $\mathbf{z} \in \text{Lin}(X \cup Q(Y))$. Consequently, $\text{Lin}(X \cup Y) \subseteq \text{Lin}(X \cup Q(Y))$.

[15] This notation was also used in Question 10 on Problem Sheet 7. (Note that in the solution to this problem, we also established that sets like $Q(Y)$ were subspaces of V.)

- Taking any vector $\mathbf{z} \in \text{Lin}(X \cup Q(Y))$, we can write $\mathbf{z} = \mathbf{x} + \mathbf{v}$ where $\mathbf{x} \in X$ and $\mathbf{v} \in Q(Y)$. But, $\mathbf{v} \in Q(Y)$ means that there exists a $\mathbf{y} \in Y$ such that $\mathsf{Q}\mathbf{y} = \mathbf{v}$, that is,

$$\mathbf{v} = \mathsf{Q}\mathbf{y} = (\mathsf{I} - \mathsf{P})\mathbf{y} = \mathbf{y} - \mathsf{P}\mathbf{y}.$$

But, $\mathsf{P}\mathbf{y} \in X$ as P is the [orthogonal] projection of all vectors in V onto X. Thus, we have

$$\mathbf{z} = \underbrace{\mathbf{x} - \mathsf{P}\mathbf{y}}_{\text{in } X} + \mathbf{y},$$

and so $\mathbf{z} \in \text{Lin}(X \cup Y)$ as $\mathbf{y} \in Y$. Consequently, $\text{Lin}(X \cup Q(Y)) \subseteq \text{Lin}(X \cup Y)$.

Hence, we can see that $\text{Lin}(X \cup Q(Y)) = \text{Lin}(X \cup Y)$ as required.[16]

To interpret this result geometrically in the case where X and Y are one-dimensional subspaces of $\mathbb{R}^3$ look at Figure 2. Clearly, the beauty of this result is that it allows us to *construct* a subspace

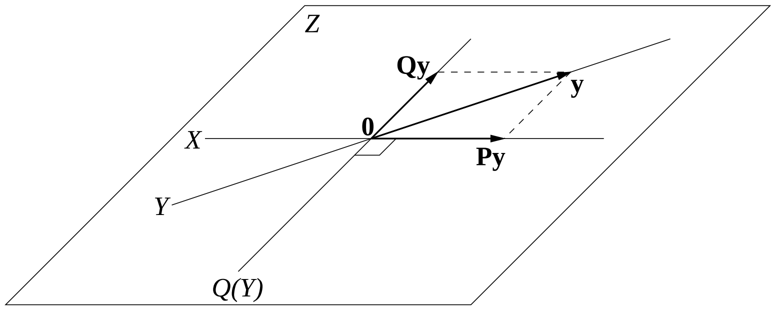

Figure 3: *This figure illustrates the result of Question 9 in $\mathbb{R}^3$. Here X and Y are one-dimensional subspaces of $\mathbb{R}^3$ and $Z = \text{Lin}(X \cup Y)$ is the plane through the origin containing X and Y. So, by the result above, $Z = \text{Lin}(X \cup Q(Y))$ too where $Q(Y)$ is a subspace containing vectors that are orthogonal to all of the vectors in X. (That is, $Q(Y) \subseteq X^\perp$.)*

$Q(Y) \subseteq \text{Lin}(X \cup Y)$ which only contains vectors that are *orthogonal* to every vector in X (i.e. $Q(Y) \subseteq X^\perp$) whilst allowing us to keep the same linear span.[17]

10. Let L and M be subspaces of the vector space V. We are asked to show that:

$$V = L \oplus M \text{ iff } L \cap M = \{\mathbf{0}\} \text{ and } \text{Lin}(L \cup M) = V.$$

(Recall that, by Theorem 2.4, if S is a set of vectors, then $\text{Lin}(S)$ is the smallest subspace that contains all of the vectors in S.) To do this, we use the result proved in the lectures, namely that:

$$V = L \oplus M \text{ iff } L \cap M = \{\mathbf{0}\} \text{ and } V = L + M,$$

i.e. we only need to show that $\text{Lin}(L \cup M) = L + M$.[18] But, this is fairly obvious since if $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_l\}$ are bases for the subspaces L and M respectively, then

$$\begin{aligned} \text{Lin}(L \cup M) &= \text{Lin}\{\mathbf{z} \,|\, \mathbf{z} \in L \text{ or } \mathbf{z} \in M\} \\ &= \left\{ \sum_{i=1}^{k} \alpha_i \mathbf{x}_i + \sum_{i=1}^{l} \beta_i \mathbf{y}_i \,|\, \alpha_i \text{ and } \beta_i \text{ are scalars} \right\} \\ &= \{\mathbf{x} + \mathbf{y} \,|\, \mathbf{x} \in L \text{ and } \mathbf{y} \in M\} \\ \therefore \quad \text{Lin}(L \cup M) &= L + M \end{aligned}$$

as required.

So, assuming that L and M are such that $L \oplus M = V$, we are now asked to prove that

[16] Notice that this result actually holds for *any* projection and not just orthogonal ones. However, the nice geometric interpretation of the result that follows does rely on P being an orthogonal projection.

[17] That is, the space spanned by $X \cup Q(Y)$ is the *same* as the space spanned by $X \cup Y$.

[18] Which is, incidentally, a result that we have taken to be intuitively obvious everywhere else!!

- $L^\perp \cap M^\perp = \{\mathbf{0}\}$.
- $[\text{Lin}(L^\perp \cup M^\perp)]^\perp = \{\mathbf{0}\}$.

We shall do each of these in turn:

Firstly, to prove that

$$L^\perp \cap M^\perp = \{\mathbf{0}\},$$

we consider any vector $\mathbf{z} \in L^\perp \cap M^\perp$, i.e.

$$\mathbf{z} \in L^\perp \text{ and } \mathbf{z} \in M^\perp.$$

Thus, by the definition of orthogonal complement, $\mathbf{z}$ must be such that

$$\forall \mathbf{x} \in L,\ \langle \mathbf{z}, \mathbf{x} \rangle = 0 \text{ and } \forall \mathbf{y} \in M,\ \langle \mathbf{z}, \mathbf{y} \rangle = 0,$$

and so, we can see that

$$\langle \mathbf{z}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{z}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle = 0 + 0 = 0,$$

i.e. since $V = L \oplus M$, $\mathbf{z}$ is orthogonal to every vector $\mathbf{x} + \mathbf{y} \in L + M = V$. But, this can only be the case if $\mathbf{z} = \mathbf{0}$ and so,

$$L^\perp \cap M^\perp = \{\mathbf{0}\},$$

as required.

Secondly, to prove that

$$[\text{Lin}(L^\perp \cup M^\perp)]^\perp = \{\mathbf{0}\},$$

we start by considering any vector $\mathbf{z} \in \text{Lin}(L^\perp \cup M^\perp)$, i.e. for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $L^\perp$ and $M^\perp$ respectively, we can write

$$\mathbf{z} = \alpha\mathbf{u} + \beta\mathbf{v},$$

where α and β are any scalars. So, by the definition of orthogonal complement, it must be the case that

$$\langle \mathbf{w}, \mathbf{z} \rangle = 0,$$

for any vector $\mathbf{w} \in [\text{Lin}(L^\perp \cup M^\perp)]^\perp$. However,

$$\langle \mathbf{w}, \mathbf{z} \rangle = 0 \implies \langle \mathbf{w}, \alpha\mathbf{u} + \beta\mathbf{v} \rangle = 0 \implies \alpha\langle \mathbf{w}, \mathbf{u} \rangle + \beta\langle \mathbf{w}, \mathbf{v} \rangle = 0,$$

and so, as this must hold for any scalars α and β, this implies that

$$\langle \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle = 0.$$

Thus, by the definition of orthogonal complement, it must be the case that $\mathbf{w} \in L$ and $\mathbf{w} \in M$, i.e. $\mathbf{w} \in L \cap M$. But, since $V = L \oplus M$, we know that $L \cap M = \{\mathbf{0}\}$, and so it must be the case that $\mathbf{w} = \mathbf{0}$. Consequently,

$$[\text{Lin}(L^\perp \cup M^\perp)]^\perp = \{\mathbf{0}\},$$

as required.

Hence, we are asked to deduce that $L^\perp \oplus M^\perp = V$, and to do this we use the result proved at the beginning of the question. So, from the lectures, we note that the orthogonal complement of any subset (and hence any subspace) of V is a subspace of V, and so we can see that

$$V = L^\perp \oplus M^\perp \text{ iff } L^\perp \cap M^\perp = \{\mathbf{0}\} \text{ and } \text{Lin}(L^\perp \cup M^\perp) = V.$$

But, as we have already established that

$$L^\perp \cap M^\perp = \{\mathbf{0}\},$$

to deduce the result we only need to establish that

$$\text{Lin}(L^\perp \cup M^\perp) = V.$$

However, we know that

$$[\text{Lin}(L^\perp \cup M^\perp)]^\perp = \{\mathbf{0}\},$$

and so as $S^{\perp\perp} = S$ (if S is a subspace of V), we have

$$\text{Lin}(L^\perp \cup M^\perp) = [\text{Lin}(L^\perp \cup M^\perp)]^{\perp\perp} = \{\mathbf{0}\}^\perp,$$

which means that

$$\text{Lin}(L^\perp \cup M^\perp) = \left\{\mathbf{x} \,\middle|\, \langle \mathbf{x}, \mathbf{y} \rangle = 0 \quad \forall \mathbf{y} \in \{\mathbf{0}\}\right\} = \{\mathbf{x} \,|\, \langle \mathbf{x}, \mathbf{0} \rangle = 0\}.$$

Thus, if we now note that

- Trivially, $\{\mathbf{x} \,|\, \langle \mathbf{x}, \mathbf{0} \rangle = 0\} \subseteq V$. (As we are working in this vector space.)
- For any $\mathbf{v} \in V$, we have $\langle \mathbf{v}, \mathbf{0} \rangle = 0$ and so $\mathbf{v} \in \{\mathbf{x} \,|\, \langle \mathbf{x}, \mathbf{0} \rangle = 0\}$. That is, $V \subseteq \{\mathbf{x} \,|\, \langle \mathbf{x}, \mathbf{0} \rangle = 0\}$.

it should be clear that

$$\text{Lin}(L^\perp \cup M^\perp) = V,$$

as required.