Further Mathematical Methods (Linear Algebra) 2002

Solutions For Problem Sheet 9

In this problem sheet, we derived a new result about orthogonal projections and used them to find least squares approximations to some simple functions. We also looked at Fourier series in theory and in practice.

1. We are given a vector $\mathbf{x} \in \mathbb{R}^n$ and told that S is the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$, i.e. $S = \text{Lin}\{\mathbf{x}\}$.[1] Now, we know that the matrix, P representing the orthogonal projection of $\mathbb{R}^n$ onto the range of A is given by $\mathsf{P} = \mathsf{A}(\mathsf{A}^t\mathsf{A})^{-1}\mathsf{A}^t$. So, if we let S be the range of A, i.e. A is the matrix with $\mathbf{x}$ as its [only] column vector, we can see that

$$\mathsf{P} = \mathbf{x}(\mathbf{x}^t\mathbf{x})^{-1}\mathbf{x}^t = \mathbf{x}(\|\mathbf{x}\|^2)^{-1}\mathbf{x}^t = \frac{\mathbf{x}\mathbf{x}^t}{\|\mathbf{x}\|^2},$$

as required.[2]

2. (a) We are asked to find an orthonormal basis for $\mathbb{P}_3^{[-\pi,\pi]}$, i.e. the vector space spanned by the vectors $\{\mathbf{1}, \mathbf{x}, \mathbf{x^2}, \mathbf{x^3}\}$, using the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx,$$

where $\mathbf{f} : x \to f(x)$, $\mathbf{g} : x \to g(x)$ for all $x \in [-\pi, \pi]$. Clearly, the set $\{\mathbf{1}, \mathbf{x}, \mathbf{x^2}, \mathbf{x^3}\}$ is a basis for $\mathbb{P}_3^{[-\pi,\pi]}$, and so we can construct an orthonormal basis for this space by using the Gram-Schmidt procedure, i.e.

- Taking $\mathbf{v}_1 = \mathbf{1}$, we get

  $$\|\mathbf{v}_1\|^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-\pi}^{\pi} 1\,dx = [x]_{-\pi}^{\pi} = 2\pi,$$

  and so we set $\mathbf{e}_1 = \mathbf{1}/\sqrt{2\pi}$.

- Taking $\mathbf{v}_2 = \mathbf{x}$, we construct the vector $\mathbf{u}_2$ where

  $$\mathbf{u}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{e}_1 \rangle \mathbf{e}_1 = \mathbf{x},$$

  since

  $$\langle \mathbf{v}_2, \mathbf{e}_1 \rangle = \frac{\langle \mathbf{x}, \mathbf{1} \rangle}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x\,dx = \frac{1}{\sqrt{2\pi}} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi} = 0.$$

  Then, we need to normalise this vector, i.e. as

  $$\|\mathbf{u}_2\|^2 = \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = \int_{-\pi}^{\pi} x^2\,dx = \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^3}{3},$$

  we set $\mathbf{e}_2 = \sqrt{\frac{3}{2\pi^3}}\,\mathbf{x}$.

[1] Note that if $\mathbf{x}$ is a unit vector, this derivation could replace part of our analysis of the $n \times n$ matrices E_i considered in Question 2 of Problem Sheet 8.

[2] Bearing in mind Footnote 1 in the Solutions for Problem Sheet 6, you may balk at this solution. However, our convention holds good here. To see this, we can re-run the argument above without assuming that $\mathbf{x}^t\mathbf{x}$ is a scalar, but treating it as a 1×1 matrix. Consider,

$$\mathsf{P} = \mathbf{x}(\mathbf{x}^t\mathbf{x})^{-1}\mathbf{x}^t = \mathbf{x}\left[\|\mathbf{x}\|^2\right]^{-1}\mathbf{x}^t.$$

But, the inverse of a 1×1 matrix $[a]$ is just $[a]^{-1} = [a^{-1}]$ as $[a][a]^{-1} = [a][a^{-1}] = [aa^{-1}] = [1]$, the 1×1 identity matrix. Thus,

$$\mathsf{P} = \mathbf{x}\left[\|\mathbf{x}\|^{-2}\right]\mathbf{x}^t.$$

However, the quantity $\left[\|\mathbf{x}\|^{-2}\right]\mathbf{x}^t$ is equivalent to having the $n \times 1$ vector $\mathbf{x}^t$ with each element multiplied by the scalar $\|\mathbf{x}\|^{-2}$. Consequently, taking out this common factor we get $\left[\|\mathbf{x}\|^{-2}\right]\mathbf{x}^t = \|\mathbf{x}\|^{-2}\mathbf{x}^t$ and hence the desired result.

- Taking $\mathbf{v}_3 = \mathbf{x^2}$, we construct the vector $\mathbf{u}_3$ where

$$\mathbf{u}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{e}_1 \rangle \mathbf{e}_1 - \langle \mathbf{v}_3, \mathbf{e}_2 \rangle \mathbf{e}_2 = \mathbf{x^2} - \frac{\pi^2}{3}\mathbf{1},$$

since

$$\langle \mathbf{v}_3, \mathbf{e}_1 \rangle = \frac{\langle \mathbf{x^2}, \mathbf{1} \rangle}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{\sqrt{2\pi}} \frac{2\pi^3}{3},$$

and,

$$\langle \mathbf{v}_3, \mathbf{e}_2 \rangle = \sqrt{\frac{3}{2\pi^3}} \langle \mathbf{x^2}, \mathbf{x} \rangle = \sqrt{\frac{3}{2\pi^3}} \int_{-\pi}^{\pi} x^3 \, dx = \sqrt{\frac{3}{2\pi^3}} \left[\frac{x^4}{4} \right]_{-\pi}^{\pi} = 0.$$

Then, we need to normalise this vector, i.e. as

$$\|\mathbf{u}_3\|^2 = \langle \mathbf{u}_3, \mathbf{u}_3 \rangle = \int_{-\pi}^{\pi} \left(x^2 - \frac{\pi^2}{3} \right)^2 dx = \int_{-\pi}^{\pi} \left(x^4 - \frac{2\pi^2}{3} x^2 + \frac{\pi^4}{9} \right) dx$$
$$= \left[\frac{x^5}{5} - \frac{2\pi^2}{9} x^3 + \frac{\pi^4}{9} x \right]_{-\pi}^{\pi} = \frac{8\pi^5}{45},$$

we set $\mathbf{e}_3 = \sqrt{\frac{5}{8\pi^5}} \, (3\mathbf{x^2} - \pi^2 \mathbf{1})$.

- Taking $\mathbf{v}_4 = \mathbf{x^3}$, we construct the vector $\mathbf{u}_4$ where

$$\mathbf{u}_4 = \mathbf{v}_4 - \langle \mathbf{v}_4, \mathbf{e}_1 \rangle \mathbf{e}_1 - \langle \mathbf{v}_4, \mathbf{e}_2 \rangle \mathbf{e}_2 - \langle \mathbf{v}_4, \mathbf{e}_3 \rangle \mathbf{e}_3 = \mathbf{x^3} - \frac{3\pi^2}{5}\mathbf{x},$$

since

$$\langle \mathbf{v}_4, \mathbf{e}_1 \rangle = \frac{\langle \mathbf{x^3}, \mathbf{1} \rangle}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^3 \, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{x^4}{4} \right]_{-\pi}^{\pi} = 0,$$

and,

$$\langle \mathbf{v}_4, \mathbf{e}_2 \rangle = \sqrt{\frac{3}{2\pi^3}} \langle \mathbf{x^3}, \mathbf{x} \rangle = \sqrt{\frac{3}{2\pi^3}} \int_{-\pi}^{\pi} x^4 \, dx = \sqrt{\frac{3}{2\pi^3}} \left[\frac{x^5}{5} \right]_{-\pi}^{\pi} = \sqrt{\frac{3}{2\pi^3}} \frac{2\pi^5}{5},$$

whilst,

$$\langle \mathbf{v}_4, \mathbf{e}_3 \rangle = \sqrt{\frac{5}{8\pi^5}} \langle \mathbf{x^3}, \mathbf{x^2} \rangle = \sqrt{\frac{5}{8\pi^5}} \int_{-\pi}^{\pi} x^5 \, dx = \sqrt{\frac{5}{8\pi^5}} \left[\frac{x^6}{6} \right]_{-\pi}^{\pi} = 0.$$

Then, we need to normalise this vector, i.e. as

$$\|\mathbf{u}_4\|^2 = \langle \mathbf{u}_4, \mathbf{u}_4 \rangle = \int_{-\pi}^{\pi} \left(x^3 - \frac{3\pi^2}{5} x \right)^2 dx = \int_{-\pi}^{\pi} \left(x^6 - \frac{6\pi^2}{5} x^4 + \frac{9\pi^4}{25} x^2 \right) dx$$
$$= \left[\frac{x^7}{7} - \frac{6\pi^2}{25} x^5 + \frac{3\pi^4}{25} x^3 \right]_{-\pi}^{\pi} = \frac{8\pi^7}{175},$$

we set $\mathbf{e}_3 = \sqrt{\frac{7}{8\pi^7}} \, (5\mathbf{x^3} - 3\pi^2 \mathbf{x})$.

Consequently, the set of vectors,

$$\left\{ \frac{1}{\sqrt{2\pi}} \mathbf{1}, \ \sqrt{\frac{3}{2\pi^3}} \mathbf{x}, \ \sqrt{\frac{5}{8\pi^5}} \, (3\mathbf{x^2} - \pi^2 \mathbf{1}), \ \sqrt{\frac{7}{8\pi^7}} \, (5\mathbf{x^3} - 3\pi^2 \mathbf{x}) \right\},$$

is an orthonormal basis for $\mathbb{P}_3^{[-\pi,\pi]}$.

Hence, to find a least squares approximation to $\sin x$ in $\mathbb{P}_3^{[-\pi,\pi]}$ we just have to evaluate the orthogonal projection of the vector **sinx** onto $\mathbb{P}_3^{[-\pi,\pi]}$, i.e.

$$\langle \mathbf{sinx}, \mathbf{e}_1 \rangle \, \mathbf{e}_1 + \langle \mathbf{sinx}, \mathbf{e}_2 \rangle \, \mathbf{e}_2 + \langle \mathbf{sinx}, \mathbf{e}_3 \rangle \, \mathbf{e}_3 + \langle \mathbf{sinx}, \mathbf{e}_4 \rangle \, \mathbf{e}_4,$$

where $\mathbf{sinx} : x \to \sin x$. To do this, it is convenient to note that using parts twice we have:

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} x^n \sin x \, dx = [-x^n \cos x]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} n x^{n-1} \cos x \, dx \\ &= [1-(-1)^n]\pi^n + n\left\{ \left[x^{n-1}\sin x\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (n-1)x^{n-2} \sin x \, dx \right\} \\ \therefore \quad I_n &= [1-(-1)^n]\pi^n - n(n-1)I_{n-2}, \end{aligned}$$

for $n \geq 2$, whilst

- If $n = 0$, we have
$$I_0 = \int_{-\pi}^{\pi} \sin x \, dx,$$
i.e. $I_0 = 0$.
- If $n = 1$, we have
$$I_1 = \int_{-\pi}^{\pi} x \sin x \, dx = [-x\cos x]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos x \, dx = 2\pi + 0,$$
i.e. $I_1 = 2\pi$.

So, we can see that the 'reduction formula' that we have just derived gives us:

- $I_2 = [1-(-1)^2]\pi^2 - 2 \times 1 \times I_0 = 0$, and
- $I_3 = [1-(-1)^3]\pi^3 - 3 \times 2 \times I_1 = 2\pi^3 - 12\pi$

Thus, we can see that looking at each of the terms that we have to evaluate we get:

- $\langle \mathbf{sinx}, \mathbf{e}_1 \rangle = \sqrt{\frac{1}{2\pi}} I_0 = 0$ and so, $\langle \mathbf{sinx}, \mathbf{e}_1 \rangle \mathbf{e}_1 = \mathbf{0}$.
- $\langle \mathbf{sinx}, \mathbf{e}_2 \rangle = \sqrt{\frac{3}{2\pi^3}} I_1 = \sqrt{\frac{3}{2\pi^3}}\, 2\pi$ and so, $\langle \mathbf{sinx}, \mathbf{e}_2 \rangle \mathbf{e}_2 = \frac{3}{2\pi^3}\, 2\pi \mathbf{x} = \frac{3}{\pi^2}\mathbf{x}$
- $\langle \mathbf{sinx}, \mathbf{e}_3 \rangle = \sqrt{\frac{5}{8\pi^5}} (3I_2 - \pi^2 I_0) = 0$ and so, $\langle \mathbf{sinx}, \mathbf{e}_3 \rangle \mathbf{e}_3 = \mathbf{0}$.
- $\langle \mathbf{sinx}, \mathbf{e}_4 \rangle = \sqrt{\frac{7}{8\pi^7}} (5I_3 - 3\pi^2 I_1) = \sqrt{\frac{7}{8\pi^7}} (10\pi^3 - 60\pi - 6\pi^3) = \sqrt{\frac{7}{8\pi^7}} (4\pi^2 - 60)$ and so,

  $\langle \mathbf{sinx}, \mathbf{e}_4 \rangle \mathbf{e}_4 = \frac{7}{2\pi^6}(\pi^2 - 15)(5\mathbf{x^3} - 3\pi^2\mathbf{x})$.

and so our desired approximation to $\sin x$ is

$$\frac{3}{\pi^2}\mathbf{x} + \frac{7}{2\pi^6}(\pi^2 - 15)(5\mathbf{x^3} - 3\pi^2\mathbf{x}),$$

which on simplifying yields

$$\frac{5}{2\pi^6}\left[(63 - 3\pi^2)\pi^2\mathbf{x} + (7\pi^2 - 105)\mathbf{x^3}\right].$$

So, to calculate the mean square error associated with this approximation, we use the result from the lectures, i.e.

$$\text{MSE} = \|\mathbf{f}\|^2 - \sum_{i=1}^{4} \langle \mathbf{f}, \mathbf{e}_i \rangle^2,$$

which as

$$\|\mathbf{f}\|^2 = \langle \mathbf{f}, \mathbf{f} \rangle = \int_{-\pi}^{\pi} \sin^2 x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [1 - \cos(2x)] \, dx = \frac{1}{2} 2\pi + 0 = \pi,$$

gives us

$$\text{MSE} = \pi - \left[0 + \frac{3}{2\pi^3}\,4\pi^2 + 0 + \frac{7}{8\pi^7}(4\pi^3 - 60\pi)^2\right] = \pi - \frac{20}{\pi} + \frac{420}{\pi^3} - \frac{3150}{\pi^5}.$$

Evaluating this expression then tells us that the mean square error for this approximation is 0.0276 to four decimal places.

(b) We are told that the first two [non-zero] terms in the Taylor series for $\sin x$ are given by

$$x - \frac{x^3}{3!},$$

for all $x \in [-\pi, \pi]$ and we are asked to find the mean square error between $\sin x$ and this approximation. This is fairly straightforward as most of the integrals that have to be evaluated have already been calculated in (a). So, using the definition of the mean square error,[3] it should be clear that:

$$\begin{aligned}
\text{MSE} &= \int_{-\pi}^{\pi} [f(x) - g(x)]^2\,dx = \int_{-\pi}^{\pi} \left[\sin x - x + \frac{x^3}{6}\right]^2 dx \\
&= \int_{-\pi}^{\pi} \left[\sin^2 x + x^2 + \frac{x^6}{36} - 2x\sin x + \frac{x^3}{3}\sin x - \frac{x^4}{3}\right] dx \\
&= \pi + \frac{2\pi^3}{3} + \frac{\pi^7}{126} - 4\pi + \frac{2\pi^3 - 12\pi}{3} - \frac{2\pi^5}{15} \\
\therefore\quad \text{MSE} &= \frac{\pi^7}{126} - \frac{2\pi^5}{15} + \frac{4\pi^3}{3} - 7\pi.
\end{aligned}$$

Evaluating this expression then tells us that the mean square error for this approximation is 2.5185 to four decimal places.

So, which of these cubics, i.e. the one calculated in (a) or the Taylor series, provides the best approximation to $\sin x$? Well, by looking at the mean square errors it should be clear that the least squares approximation calculated in (a) gives a *better* approximation to $\sin x$ than the Taylor series. Indeed, the mean square error for the Taylor series is roughly 100 times bigger than the mean square error for the result calculated in part (a). To put this all into perspective, these results are illustrated in Figure 1.

3. We are asked to find least squares approximations to two functions over the interval $[0, 1]$ using the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)\,dx,$$

where $\mathbf{f} : x \to f(x)$ and $\mathbf{g} : x \to g(x)$ for all $x \in [0, 1]$. We shall do each of these in turn:

Firstly, we are asked to find a least squares approximation to x by a function of the form $a + b\,\mathrm{e}^x$. That is, we need to find the orthogonal projection of the vector $\mathbf{x}$ onto the subspace spanned by the vectors $\mathbf{1}$ and $\mathbf{e^x}$ where $\mathbf{e^x} : x \to \mathrm{e}^x$. So, we note that the set of vectors $\{\mathbf{1}, \mathbf{e^x}\}$ is linearly independent since, for any $x \in [0, 1]$, the Wronskian for these functions is given by:

$$W(x) = \begin{vmatrix} 1 & \mathrm{e}^x \\ 0 & \mathrm{e}^x \end{vmatrix} = \mathrm{e}^x \neq 0,$$

i.e. this set of vectors gives us a basis for the subspace that we are going to orthogonally project onto. However, to do the orthogonal projection, we need an orthonormal basis for this subspace and to get this, we use the Gram-Schmidt procedure:

[3] Notice that we can *not* use the nice mean square error formula which we used in (a) here. This is because this Taylor series is not written in terms of an orthonormal basis.

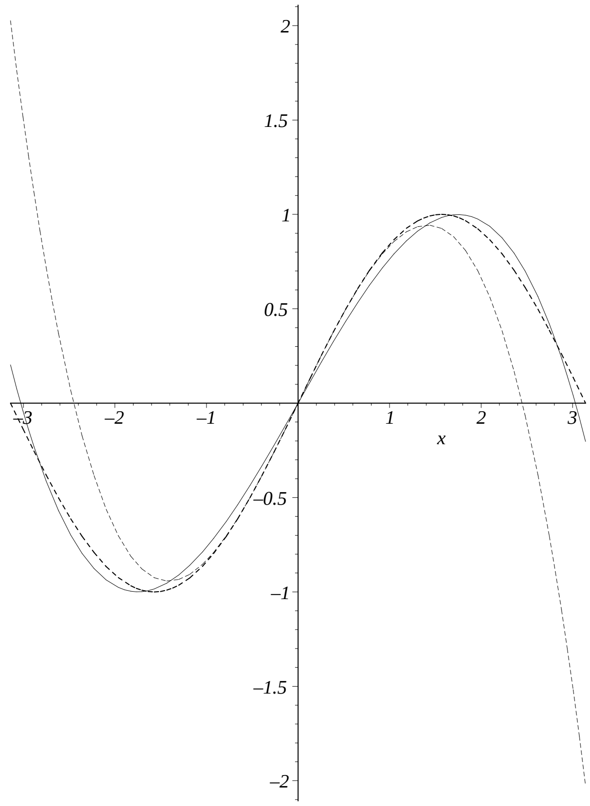


Figure 1: *This figure illustrates the function* $\sin x$ *(i.e. '- - -'), the first two [non-zero] terms in the Taylor series for* $\sin x$ *(i.e. '- - -') and the least squares approximation for* $\sin x$ *calculated in part (a) (i.e. '——'). Notice that the Taylor series gives a very good approximation to* $\sin x$ *for* $|x| \lesssim 1$*, but then starts to deviate rapidly from this function (we should expect this since we know that the Taylor series for* $\sin x$ *is a good approximation for 'small'* x*). However, the least squares approximation for* $\sin x$ *calculated in part (a) gives a good approximation for all values of* x *in the interval* $[-\pi, \pi]$.

- Taking $\mathbf{v}_1 = \mathbf{1}$, we get

$$\|\mathbf{v}_1\|^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_0^1 1\, dx = [x]_0^1 = 1,$$

  and so we set $\mathbf{e}_1 = \mathbf{1}$.

- Taking $\mathbf{v}_2 = \mathbf{e^x}$, we construct the vector $\mathbf{u}_2$ where

$$\mathbf{u}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{e}_1 \rangle \mathbf{e}_1 = \mathbf{e^x} - (\mathrm{e} - 1)\, \mathbf{1},$$

  since

$$\langle \mathbf{v}_2, \mathbf{e}_1 \rangle = \langle \mathbf{e^x}, \mathbf{1} \rangle = \int_0^1 \mathrm{e}^x\, dx = [\mathrm{e}^x]_0^1 = \mathrm{e} - 1.$$

  Then, we need to normalise this vector, i.e. as

$$\|\mathbf{u}_2\|^2 = \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = \int_0^1 [\mathrm{e}^x - (\mathrm{e}-1)]^2\, dx = \int_0^1 \left[\mathrm{e}^{2x} - 2(\mathrm{e}-1)\mathrm{e}^x + (\mathrm{e}-1)^2\right] dx$$
$$= \left[\frac{\mathrm{e}^{2x}}{2} - 2(\mathrm{e}-1)\mathrm{e}^x + (\mathrm{e}-1)^2 x\right]_0^1 = \left[\frac{\mathrm{e}^2}{2} - 2(\mathrm{e}-1)\mathrm{e} + (\mathrm{e}-1)^2\right] - \left[\frac{1}{2} - 2(\mathrm{e}-1)\right]$$

  i.e. $\|\mathbf{u}_2\|^2 = \frac{1}{2}(3-\mathrm{e})(\mathrm{e}-1)$,

  which is a positive quantity since $1 < \mathrm{e} < 3$, we set $\mathbf{e}_2 = \sqrt{\frac{2}{\alpha}}[\mathbf{e^x} - (\mathrm{e}-1)\,\mathbf{1}]$ where $\alpha = (3-\mathrm{e})(\mathrm{e}-1)$.

Consequently, the set of vectors,

$$\left\{ \mathbf{1},\ \sqrt{2}\frac{\mathbf{e^x} - (\mathrm{e}-1)\,\mathbf{1}}{\sqrt{(3-\mathrm{e})(\mathrm{e}-1)}} \right\},$$

is an orthonormal basis for the subspace spanned by the vectors $\{\mathbf{1}, \mathbf{e^x}\}$.

Hence, to find a least squares approximation to x in this subspace we just have to evaluate

$$\langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \mathbf{e}_2,$$

where $\mathbf{x} : x \to x$. To do this, we note that

- $\langle \mathbf{x}, \mathbf{1} \rangle = \int_0^1 x\, dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2}$ and so, $\langle \mathbf{x}, \mathbf{e}_1 \rangle = \frac{1}{2}$.
- $\langle \mathbf{x}, \mathbf{e^x} \rangle = \int_0^1 x\mathrm{e}^x\, dx = [x\mathrm{e}^x]_0^1 - \int_0^1 \mathrm{e}^x\, dx = \mathrm{e} - (\mathrm{e}-1) = 1$ and so, $\langle \mathbf{x}, \mathbf{e}_2 \rangle = \sqrt{\frac{2}{\alpha}}\left[\langle \mathbf{x}, \mathbf{e^x} \rangle - (\mathrm{e}-1)\langle \mathbf{x}, \mathbf{1} \rangle \right] = \sqrt{\frac{2}{\alpha}}\left[\frac{3-\mathrm{e}}{2} \right]$

and so our desired approximation to x is

$$\frac{1}{2}\mathbf{1} + \frac{\mathbf{e^x} - (\mathrm{e}-1)\,\mathbf{1}}{\mathrm{e}-1},$$

which on simplifying yields,

$$-\frac{1}{2}\mathbf{1} + \frac{\mathbf{e^x}}{\mathrm{e}-1}.$$

So, using the form given in the question, the least squares approximation to x over the interval $[0, 1]$ has $a = -1/2$ and $b = 1/(\mathrm{e}-1)$.

Secondly, we are asked to find a least squares approximation to e^x by a function of the form $a + bx$. That is, we need to find the orthogonal projection of the vector $\mathbf{e^x}$ onto the subspace spanned by the vectors $\mathbf{1}$ and $\mathbf{x}$. However, the set of vectors $\{\mathbf{1}, \mathbf{x}\}$ is actually a basis for the vector space $\mathbb{P}_1^{[0,1]}$ and we know from Question 6 of Problem Sheet 3 that an orthonormal basis for this vector space is

$$\left\{ \mathbf{1},\ \sqrt{3}(2\mathbf{x} - \mathbf{1}) \right\}.$$

Hence, to find the required least squares approximation to e^x we just have to evaluate

$$\langle \mathbf{e^x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{e^x}, \mathbf{e}_2 \rangle \mathbf{e}_2,$$

where $\mathbf{e}_1 = \mathbf{1}$ and $\mathbf{e}_2 = \sqrt{3}(2\mathbf{x} - \mathbf{1})$. To do this, we note that

- $\langle \mathbf{e^x}, \mathbf{1} \rangle = \int_0^1 \mathrm{e}^x\, dx = [\mathrm{e}^x]_0^1 = \mathrm{e} - 1$ and so, $\langle \mathbf{e^x}, \mathbf{e}_1 \rangle = \mathrm{e} - 1$.
- $\langle \mathbf{e^x}, \mathbf{x} \rangle = \int_0^1 x\mathrm{e}^x\, dx = 1$ (from above) and so, $\langle \mathbf{e^x}, \mathbf{e}_2 \rangle = \sqrt{3}(2\langle \mathbf{e^x}, \mathbf{x} \rangle - \langle \mathbf{e^x}, \mathbf{1} \rangle) = \sqrt{3}(3 - \mathrm{e})$.

and so our desired approximation to e^x is

$$(\mathrm{e}-1)\mathbf{1} + 3(3-\mathrm{e})(2\mathbf{x} - \mathbf{1}),$$

which on simplifying yields,

$$2(2\mathrm{e}-5)\mathbf{1} + 6(3-\mathrm{e})\mathbf{x}.$$

So, using the form given in the question, the least squares approximation to e^x over the interval $[0, 1]$ has $a = 2(2\mathrm{e}-5)$ and $b = 6(3-\mathrm{e})$.

4. We are asked to find a vector $\mathbf{q} \in \mathbb{P}_2^{\mathbb{R}}$ such that

$$p(1) = \langle \mathbf{p}, \mathbf{q} \rangle,$$

for every vector $\mathbf{p} : x \to p(x)$ in $\mathbb{P}_2^{\mathbb{R}}$ where the inner product defined on this vector space is given by

$$\langle \mathbf{p}, \mathbf{q} \rangle = \sum_{i=-1}^{1} p(i)q(i).$$

(Notice that this is an instantiation of the inner product given in Question 4 of Problem Sheet 3 where in this case, $-1, 0, 1$ are the three 'fixed and distinct real numbers' required to define it.) So, we need to find a quadratic $\mathbf{q}(x)$ [say $ax^2 + bx + c$] such that its inner product with *any* other quadratic $\mathbf{p}(x)$ [say $\alpha x^2 + \beta x + \gamma$], gives $p(1)$ [which in this case is $\alpha + \beta + \gamma$]. So, taking $\mathbf{p}$ and $\mathbf{q}$ as suggested we have

$$\langle \mathbf{p}, \mathbf{q} \rangle = \sum_{i=-1}^{1} p(i)q(i) = \underbrace{(a-b+c)(\alpha-\beta+\gamma)}_{i=-1} + \underbrace{(0+0+c)(0+0+\gamma)}_{i=0} + \underbrace{(a+b+c)(\alpha+\beta+\gamma)}_{i=1}$$

and this must equal $p(1)$, i.e. $\alpha + \beta + \gamma$. So, to find $\mathbf{q}$ we equate the coefficients of the terms in α, β and γ to get

$$\begin{array}{rcr} (a-b+c)+0+(a+b+c)=1 & & 2a{+}2c=1 \\ -(a-b+c)+0+(a+b+c)=1 & \Longrightarrow & 2b=1 \\ (a-b+c)+c+(a+b+c)=1 & & 2a{+}3c=1 \end{array}$$

which on solving gives us $a = b = 1/2$ and $c = 0$. Thus, the required quadratic $\mathbf{q}$ is given by

$$\mathbf{q}(x) = \frac{x^2 + x}{2}.$$

It may surprise you that there is only *one* quadratic that does this job, or indeed that there *is* one at all. But, there is actually some theory that guarantees its existence and uniqueness (although, we do not cover it in this course).

Notice that we can check that this answer is correct since, for any quadratic $\mathbf{p}(x) = ax^2 + bx + c$, it gives

$$\langle \mathbf{p}, \mathbf{q} \rangle = \underbrace{(a-b+c)(\tfrac{1}{2}-\tfrac{1}{2})}_{i=-1} + \underbrace{(0+0+c)(0+0)}_{i=0} + \underbrace{(a+b+c)(\tfrac{1}{2}+\tfrac{1}{2})}_{i=1} = a+b+c = p(1),$$

as desired.

The rest of the questions on this problem sheet are intended to develop your intuitions as to what a Fourier series *is* in terms of orthogonal projections onto subspaces. To do this, we start by considering an example where the function we are dealing with lies in $\mathrm{Lin}(G_n)$ for some n.[4] Indeed, Question 9 gives us a possible application of this kind of situation.

5. We are asked to consider the function $\cos(3x)$ defined over the interval $[-\pi, \pi]$. To find the Fourier series of orders 2, 3 and 4 that represent this function we evaluate integrals of the form

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(3x)\cos(kx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\cos(3+k)x + \cos(3-k)x\}\,dx,$$

where if $k = 3$, we have

$$a_3 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\cos(6x) + 1\}\,dx = 0 + \frac{2\pi}{2\pi} = 1,$$

and if $k \neq 3$, we get $a_k = 0$. Also, as $\cos(3x)$ is an even function, the b_k coefficients will be zero too. Thus, we find that the required Fourier series are:

[4] For simplicity, we shall deal with a cases where the required n is small!

Order	Fourier series	MSE
2	0	π
3	$\cos(3x)$	0
4	$\cos(3x)$	0

where the mean square error for the second order Fourier series is given by

$$\int_{-\pi}^{\pi} \cos^2(3x)dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(6x)+1]\, dx = \frac{2\pi}{2} = \pi,$$

and the third and fourth order Fourier series have a mean square error of zero. Which tells us [unsurprisingly, perhaps] that *the* Fourier series for $\cos(3x)$ *is* $\cos(3x)$ and that $\cos(3x)$ is in $\text{Lin}(G_3)$ and $\text{Lin}(G_4)$, but not in $\text{Lin}(G_2)$ (as one would expect from the mean square errors).

Other Problems.

Here we derived the formulae for calculating Fourier series. We then saw how to apply these formulae by finding the Fourier series of some simple functions.

Warning: The remaining solutions will make prodigious use of the following facts about sines and cosines:

- If $r \in \mathbb{N}$, then $\sin(r\pi) = 0$ and $\cos(r\pi) = (-1)^r$.
- The sine function is odd, i.e. $\sin(-x) = -\sin(x)$, whereas the cosine function is even, i.e. $\cos(-x) = \cos(x)$.
- The integral of an odd function over the interval $[-\pi, \pi]$ is zero.
- The following trigonometric identities:

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right),$$
$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right),$$
$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right),$$
$$-\cos\theta + \cos\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right),$$

  from which, among other things, the 'double angle' formulae follow.

You should bear them in mind when looking at many of the integrals that are evaluated below!

6. We are asked to consider the subset of $\mathbb{F}^{[-\pi,\pi]}$ given by $G_n = \{\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_n, \mathbf{g}_{n+1}, \ldots, \mathbf{g}_{2n}\}$ where

$$\mathbf{g}_0(x) = \frac{1}{\sqrt{2\pi}},$$

and

$$\mathbf{g}_1(x) = \frac{1}{\sqrt{\pi}}\cos(x),\ \mathbf{g}_2(x) = \frac{1}{\sqrt{\pi}}\cos(2x), \ldots,\ \mathbf{g}_n(x) = \frac{1}{\sqrt{\pi}}\cos(nx),$$

whereas, (for convenience we relabel these)

$$\mathbf{h}_1(x) = \mathbf{g}_{n+1}(x) = \frac{1}{\sqrt{\pi}}\sin(x),\ \mathbf{h}_2(x) = \mathbf{g}_{n+2}(x) = \frac{1}{\sqrt{\pi}}\sin(2x), \ldots,\ \mathbf{h}_n(x) = \mathbf{g}_{2n}(x) = \frac{1}{\sqrt{\pi}}\sin(nx).$$

To show that G_n is an orthonormal set when using the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx,$$

we have to establish that they satisfy the orthonormality condition, i.e.

$$\langle \mathbf{g}_i, \mathbf{g}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ \\ 0 & \text{if } i \neq j \end{cases}$$

Firstly, we can see that the vector $\mathbf{g}_0$ satisfies this condition, as

$$\langle \mathbf{g}_0, \mathbf{g}_0 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = 1,$$

and so it is unit; whilst for $1 \leq k \leq n$, we have

$$\langle \mathbf{g}_0, \mathbf{g}_k \rangle = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} \cos(kx) dx = \frac{1}{\sqrt{2}\pi} \left[\frac{\sin(kx)}{k} \right]_{-\pi}^{\pi} = 0,$$

and

$$\langle \mathbf{g}_0, \mathbf{h}_k \rangle = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} \sin(kx) dx = \frac{1}{\sqrt{2}\pi} \left[-\frac{\cos(kx)}{k} \right]_{-\pi}^{\pi} = 0,$$

and so the vector $\mathbf{g}_0$ is orthogonal to the other vectors in G_n. Secondly, note that for $1 \leq k, l \leq n$,

$$\langle \mathbf{g}_k, \mathbf{g}_l \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(lx) \cos(kx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\cos(k+l)x + \cos(k-l)x\} \, dx,$$

and so if $k \neq l$, then

$$\langle \mathbf{g}_k, \mathbf{g}_l \rangle = \frac{1}{2\pi} \left[\frac{\sin(k+l)x}{k+l} + \frac{\sin(k-l)x}{k-l} \right]_{-\pi}^{\pi} = 0,$$

whereas if $k = l$, we have

$$\langle \mathbf{g}_k, \mathbf{g}_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\cos(2kx) + 1\} \, dx = \frac{1}{2\pi} \left[\frac{\sin(2kx)}{2k} + x \right]_{-\pi}^{\pi} = 1.$$

Thus, we can see that, for $1 \leq k \leq n$, the vectors $\mathbf{g}_k$ are unit and mutually orthogonal. Thirdly, using a similar calculation, we note that for $n \leq k, l \leq n$,

$$\langle \mathbf{h}_k, \mathbf{h}_l \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kx) \sin(lx) dx = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \{\cos(k+l)x - \cos(k-l)x\} \, dx,$$

and so if $k \neq l$, then

$$\langle \mathbf{h}_k, \mathbf{h}_l \rangle = -\frac{1}{2\pi} \left[\frac{\sin(k+l)x}{k+l} - \frac{\sin(k-l)x}{k-l} \right]_{-\pi}^{\pi} = 0,$$

whereas if $k = l$, we have

$$\langle \mathbf{h}_k, \mathbf{h}_k \rangle = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \{\cos(2kx) - 1\} \, dx = -\frac{1}{2\pi} \left[\frac{\sin(2kx)}{2k} - x \right]_{-\pi}^{\pi} = 1.$$

Thus, we can see that, for $1 \leq k \leq n$, the vectors $\mathbf{h}_k$ are unit and orthogonal. Lastly, taking $1 \leq k, l \leq n$, we can see that

$$\langle \mathbf{g}_k, \mathbf{h}_l \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx) \sin(lx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\sin(k+l)x - \sin(k-l)x\} \, dx,$$

and so if $k \neq l$, then

$$\langle \mathbf{g}_k, \mathbf{h}_l \rangle = \frac{1}{2\pi} \left[-\frac{\cos(k+l)x}{k+l} + \frac{\cos(k-l)x}{k-l} \right]_{-\pi}^{\pi} = 0,$$

whereas if $k = l$, we have

$$\langle \mathbf{g}_k, \mathbf{h}_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(2kx) dx = \left[-\frac{\cos(2kx)}{2k} \right]_{-\pi}^{\pi} = 0.$$

Thus, for $1 \leq k, l \leq n$, the vectors $\mathbf{g}_k$ and $\mathbf{h}_l$ are mutually orthogonal. Consequently, we have shown that the vectors in G_n are orthonormal, as required.

Further, we are asked to show that any trigonometric polynomial of order n or less can be represented by a vector in $\text{Lin}(G_n)$. This is clearly the case as a trigonometric polynomial of degree n or less is of the form

$$c_0 + c_1 \cos(x) + \cdots + c_n \cos(nx) + d_1 \sin(x) + \cdots + d_n \sin(nx),$$

and this can therefore be represented by the vector

$$\sqrt{2\pi} c_0 \mathbf{g}_0 + \sqrt{\pi} c_1 \mathbf{g}_1 + \cdots + \sqrt{\pi} c_n \mathbf{g}_n + \sqrt{\pi} d_1 \mathbf{h}_1 + \cdots + \sqrt{\pi} d_n \mathbf{h}_n,$$

which is in $\text{Lin}(G_n)$, as required.

Note: In this question, we have established some orthogonality relations which may be useful later. For convenience, we list them here for easy reference:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx) \cos(lx) dx = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kx) \sin(lx) dx = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx) \sin(lx) dx = 0,$$

where k and l are positive integers. Further, just in case you haven't twigged, we note that

$$\int_{-\pi}^{\pi} \sin(kx) dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \cos(kx) dx = 0,$$

when, again, k is a positive integer.

7. Let us suppose that the vector $\mathbf{f} \in \mathbb{F}^{[-\pi,\pi]}$ represents a function that is not in $\text{Lin}(G_n)$, using the result from the lectures we can see that the orthogonal projection of $\mathbf{f}$ onto $\text{Lin}(G_n)$ is just[5]

$$\mathsf{P}\mathbf{f} = \sum_{k=0}^{2n} \langle \mathbf{f}, \mathbf{g}_k \rangle \mathbf{g}_k.$$

As this series represents the orthogonal projection, we know (from the lectures) that it will minimise the quantity given by $\|\mathbf{f} - \mathbf{g}\|$ where $\mathbf{g} \in \text{Lin}(G_n)$. That is, it minimises the quantity

$$\|\mathbf{f} - \mathbf{g}\|^2 = \langle \mathbf{f} - \mathbf{g}, \mathbf{f} - \mathbf{g} \rangle = \int_{-\pi}^{\pi} [f(x) - g(x)]^2 \, dx,$$

where $\mathbf{f} : x \to f(x)$ and $\mathbf{g} : x \to g(x)$.

Further, we must show that this series can be written as

$$\frac{a_0}{2} + \sum_{k=1}^{n} \left\{ a_k \cos(kx) + b_k \sin(kx) \right\},$$

[5]Clearly, as $G_n \subseteq \mathbb{F}^{[-\pi,\pi]}$ and we are considering the subspace given by $\text{Lin}(G_n)$. (This *is* a subspace by Theorem 2.4.)

where the coefficients are given by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \quad \text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx.$$

To do this, we note that

$$\mathsf{P}\mathbf{f}(x) = \sum_{k=0}^{2n}\langle \mathbf{f}, \mathbf{g}_k\rangle \mathbf{g}_k = \langle \mathbf{f}, \mathbf{g}_0\rangle \mathbf{g}_0 + \sum_{k=1}^{n}\langle \mathbf{f}, \mathbf{g}_k\rangle \mathbf{g}_k + \sum_{k=n+1}^{2n}\langle \mathbf{f}, \mathbf{g}_k\rangle \mathbf{g}_k,$$

and rewriting this in terms of functions we get[6]

$$\begin{aligned}\mathsf{P}\mathbf{f} = \left\{\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(u)du\right\}\frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n}\left\{\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(u)\cos(ku)du\right\}\frac{1}{\sqrt{\pi}}\cos(kx)\\ + \sum_{k=1}^{n}\left\{\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(u)\sin(ku)du\right\}\frac{1}{\sqrt{\pi}}\sin(kx),\end{aligned}$$

which gives the required result when we replace the integrals with the appropriate coefficients.[7] Indeed, as noted in the question, this is the Fourier series of order n representing $f(x)$.

As it turns out, this series can sometimes be simplified due to the fact that the integral of an odd function over the range $[-\pi, \pi]$ is zero.[8] So, if $f(x)$ is an odd (even) function, then as the product of an odd (even) function and an even (odd) function is an odd function, we can see that the a_k (b_k) coefficients will be zero. Thus, a useful result is that the Fourier series of order n will be

$$\sum_{k=1}^{n} b_k \sin(kx) \quad \text{if } f(x) \text{ is odd, and}$$

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) \quad \text{if } f(x) \text{ is even.}$$

8. Following on from Question 5, when we are asked to find the Fourier series of order n representing the function $\sin(3x)$ over the interval $[-\pi, \pi]$, it should be obvious (i.e. there should be no need to integrate!) that it is just $\sin(3x)$ if $n \geq 3$ and zero if $0 \leq n \leq 2$. The corresponding mean square errors are zero if $n \geq 3$ and

$$\int_{-\pi}^{\pi} \sin^2(3x)dx = \frac{1}{2}\int_{-\pi}^{\pi}[1 - \cos(6x)]dx = \frac{2\pi}{2} = \pi,$$

if $0 \leq n \leq 2$. But, if you are not convinced, we can verify this result by calculating the coefficients. In this case, because $\sin(3x)$ is an odd function the a_k are zero and so we just have to evaluate (for $k > 0$) an integral of the form

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(3x)\sin(kx)dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\cos(3+k)x + \cos(3-k)x\}\,dx,$$

where if $k = 3$, we have

$$b_3 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{\cos(6x) + 1\}\,dx = 0 + \frac{2\pi}{2\pi} = 1,$$

and if $k \neq 3$, we get $a_k = 0$, as before.

[6] When writing the integrals that correspond to the inner products, we use the dummy variable 'u' to replace the variable 'x' to avoid confusion.

[7] Notice that the coefficient a_0 is a special case of the coefficient a_k (as if we set $k = 0$ in the formula for a_k, $\cos(kx) = 1$). This little simplification is the reason for the 'extra' factor of a half in the first term of this series.

[8] Again, recall that an odd function is such that $f(-x) = -f(x)$ and an even function is such that $f(-x) = f(x)$.

Note: In case you hadn't guessed, this all works because the vectors being used are linearly independent[9] and so the series that we find is unique. Thus, whether we calculate a Fourier series by calculating the coefficients, or by some other method (say, using trigonometric identities), we will always get the same result! To illustrate this, let us consider a *bonus* question!

Question: Find *the* Fourier series for $\cos^3 x$ and $\sin^3 x$ by evaluating the coefficients. Further, amaze your friends by doing this in four lines using complex numbers!

Answer: As $\cos^3 x$ is an even function, the b_k coefficients are zero and so we only need to find the a_k, i.e.

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^3 x \cos(kx) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x \cos x \cos(kx) dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \{\cos(2x) + 1\}\{\cos(k+1)x + \cos(k-1)x\}\, dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \{\cos(2x)\cos(k+1)x + \cos(2x)\cos(k-1)x + \cos(k+1)x + \cos(k-1)x\}\, dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left\{\frac{1}{2}[\cos(k+3)x + \cos(1-k)x] + \frac{1}{2}[\cos(k+1)x + \cos(3-k)x] \right. \\
&\qquad\qquad\qquad \left. + \cos(k+1)x + \cos(k-1)x \right\} dx \\
\therefore \quad a_k &= \frac{1}{8\pi}\int_{-\pi}^{\pi} \{\cos(k+3)x + \cos(k-3)x + 3\cos(k+1)x + 3\cos(k-1)x\}\, dx.
\end{aligned}$$

Thus, for $k = 1$ and $k = 3$ we get $a_1 = 3/4$ and $a_3 = 1/4$ respectively, whereas for other non-negative values of k we get $a_k = 0$. Consequently, the Fourier series for $\cos^3 x$ is

$$\frac{1}{4}[3\cos x + \cos(3x)].$$

Similarly, for $\sin^3 x$, an odd function, the a_k coefficients are zero and so we only need to find the b_k, i.e.

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3 x \cos(kx) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x \sin x \sin(kx) dx \\
&= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \{1 - \cos(2x)\}\{\cos(k+1)x - \cos(k-1)x\}\, dx \\
&= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \{\cos(k+1)x - \cos(k-1)x - \cos(2x)\cos(k+1)x + \cos(2x)\cos(k-1)x\}\, dx \\
&= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \left\{\cos(k+1)x - \cos(k-1)x - \frac{1}{2}[\cos(k+3)x + \cos(1-k)x] \right. \\
&\qquad\qquad\qquad \left. + \frac{1}{2}[\cos(k+1)x + \cos(3-k)x] \right\} dx \\
\therefore \quad b_k &= \frac{1}{8\pi}\int_{-\pi}^{\pi} \{\cos(k+3)x - \cos(k-3)x - 3\cos(k+1)x + 3\cos(k-1)x\}\, dx.
\end{aligned}$$

Thus, for $k = 1$ and $k = 3$ we get $b_1 = 3/4$ and $b_3 = -1/4$ respectively, whereas for other positive values of k we get $b_k = 0$. Consequently, the Fourier series for $\sin^3 x$ is

$$\frac{1}{4}[3\sin x - \sin(3x)].$$

These results should look familiar as they are just the 'triple angle' identities for sines and cosines.

An alternative way to derive them would be to use complex numbers, and it is convenient for us to pause for a moment to note some useful results:

[9] Recall that in Question 5 of Problem Sheet 3, we established that orthogonal vectors are linearly independent.

We can write complex numbers in their exponential form, i.e.

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta,$$

which allows us to write

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

This form is particularly useful when used in conjunction with De Moivre's theorem, which tells us that

$$(e^{i\theta})^n = (\cos\theta \pm i\sin\theta)^n = \cos(n\theta) \pm i\sin(n\theta).$$

In what follows we use these ideas together with a rather nice substitution, namely

$$z = e^{i\theta}.$$

(Notice that this implies that $1/z = z^{-1} = e^{-i\theta}$.)

So, to proceed we consider the following identity[10]

$$\left(z \pm \frac{1}{z}\right)^3 = z^3 \pm 3z^2\frac{1}{z} + 3z\frac{1}{z^2} \pm \frac{1}{z^3} = z^3 \pm \frac{1}{z^3} \pm 3\left(z \pm \frac{1}{z}\right).$$

Now, if we let $z = e^{ix}$, and take the '+' we get

$$2^3\cos^3 x = 2\cos 3x + 3(2\cos x) \quad \Longrightarrow \quad 4\cos^3 x = \cos 3x + 3\cos x,$$

whereas, taking the '$-$' we find

$$(2i)^3\sin^3 x = 2i\sin 3x - 3(2i\sin x) \quad \Longrightarrow \quad -4\sin^3 x = \sin 3x - 3\sin x,$$

which are the desired results.

9. We are asked to show that

$$\frac{1}{2} + \cos(x) + \cos(2x) + \cdots + \cos(nx) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x},$$

without integrating to find the coefficients and assuming that $x \neq 2r\pi$ where $r \in \mathbb{Z}$. To do this we consider the series

$$S_n = 1 + e^{ix} + e^{2ix} + \cdots + e^{inx},$$

which is a geometric progression, and so summing this we get

$$S_n = \frac{1-e^{i(n+1)x}}{1-e^{ix}},$$

provided that $x \neq 2r\pi$ (as if this were the case, $e^{ix} = 1$). Multiplying the top and bottom of this expression by $e^{-\frac{1}{2}ix}$ gives

$$\begin{aligned} S_n &= \frac{e^{-\frac{1}{2}ix} - e^{i(n+\frac{1}{2})x}}{e^{-\frac{1}{2}ix} - e^{\frac{1}{2}ix}} \\ &= \frac{[\cos(\frac{1}{2}x) - i\sin(\frac{1}{2}x)] - [\cos(n+\frac{1}{2})x + i\sin(n+\frac{1}{2})x]}{-2i\sin(\frac{1}{2}x)} \\ S_n &= \frac{i[\cos(\frac{1}{2}x) - \cos(n+\frac{1}{2})x] + [\sin(\frac{1}{2}x) + \sin(n+\frac{1}{2})x]}{2\sin(\frac{1}{2}x)}, \end{aligned}$$

[10] To get this, use the Binomial Theorem (remember?) as I have, or just expand out the brackets. Either way this should be obvious!

and taking the real part of this expression yields

$$1 + \cos(x) + \cos(2x) + \cdots + \cos(nx) = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})x}{2\sin(\frac{1}{2}x)},$$

which gives the desired result. If the condition that $x \neq 2r\pi$ with $r \in \mathbb{N}$ is violated, then the left-hand-side of this expression becomes

$$\frac{1}{2} + \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = \frac{1}{2} + n.$$

But, the right-hand-side is indeterminate and so we need to evaluate

$$\begin{aligned}
\lim_{x\to 2r\pi} \frac{\sin(n+\frac{1}{2})x}{2\sin(\frac{1}{2}x)} &= \lim_{x\to 2r\pi} \frac{(n+\frac{1}{2})\cos(n+\frac{1}{2})x}{\cos(\frac{1}{2}x)} && \text{:Using l'Hôpital's rule.} \\
&= \frac{(n+\frac{1}{2})\cos(2n+1)r\pi}{\cos(r\pi)} \\
&= \frac{(n+\frac{1}{2})(-1)^{(2n+1)r}}{(-1)^r} \\
\therefore \quad \lim_{x\to 2r\pi} \frac{\sin(n+\frac{1}{2})x}{2\sin(\frac{1}{2}x)} &= n + \frac{1}{2}.
\end{aligned}$$

Thus, the result still holds if $x = 2r\pi$ with $r \in \mathbb{N}$.

10. We are asked to find the Fourier series of order n of three functions which are defined in the interval $[-\pi, \pi]$. Further, we must calculate the mean square error in each case.

The first function to consider is $f(x) = \pi - x$. To find the required Fourier series, we have to calculate the coefficients by evaluating integrals of the form

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)\cos(kx)dx,$$

(see Question 7) and so for $k = 0$, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)dx = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = 2\pi,$$

whereas, for $k \neq 0$, we have

$$a_k = \frac{1}{\pi}\left\{\left[(\pi - x)\frac{\sin(kx)}{k}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\sin(kx)}{k}dx\right\} = 0.$$

Also, for $k \neq 0$, we have to evaluate integrals of the form

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)\sin(kx)dx \\
&= \frac{1}{\pi}\left\{\left[(\pi - x)\frac{-\cos(kx)}{k}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\cos(kx)}{k}dx\right\} \\
&= \frac{1}{\pi}\left[0 - 2\pi\frac{-\cos(-k\pi)}{k}\right] - 0 \\
\therefore \quad b_k &= 2\frac{(-1)^k}{k},
\end{aligned}$$

(again, see Question 7). Thus, the Fourier series of order n representing the function $\pi - x$ is

$$\pi + 2\sum_{k=1}^{n} \frac{(-1)^k}{k}\sin(kx).$$

Using this, the mean square error between a function, $f(x)$ and the corresponding Fourier series of order n, $g(x)$ is given by

$$\text{MSE} = \int_{-\pi}^{\pi} [f(x) - g(x)]^2 \, dx,$$

and so in this case we have

$$\text{MSE} = \int_{-\pi}^{\pi} \left[x + 2\sum_{k=1}^{n} \frac{(-1)^k}{k} \sin(kx) \right]^2 dx.$$

This looks a bit tricky, but squaring the bracket you should be able to convince yourself that this gives

$$\text{MSE} = \int_{-\pi}^{\pi} \left[x^2 + 4x\sum_{k=1}^{n} \frac{(-1)^k}{k} \sin(kx) + 4\sum_{k=1}^{n}\sum_{\substack{l=1 \\ (l\neq k)}}^{n} \frac{(-1)^{k+l}}{kl} \sin(kx)\sin(lx) \right.$$
$$\left. + 4\sum_{k=1}^{n} \frac{(-1)^{2k}}{k^2} \sin^2(kx) \right] dx.$$

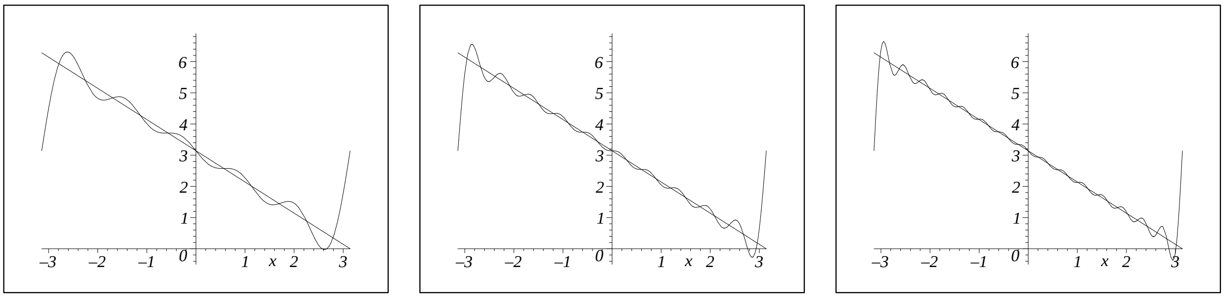
But, noting that

$$\int_{-\pi}^{\pi} x\sin(kx)dx = \left[x\frac{-\cos(kx)}{k} \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(kx)}{k} dx = -2\pi\frac{(-1)^k}{k},$$

as well as our earlier results, we can see that this is just

$$\text{MSE} = \frac{2}{3}\pi^3 - 8\pi\sum_{k=1}^{n} \frac{(-1)^{2k}}{k^2} + 0 + 4\pi\sum_{k=1}^{n} \frac{(-1)^{2k}}{k^2} = \frac{2}{3}\pi^3 - 4\pi\sum_{k=1}^{n} \frac{1}{k^2}.$$

This Fourier series is illustrated in Figure 2.



5 terms *10 terms* *15 terms*

Figure 2: *The function $\pi - x$ and its Fourier series of order 5, 10 and 15 respectively. (Notice that the Fourier series of order n for $\pi - x$ behaves unusually as $x \to \pm\pi$. There is a reason for this, but we will not discuss it in this course.)*

The second function to consider is $f(x) = x^2$. To find the required Fourier series, we have to calculate the coefficients by evaluating integrals of the form

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(kx)dx,$$

(see Question 7) and so for $k = 0$, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^2,$$

whereas, for $k \neq 0$, we have

$$\begin{aligned}
a_k &= \frac{1}{\pi}\left\{\left[x^2\frac{\sin(kx)}{k}\right]_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi} x\frac{\sin(kx)}{k}dx\right\} \\
&= 0 - \frac{2}{\pi k}\left\{\int_{-\pi}^{\pi} x\sin(kx)dx\right\} \\
&= -\frac{2}{\pi k}\left\{\left[x\frac{-\cos(kx)}{k}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(kx)}{k}dx\right\} \\
&= -\frac{2}{\pi k}\left[-2\pi\frac{(-1)^k}{k}\right] + 0 \\
\therefore\quad a_k &= 4\frac{(-1)^k}{k^2}.
\end{aligned}$$

Also, as $f(x) = x^2$ is an even function, we know (from the last part of Question 7) that the b_k (for $1 \leq k \leq n$) will be zero. Thus, the Fourier series of order n representing the function x^2 is

$$\frac{\pi^2}{3} + 4\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\cos(kx).$$

Using this, the mean square error between a function, $f(x)$ and the corresponding Fourier series of order n, $g(x)$ is given by

$$\text{MSE} = \int_{-\pi}^{\pi}[f(x) - g(x)]^2\,dx,$$

and so in this case we have

$$\text{MSE} = \int_{-\pi}^{\pi}\left[x^2 - \frac{\pi^2}{3} - 4\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\cos(kx)\right]^2 dx.$$

This [again,] looks a bit tricky, but squaring the bracket you should be able to convince yourself that this gives

$$\begin{aligned}
\text{MSE} = \int_{-\pi}^{\pi}\Bigg[&\left(x^2 - \frac{\pi^2}{3}\right)^2 - 8\left(x^2 - \frac{\pi^2}{3}\right)\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\cos(kx) \\
&+ 16\sum_{k=1}^{n}\sum_{\substack{l=1 \\ (l\neq k)}}^{n}\frac{(-1)^{k+l}}{k^2l^2}\cos(kx)\cos(lx) + 16\sum_{k=1}^{n}\frac{(-1)^{2k}}{k^4}\cos^2(kx)\Bigg]dx.
\end{aligned}$$

But, noting that

$$\begin{aligned}
\int_{-\pi}^{\pi}\left(x^2 - \frac{\pi^2}{3}\right)^2 dx = \int_{-\pi}^{\pi}\left(x^4 - 2\frac{\pi^2}{3}x^2 + \frac{\pi^4}{9}\right)dx &= \left[\frac{x^5}{5} - 2\frac{\pi^2}{3}\frac{x^3}{3} + \frac{\pi^4}{9}x\right]_{-\pi}^{\pi} \\
&= \frac{2}{5}\pi^5 - 4\frac{\pi^2}{3}\frac{\pi^3}{3} + 2\frac{\pi^4}{9}\pi = \frac{8}{45}\pi^5,
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\pi}^{\pi}\left(x^2 - \frac{\pi^2}{3}\right)\cos(kx)dx &= \left[\left(x^2 - \frac{\pi^2}{3}\right)\frac{\sin(kx)}{k}\right]_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi}x\frac{\sin(kx)}{k}dx \\
&= 0 - \frac{2}{k}\left\{\left[x\frac{-\cos(kx)}{k}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(kx)}{k}dx\right\} \\
&= \frac{2}{k}2\pi\frac{(-1)^k}{k} + 0 \\
\therefore\quad \int_{-\pi}^{\pi}\left(x^2 - \frac{\pi^2}{3}\right)\cos(kx)dx &= 4\pi\frac{(-1)^k}{k^2},
\end{aligned}$$

as well as our earlier results, we can see that this is just

$$\text{MSE} = \frac{8}{45}\pi^5 - 32\pi \sum_{k=1}^{n} \frac{(-1)^{2k}}{k^4} + 0 + 16\pi \sum_{k=1}^{n} \frac{(-1)^{2k}}{k^4} = \frac{8}{45}\pi^5 - 16\pi \sum_{k=1}^{n} \frac{1}{k^4}.$$

This Fourier series is illustrated in Figure 3.

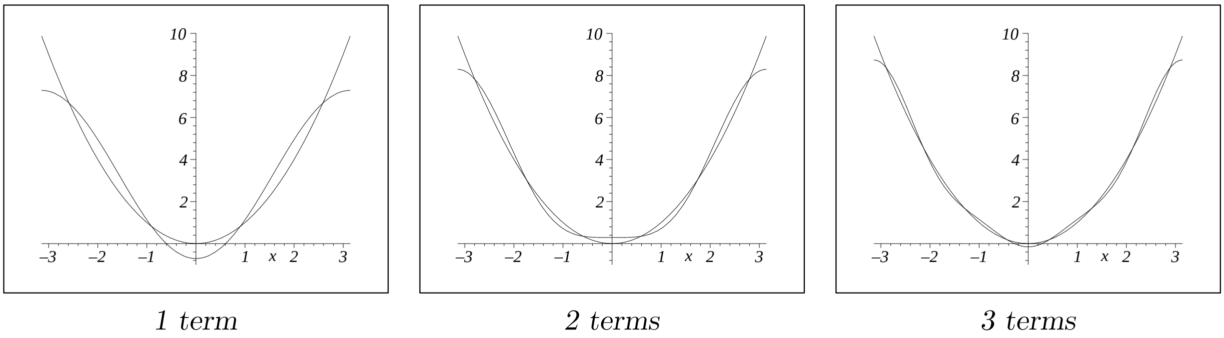


Figure 3: *The function x^2 and its Fourier series of order 1, 2 and 3 respectively.*

The third function to consider is $f(x) = |x|$. To find the required Fourier series, we have to calculate the coefficients by evaluating integrals of the form

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(kx) dx,$$

(see Question 7) and so for $k = 0$, we have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} (-x) dx + \int_{0}^{\pi} x dx \right\} = \frac{1}{\pi} \left\{ \left[-\frac{x^2}{2} \right]_{-\pi}^{0} + \left[\frac{x^2}{2} \right]_{0}^{\pi} \right\} = \pi,$$

whereas, for $k \neq 0$, we have

$$\begin{aligned} a_k &= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} (-x) \cos(kx) dx + \int_{0}^{\pi} x \cos(kx) dx \right\} \\ &= \frac{1}{\pi} \left\{ \left[-x \frac{\sin(kx)}{k} \right]_{-\pi}^{0} + \int_{-\pi}^{0} \frac{\sin(kx)}{k} dx + \left[x \frac{\sin(kx)}{k} \right]_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin(kx)}{k} dx \right\} \\ &= \frac{1}{\pi} \left\{ 0 + \int_{-\pi}^{0} \frac{\sin(kx)}{k} dx + 0 - \int_{0}^{\pi} \frac{\sin(kx)}{k} dx \right\} \\ &= \frac{1}{\pi} \left\{ \left[\frac{-\cos(kx)}{k^2} \right]_{-\pi}^{0} - \left[\frac{-\cos(kx)}{k^2} \right]_{0}^{\pi} \right\} \\ &= \frac{1}{\pi k^2} \left\{ \left[-1 + (-1)^k \right] - \left[-(-1)^k + 1 \right] \right\} \\ \therefore \quad a_k &= \frac{2}{\pi} \frac{(-1)^k - 1}{k^2}. \end{aligned}$$

Also, as $f(x) = |x|$ is an even function, we know (from the last part of Question 7) that the b_k (for $1 \leq k \leq n$) will be zero. Thus, the Fourier series of order n representing the function $|x|$ is

$$\frac{\pi}{2} + \frac{2}{\pi} \sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos(kx).$$

Using this, the mean square error between a function, $f(x)$ and the corresponding Fourier series of order n, $g(x)$ is given by

$$\text{MSE} = \int_{-\pi}^{\pi} [f(x) - g(x)]^2 \, dx,$$

and so in this case we have

$$\text{MSE} = \int_{-\pi}^{\pi} \left[|x| - \frac{\pi}{2} - \frac{2}{\pi} \sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos(kx) \right]^2 dx.$$

This [again,] looks a bit tricky, but squaring the bracket you should be able to convince yourself that this gives

$$\begin{aligned} \text{MSE} = \int_{-\pi}^{\pi} \Bigg[&\left(|x| - \frac{\pi}{2} \right)^2 - \frac{4}{\pi} \left(|x| - \frac{\pi}{2} \right) \sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos(kx) \\ &+ \frac{4}{\pi^2} \sum_{k=1}^{n} \sum_{\substack{l=1 \\ (l \neq k)}}^{n} \frac{(-1)^k - 1}{k^2} \frac{(-1)^l - 1}{l^2} \cos(kx)\cos(lx) + \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{[(-1)^k - 1]^2}{k^4} \cos^2(kx) \Bigg] dx. \end{aligned}$$

But, noting that

$$\begin{aligned} \int_{-\pi}^{\pi} \left(|x| - \frac{\pi}{2} \right)^2 dx &= \int_{-\pi}^{\pi} \left(|x|^2 - 2\frac{\pi}{2}|x| + \frac{\pi^2}{4} \right) dx = \int_{-\pi}^{\pi} \left(x^2 + \frac{\pi^2}{4} \right) - \pi \int_{-\pi}^{0} (-x) dx - \pi \int_{0}^{\pi} x dx \\ &= \left[\frac{x^3}{3} + \frac{\pi^2}{4} x \right]_{-\pi}^{\pi} + \pi \left[\frac{x^2}{2} \right]_{-\pi}^{0} - \pi \left[\frac{x^2}{2} \right]_{0}^{\pi} = 2 \left[\frac{\pi^3}{3} + \frac{\pi^3}{4} \right] - \frac{\pi^3}{2} - \frac{\pi^3}{2} = \frac{\pi^3}{6}, \end{aligned}$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} \left(|x| - \frac{\pi}{2} \right) \cos(kx) dx &= \int_{-\pi}^{0} \left(-x - \frac{\pi}{2} \right) \cos(kx) dx + \int_{0}^{\pi} \left(x - \frac{\pi}{2} \right) \cos(kx) dx \\ &= - \left[\left(x + \frac{\pi}{2} \right) \frac{\sin(kx)}{k} \right]_{-\pi}^{0} + \int_{-\pi}^{0} \frac{\sin(kx)}{k} dx \\ &\quad + \left[\left(x - \frac{\pi}{2} \right) \frac{\sin(kx)}{k} \right]_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin(kx)}{k} dx \\ &= 0 + \left[\frac{-\cos(kx)}{k^2} \right]_{-\pi}^{0} + 0 - \left[\frac{-\cos(kx)}{k^2} \right]_{0}^{\pi} \\ \therefore \quad \int_{-\pi}^{\pi} \left(|x| - \frac{\pi}{2} \right) \cos(kx) dx &= \frac{2}{k^2} \left[(-1)^k - 1 \right], \end{aligned}$$

as well as our earlier results, we can see that this is just

$$\text{MSE} = \frac{\pi^3}{6} - \frac{8}{\pi} \sum_{k=1}^{n} \frac{[(-1)^k - 1]^2}{k^4} + 0 + \frac{4}{\pi} \sum_{k=1}^{n} \frac{[(-1)^k - 1]^2}{k^4} = \frac{\pi^3}{6} - \frac{4}{\pi} \sum_{k=1}^{n} \frac{[(-1)^k - 1]^2}{k^4}.$$

This Fourier series is illustrated in Figure 4.

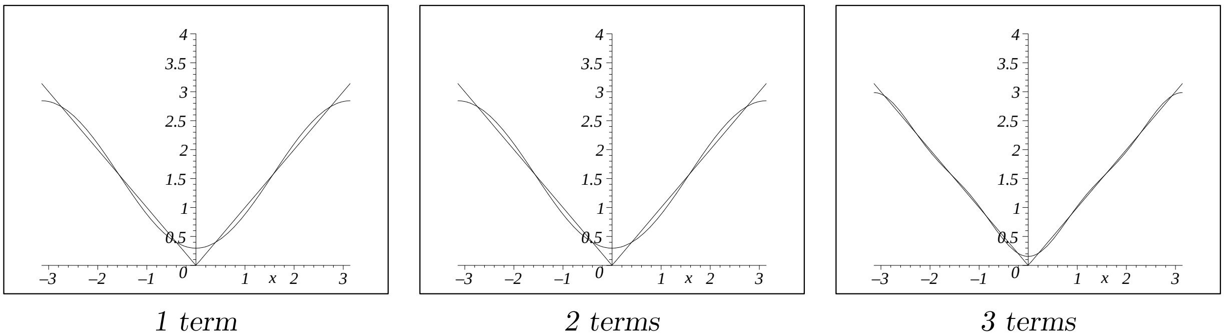


Figure 4: *The function $|x|$ and its Fourier series of order 1, 2 and 3 respectively. (Notice that the Fourier series of orders 1 and 2 are the same here as the second order term is zero!)*

Note: You may wonder why we have bothered to calculate the mean square error in this question (as it is not normally calculated, or even introduced, in methods courses). We take this opportunity to note the following facts:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$
$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90},$$
$$\sum_{k=1}^{\infty} \frac{[(-1)^k - 1]^2}{k^4} = \frac{\pi^4}{24}.$$

Thus, if we let $n \to \infty$ in the above results, the mean square errors will tend to zero. This indicates that if we take *the* Fourier series of these functions (i.e. their representation in terms of a trigonometric polynomial of 'infinite' degree), we will have an 'exact' expression for them in terms of sines and cosines. Indeed, this seems to imply that these functions are *in* the subspace of $\mathbb{F}^{[-\pi,\pi]}$ spanned by the vectors in G_∞. This idea is explored further (in a more transparent context) in Questions 5 and 8. Howvever, you may care to take a look at the illustrations of these Fourier series that are given in Figures 2, 3 and 4. (Notice how the Fourier series gives a better approximation to the function in question as we increase the number of terms!) Also observe how the mean square error is much easier to find if we use the result derived in the lectures (as we did in Question 2) instead of just evaluating the appropriate norm (as we did in Questions 5, 8 and 10)!

Remark: There is, of course, much more to Fourier series than we have covered here. Once again, you should note that *the* Fourier series of a function is the series that we get as n (the 'order' in our exposition) tends to infinity.