Mixing Colour(ing)s in Graphs

JAN VAN DEN HEUVEL

includes joint work with LUIS CERECEDA (LSE)
and MATTHEW JOHNSON (Durham)

after an idea by HAJO BROERSMA (Durham)

Centre for Discrete and Applicable Mathematics
Department of Mathematics

London School of Economics and Political Science

First definitions

- *graph* $G = (V, E)$: finite, simple, no loops, n vertices

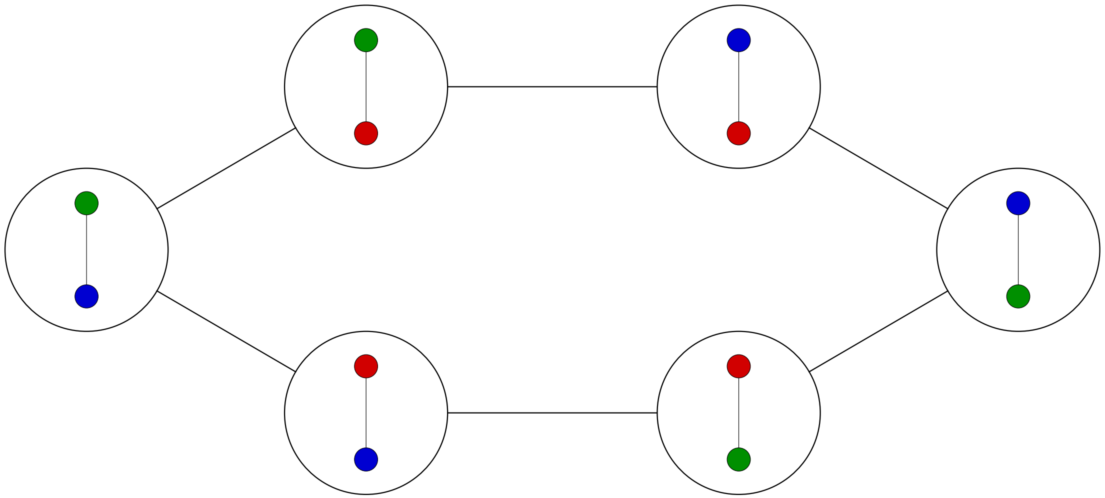
- *k-colouring* of G : proper vertex-colouring using colours from $\{1, 2, \ldots, k\}$
  - we always assume $k \geq \chi(G)$
  - we use $\alpha, \beta, \ldots$ to indicate k-colourings

- *k-colour graph* $\mathcal{C}(G; k)$
  - vertices are the k-colourings of G
  - two k-colourings are adjacent if they differ in the colour on exactly one vertex of G

Some examples

- 2-colour graph for K_2:

- 3-colour graph for K_2:



Central question

- *k-colour graph* $\mathcal{C}(G;k)$: two k-colourings are adjacent
  if they differ in the colour on exactly one vertex of G

General question

- Given G and k, what can we say about $\mathcal{C}(G;k)$?

In particular

- for what G and k is $\mathcal{C}(G;k)$ connected ?
  - intuitively : can we go between any two k-colourings by recolouring one vertex at the time ?

Terminology : $\mathcal{C}(G;k)$ is connected $\iff$ G is k-mixing

Research on $\mathcal{C}(G;k)$

- little research in pure graph theory

- related to work in <u>theoretical physics</u> on
  Glauber dynamics of k-state anti-ferromagnetic Potts models at zero temperature

- related to work in <u>theoretical computer science</u> on
  Markov chain Monte Carlo methods for generating random k-colourings

The Markov chain of k-colourings

define the Markov chain $\underline{\mathcal{M}(G;k)}$ as follows :

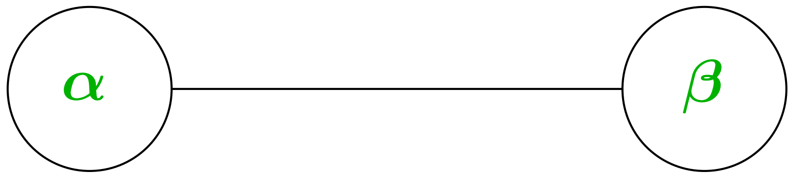
- states are the k-colourings of G
- transitions from a state (colouring) α :
  - choose a vertex v uniformly at random
  - choose a colour $c \in \{1, \ldots, k\}$ uniformly at random
  - recolour vertex v with colour c if possible

    (i.e., must stay a proper colouring)

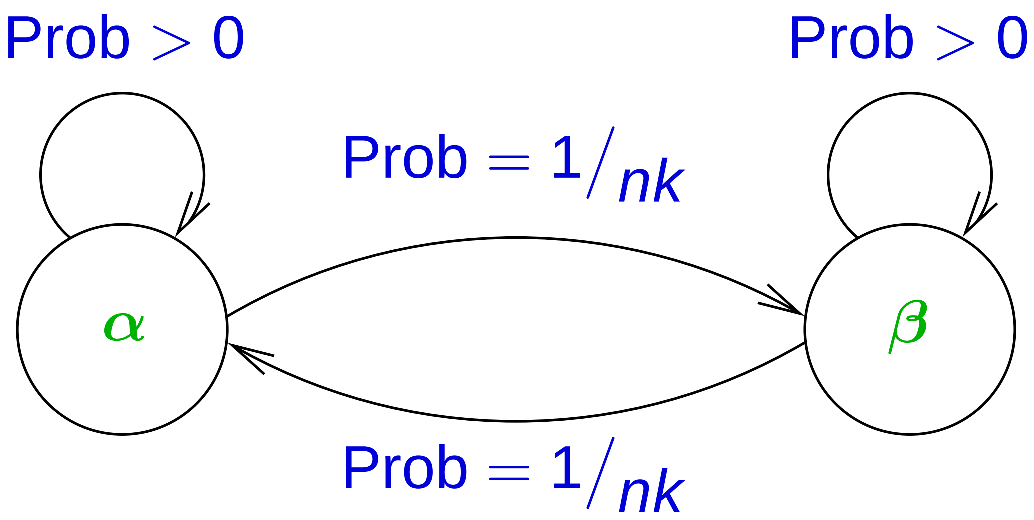
    $\Longrightarrow$ make this new k-colouring the new state
  - otherwise, the state remains the same colouring α

The colour graph and the Markov chain

- edge in the colour graph $\mathcal{C}(G;k)$:



- gives in the Markov Chain $\mathcal{M}(G;k)$:



A little Markov chain theory

- $\mathcal{M}(G;k)$ <u>irreducible</u> $\iff$ $\mathcal{C}(G;k)$ connected
- $\mathcal{M}(G;k)$ <u>aperiodic</u> (since $\text{Prob}(\alpha,\alpha) > 0$)

hence : $\mathcal{C}(G;k)$ connected $\implies$ $\mathcal{M}(G;k)$ <u>ergodic</u>

with : unique <u>stationary distribution</u> $\pi \equiv 1/_{\#\ k\text{-colourings}}$

<u>**which means**</u> :

- starting at any k-colouring α,
  walking through the Markov chain long enough,
  the final state can be any k-colouring
  with (almost) equal probability

Rapid mixing in Markov chains

Problem:

- how long is "long enough"?
- G is rapidly mixing for k colours:

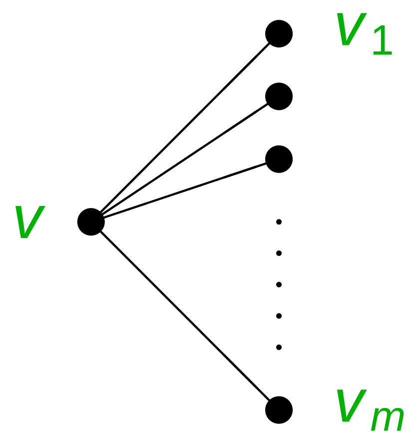
  starting at any k-colouring α, a state "close" to stationarity is reached after a number of steps that is polynomial in n

rapid mixing gives:

- a "feasible" way to obtain (almost) uniformly random samples of k-colourings
- a way to approximately count the number of k-colourings of a graph G

Example: mixing but not rapidly mixing

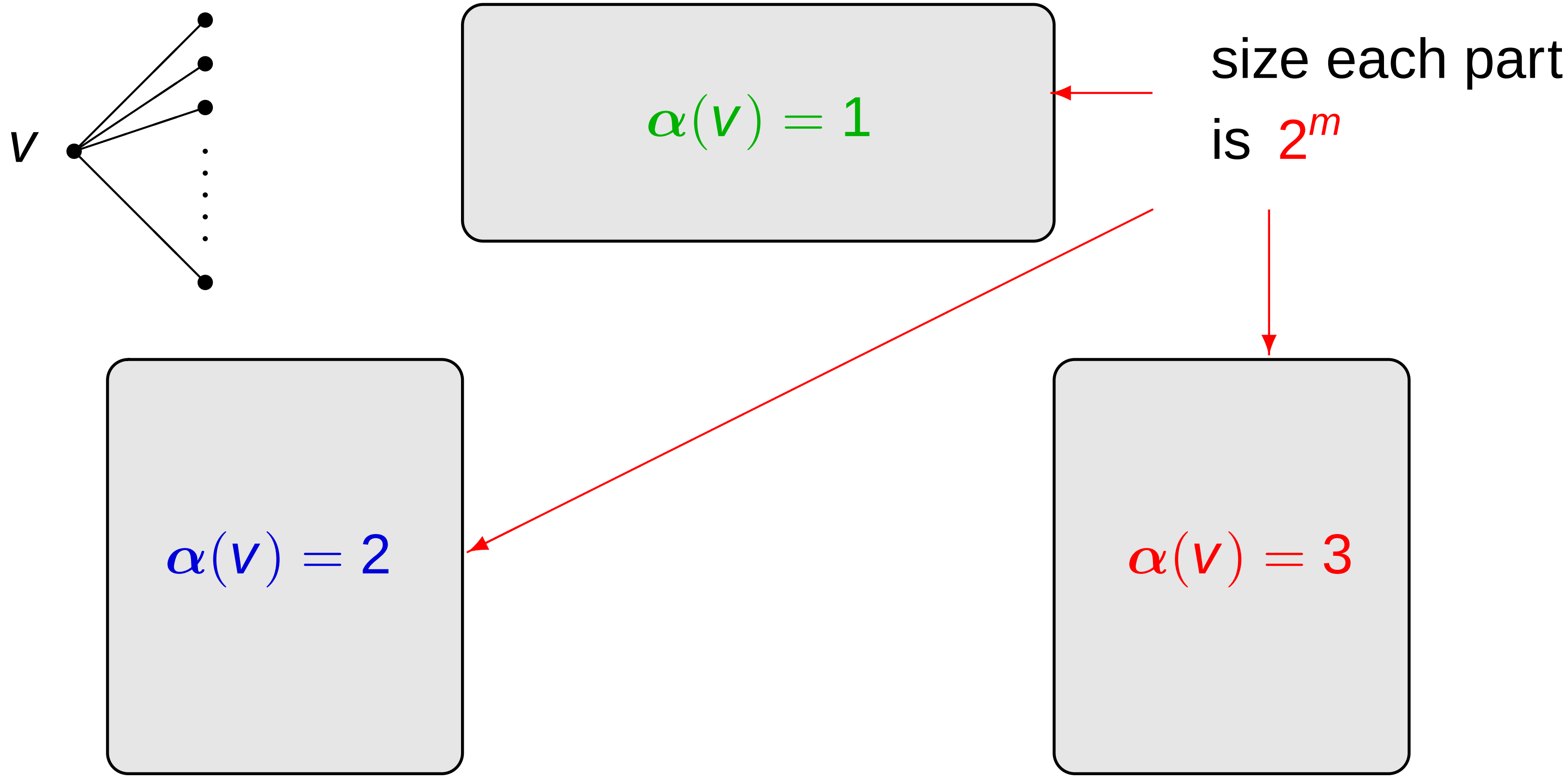
- take $k = 3$ and $G = K_{1,m}$:



- in the corresponding Markov chain $\mathcal{M}(K_{1,m}; 3)$, group 3-colourings according to the colour of v

Example : mixing but not rapidly mixing

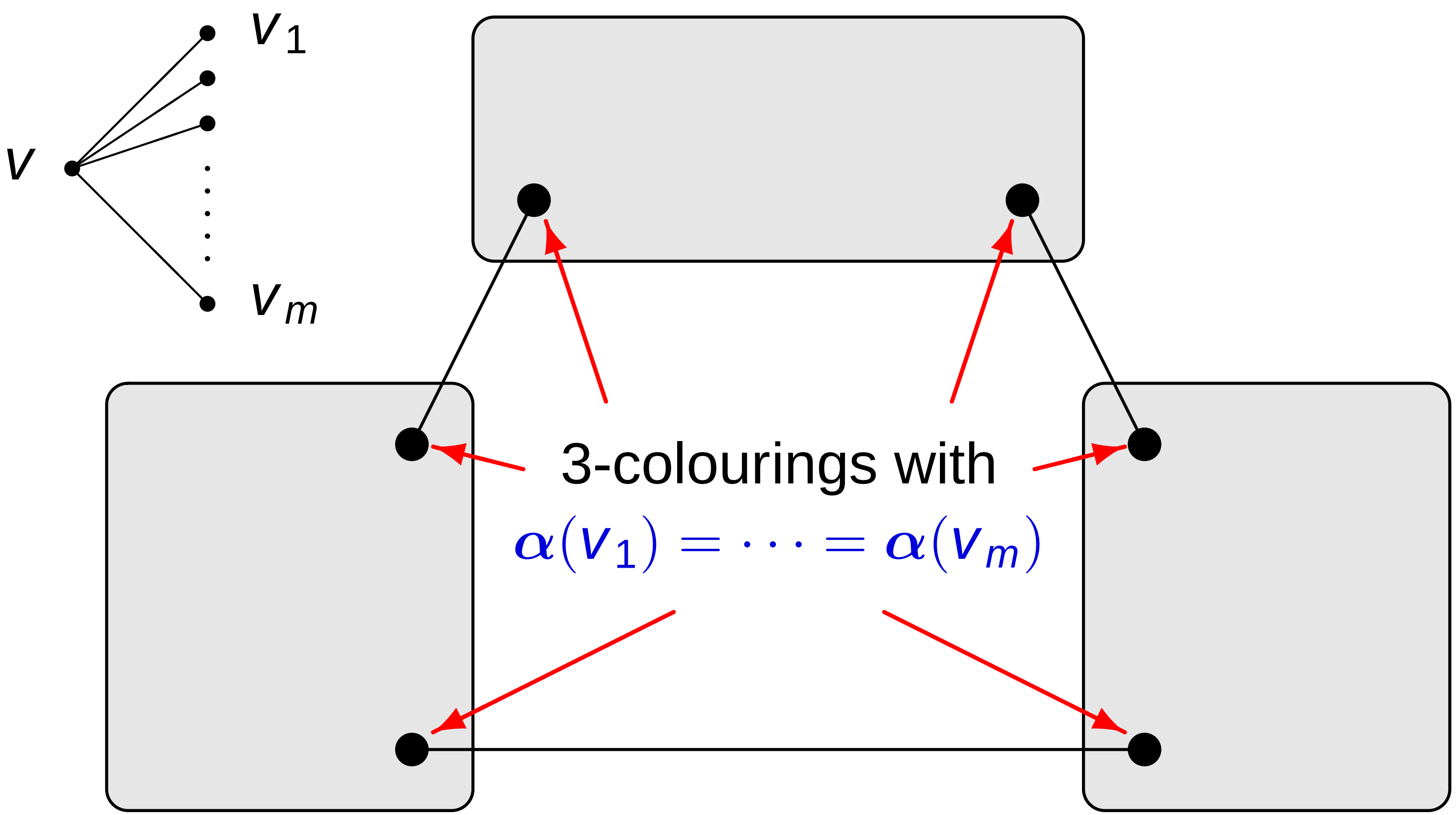
- the Markov chain $\mathcal{M}(K_{1,m};3)$:



each part has the structure of a hypercube

Example : mixing but not rapidly mixing

- the Markov chain $\mathcal{M}(K_{1,m};3)$:



Example : mixing but not rapidly mixing

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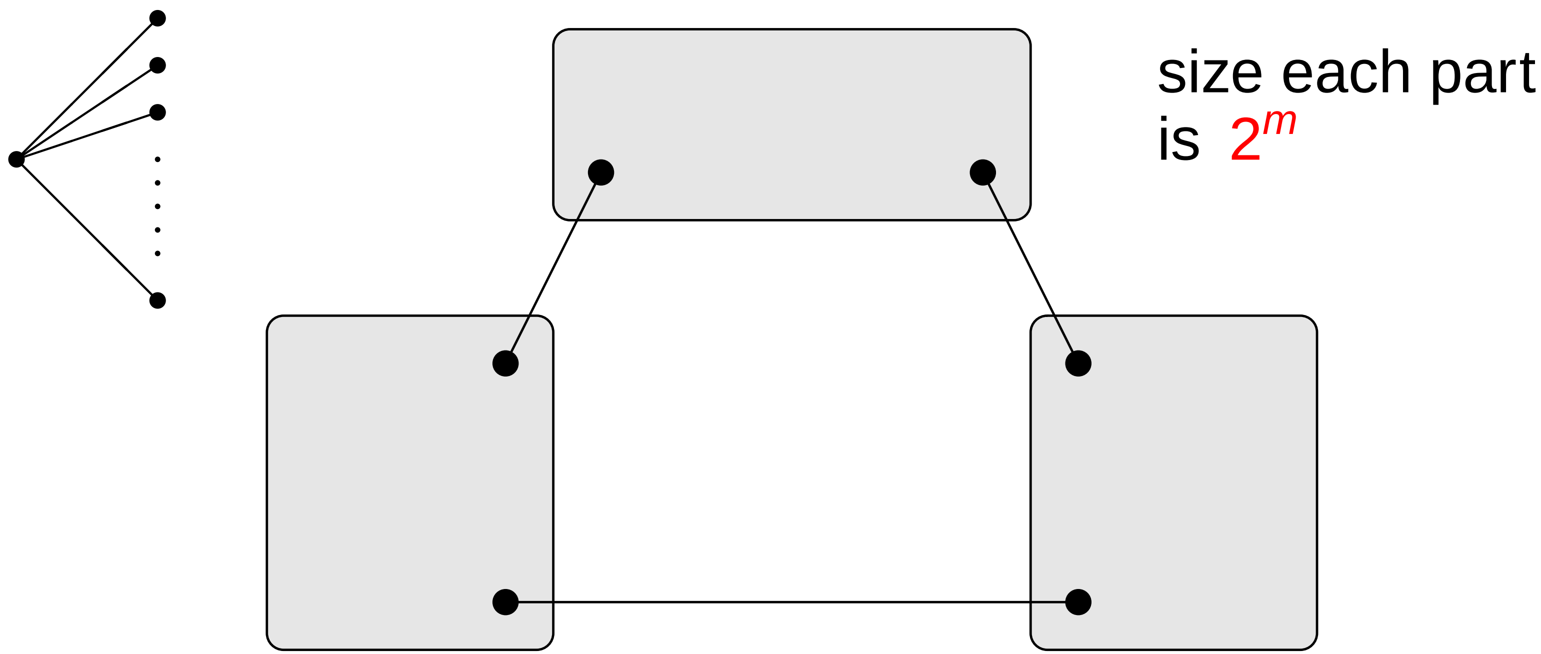


- bottlenecks : $K_{1,m}$ is not rapidly mixing for $k = 3$
- can be extended to not rapidly mixing
  for $k = O(m^{1-\varepsilon})$ for any $\varepsilon > 0$ (Łuczak & Vigoda, 2005)

Some positive results on rapid mixing

- $\underline{\Delta(G)}$: maximum degree of G

- $k > 2\,\Delta(G) \implies G$ rapidly mixing for k (Jerrum, 1995)
- $k > 1.8\,\Delta(G) \implies G$ rapidly mixing for k (Vigoda, 1999)

Towards mixing :

- $k \geq \Delta(G) + 2 \implies G$ is k-mixing (“well-known”)

Open :

- $k \geq \Delta(G) + 2 \implies G$ rapidly mixing for k colours ?

A better bound for mixing

- $\delta(G)$: minimum degree of G
- $\mathrm{col}(G)$: colouring number (degeneracy, maximin degree)

  $=$ maximum minimum degree of a subgraph of G

$$\mathrm{col}(G) = \max\{\,\delta(H) \mid H \subseteq G\,\}$$

Property :

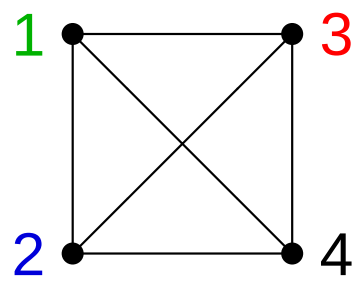
- $k \geq \mathrm{col}(G) + 2 \implies G$ is k-mixing (Dyer, et al., 2004)

e.g. :

- T a tree $\implies \mathrm{col}(T) = 1 \implies T$ is 3-mixing

Extremal graphs

- "boring" extremal graph : complete graph K_m
  - $\Delta(K_m) = \mathrm{col}(K_m) = m - 1$
  - all m-colourings look the same :
  - no vertex can change colour

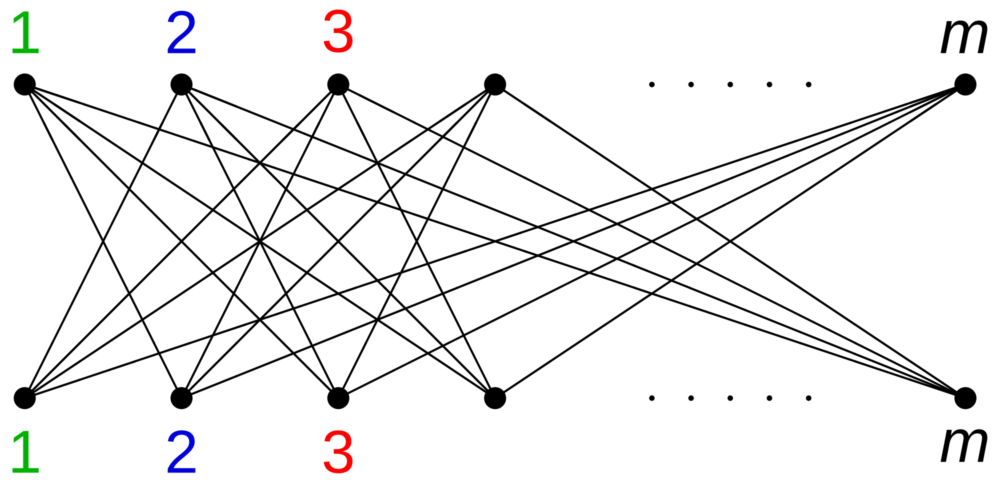


Terminology

- frozen k-colouring : colouring in which no vertex can change colour
  - frozen colourings form isolated vertices in $\mathcal{C}(G; k)$
  - immediately mean G is not k-mixing

More interesting extremal graphs

- graph L_m: $K_{m,m}$ minus a perfect matching ($m \geq 3$)
  - $\boldsymbol{\Delta}(L_m) = \mathrm{col}(L_m) = m - 1$



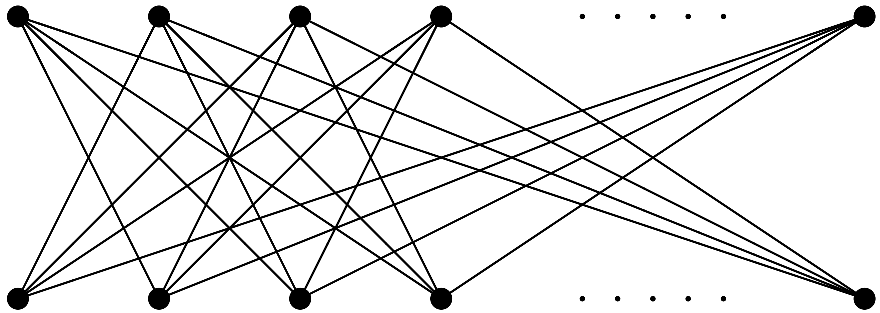
  - has frozen m-colourings – hence L_m is not m-mixing

so:

graph with $\chi(G) = 2$ can be non-k-mixing for arbitrarily large k

More interesting properties of L_m

- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m-1$
  - suppose L_m coloured with $k \leq m-1$ colours



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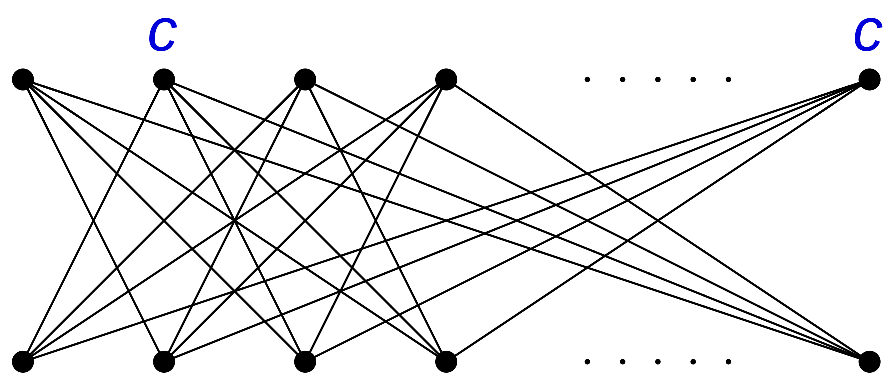

  - some colour c must appear more than once on the top

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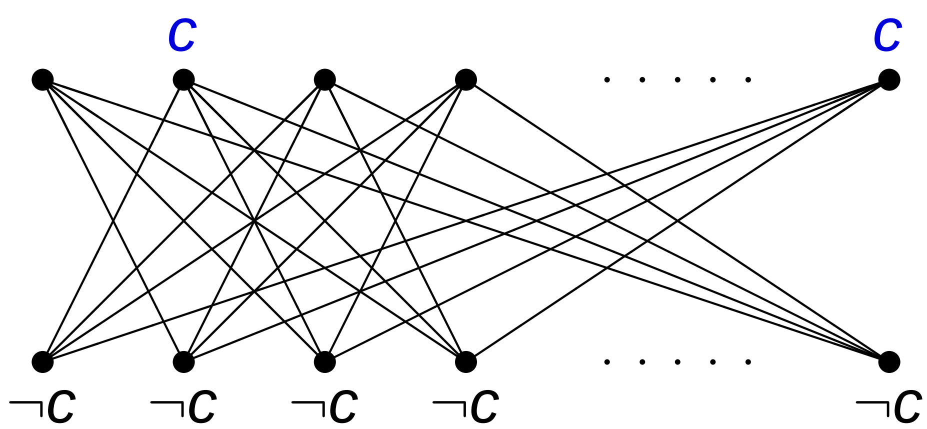

  - that colour c can't appear among the bottom vertices

More interesting properties of L_m

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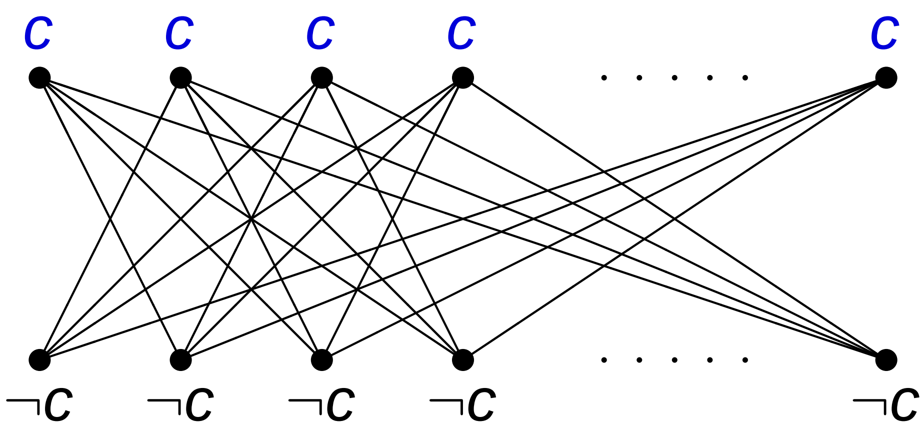
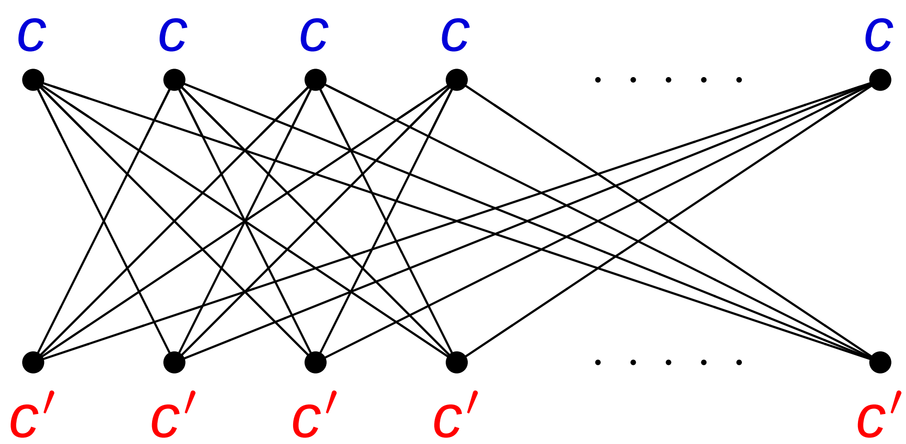


  - all vertices on the top can all be recoloured to c

More interesting properties of L_m

- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m - 1$
  - suppose L_m coloured with $k \leq m - 1$ colours



  - then the bottom can be recoloured to some $c' \neq c$

More interesting properties of L_m

- non-k-mixing for $k = m$ colours
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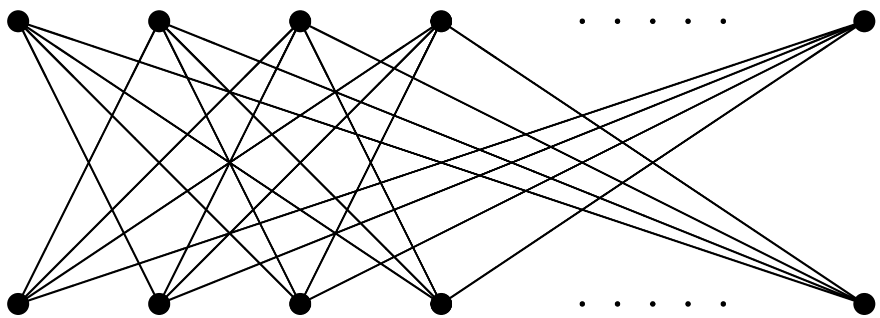
  - suppose L_m coloured with $k \le m-1$ colours



  - so any colouring is connected to a 2-colouring
  - easy to see that all these 2-colourings are connected

More interesting properties of L_m

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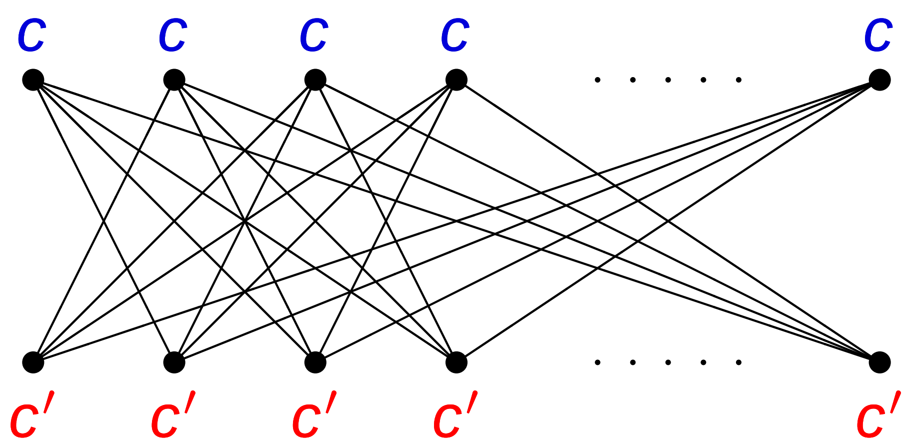


  - so any colouring is connected to a 2-colouring
  - easy to see that all these 2-colourings are connected

so : mixing is not a monotone property

Really surprising properties of L_m

Theorem

- fix constant $\boldsymbol{\lambda} > 0$
- $k \geq \boldsymbol{\lambda} m$, $k \neq m \implies L_m$ rapidly mixing for k colours

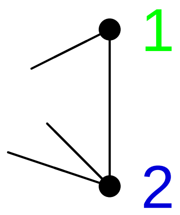
- in particular :
  - L_m is not mixing for m colours
  - L_m is rapidly mixing for $m - 1$ colours

so : rapid mixing is not a monotone property

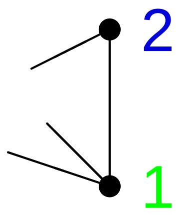
- first known class of graphs with this property

Small values of k

- smallest possible is $k = \chi(G)$
- $\chi(G) = 1$: graph without edges – boring
- $\chi(G) = 2$: bipartite graph with at least one edge
  - not-mixing for $k = 2$:



can't become



- $\chi(G) = 3$: 3-colourable graph with at least one odd cycle

The case $k = \chi(G) = 3$

- cycle C_3 has six 3-colourings, all frozen

  $\Longrightarrow \quad C_3$ is not 3-mixing

- cycle C_5 has 30 3-colourings, none of them frozen
  - the colour graph $\mathcal{C}(C_5; 3)$ is formed of two 15-cycles

  $\Longrightarrow \quad C_5$ is not 3-mixing

Theorem

- $\chi(G) = 3 \quad \Longrightarrow \quad G$ is not 3-mixing

Proof looks at 3-colourings of cycles

- suppose α a 3-colouring of a graph G and C a cycle in G
  - choose an orientation $\overrightarrow{C}$
  - weight of an arc of $\overrightarrow{C}$:

$$\boxed{+\mathbf{1}} \quad \text{if} \quad 1 \longrightarrow 2 \quad \text{or} \quad 2 \longrightarrow 3 \quad \text{or} \quad 3 \longrightarrow 1$$

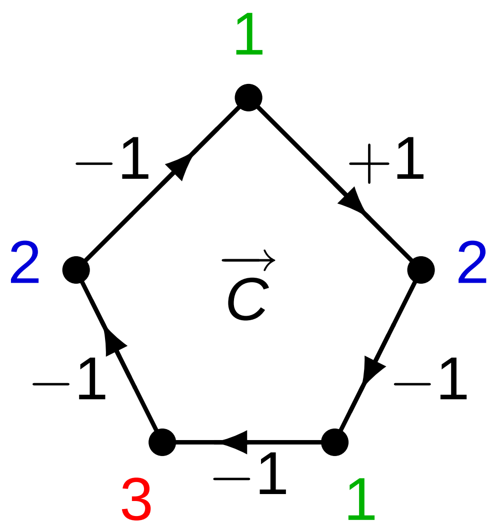
$$\boxed{-\mathbf{1}} \quad \text{if} \quad 2 \longrightarrow 1 \quad \text{or} \quad 3 \longrightarrow 2 \quad \text{or} \quad 1 \longrightarrow 3$$

  - weight of the oriented cycle :

$$w(\overrightarrow{C};\alpha) \;=\; \text{sum of weights of arcs}$$

Proof looks at 3-colourings of cycles

- Example :

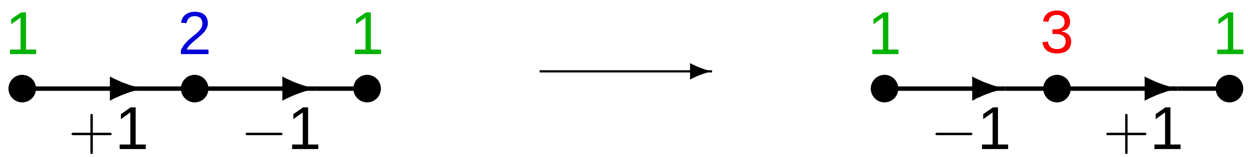


$$w(\overrightarrow{C};\alpha) = -3$$

Weights of 3-colourings of cycles

- recolour one vertex to obtain β from α

e.g.:



$$\Longrightarrow \quad w(\overrightarrow{C};\alpha) = w(\overrightarrow{C};\beta)$$

Property

- α and β connected by a path in $\mathcal{C}(G;3)$

$$\Longrightarrow \quad w(\overrightarrow{C};\alpha) = w(\overrightarrow{C};\beta)$$

Weights of 3-colourings of cycles

- given 3-colouring α, form α^* by swapping colours 1 and 2

  $\Longrightarrow$ all arcs change sign

  $\Longrightarrow$ $w(\overrightarrow{C};\alpha^*) = -w(\overrightarrow{C};\alpha)$

so:

- C odd cycle $\Longrightarrow$ $w(\overrightarrow{C};\alpha) \neq 0$

  $\Longrightarrow$ $w(\overrightarrow{C};\alpha^*) \neq w(\overrightarrow{C};\alpha)$

  $\Longrightarrow$ α and α^* not connected in $\mathcal{C}(G;3)$

  $\Longrightarrow$ $\mathcal{C}(G;3)$ not connected

Larger values of k

- $\chi(G) = 2 \quad \Longrightarrow \quad G$ is not 2-mixing
- $\chi(G) = 3 \quad \Longrightarrow \quad G$ is not 3-mixing

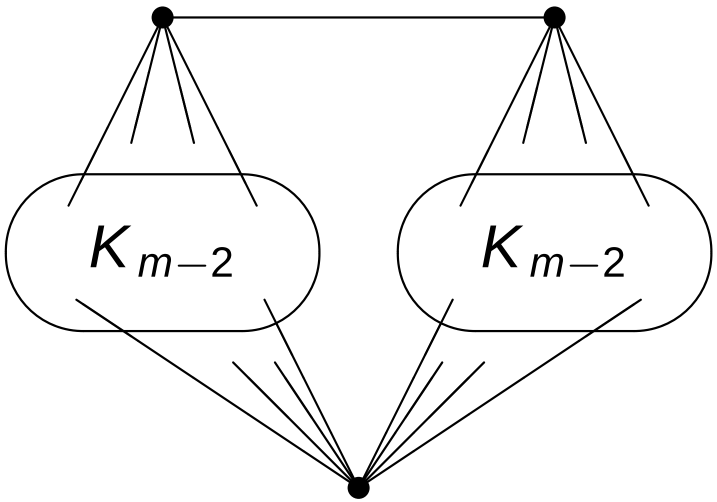
- What about $k \geq 4$?

- complete graph K_k has frozen k-colourings

  so: G has K_k as a subgraph $\quad \Longrightarrow \quad G$ not k-mixing

Larger values of k

- Hajos' graph H_m $\quad (m \geq 3)$



- has $\chi(H_m) = m$

- and is m-mixing for $m \geq 4$

Graphs with prescribed mixing behaviour

- L_m : $\chi(L_m) = 2$, mixing for $k > 2,\ k \neq m$
- K_m : $\chi(K_m) = m$, mixing for $k > m$
- H_m : $\chi(H_m) = m$, mixing for $k \geq m$ $(\,m \geq 4\,)$
- G 2-chromatic $\Longrightarrow$ not 2-mixing
- G 3-chromatic $\Longrightarrow$ not 3-mixing

allows to characterise

- integers q and sets S
  such that there is a graph G
  with $\chi(G) = q$ and k-mixing if and only if $k \notin S$

Decision problems

Input : graph G and integer k

Question : is G k-mixing ?

- probably very hard, since finding one k-colouring of a graph G is very hard, even if we know $k \geq \chi(G)$

Maybe easier :

Input : bipartite graph G and integer k

Question : is G k-mixing ?

Is a given bipartite graph k-mixing ?

- trivial for $k = 2$ ("yes" if and only if G has no edges)

necessary for $k = 3$:

- for all 3-colourings α and cycles C in G: $w(\overrightarrow{C};\alpha) = 0$

Theorem

- the condition is also **sufficient** for a graph to be 3-mixing

- **so** : deciding 3-mixing for bipartite graphs is in coNP

- certificate for non-3-mixing :
  a 3-colouring α and cycle C in G with $w(\overrightarrow{C};\alpha) = 0$

A structural certificate for bipartite non-3-mixing

- pinch of two vertices at distance 2 :

- G pinchable to H : sequence of pinches changes G to H

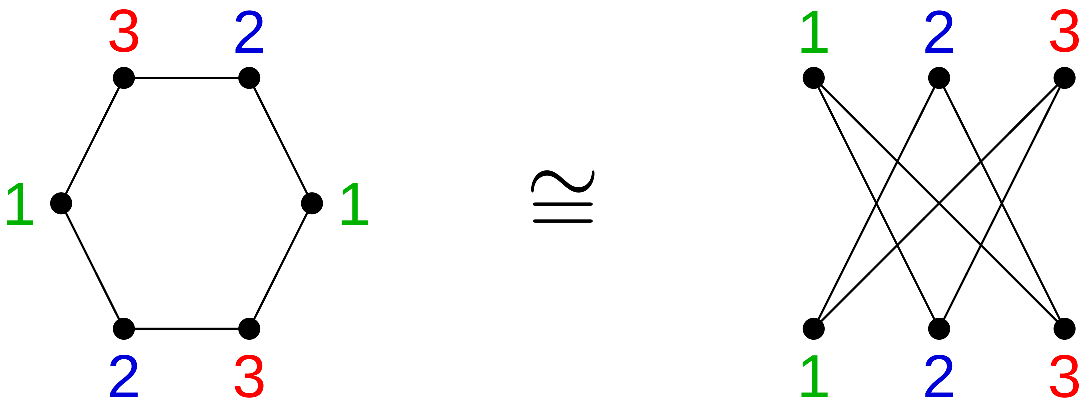
Theorem

- connected bipartite G is non-3-mixing

  $\iff$ G is pinchable to a chordless 6-cycle

Why the 6-cycle ?

- $C_6 \cong L_3$ – so C_6 is non-3-mixing



- **note** : C_4 is 3-mixing

Results / open problems for bipartite mixing

Theorem

- deciding 3-mixing for bipartite graphs is in coNP
- deciding 3-mixing for bipartite graphs
  is polynomial for planar graphs

open

- is deciding 3-mixing for bipartite graph
  polynomial or coNP-complete or $\ldots$?

- what happens for $k \geq 4$?

A decision problem for general graphs

- call k-colourings α and β connected :

  if there is a path in $\mathcal{C}(G;k)$ from α to β

Input : graph G, integer k, two k-colourings α and β

Question : are α and β connected ?

- this question might be doable for any k

Connected 3-colourings

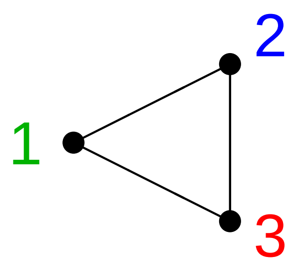
- **necessary**

  for two 3-colourings $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to be connected :

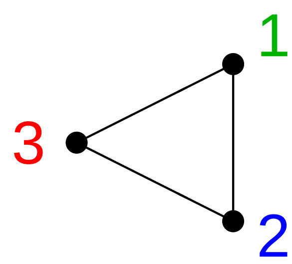
  - for all cycles C in G we must have

$$w(\overrightarrow{C};\boldsymbol{\alpha}) = w(\overrightarrow{C};\boldsymbol{\beta})$$

- but **not sufficient** :



$\boldsymbol{\alpha}$



$\boldsymbol{\beta}$

Connected 3-colourings

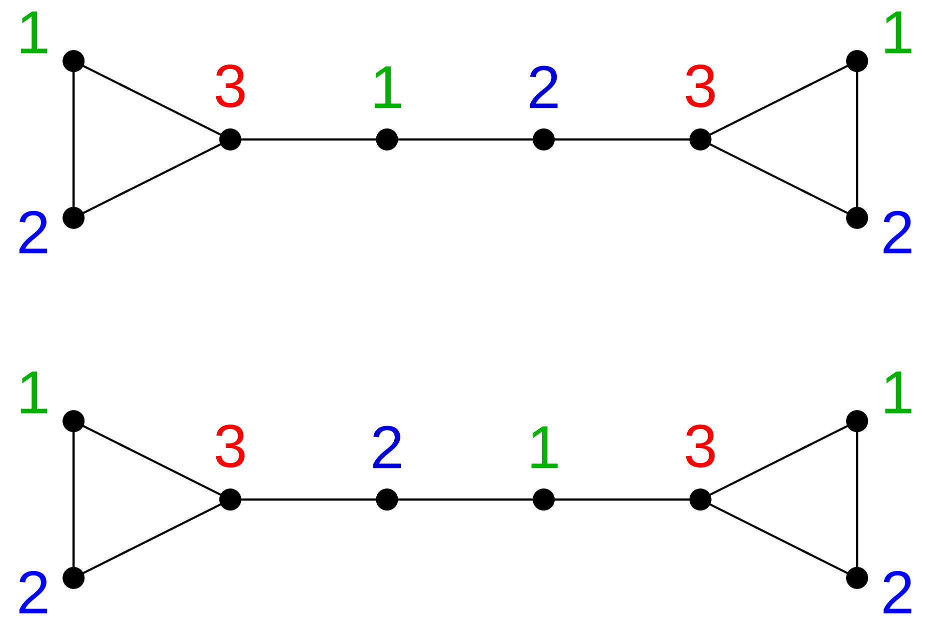
- frozen vertex of a colouring :

  vertex that never can change colour

- frozen vertices can appear outside cycles :

α

β



Connected 3-colourings

- **necessary**

  for two 3-colourings α and β to be connected :

  - all cycles C must satisfy $w(\overrightarrow{C}; \alpha) = w(\overrightarrow{C}; \beta)$
  - frozen vertices in α must be frozen also in β and must have the same colour in both

Theorem

- the conditions above are also **sufficient**
- the conditions can be checked in polynomial time

- **and again** : no idea what to do for $k \geq 4$

Mixing of planar graphs

- suppose G is planar

  how many colours k are needed to be sure G is k-mixing ?

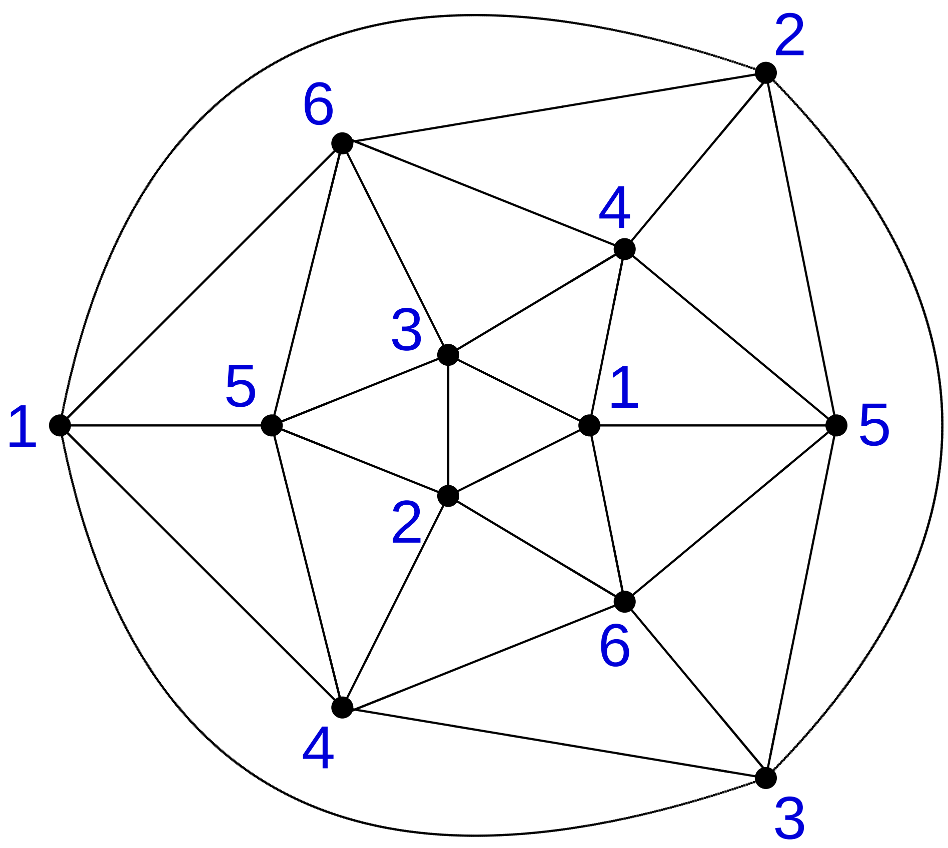
- G planar $\implies$ $\mathrm{col}(G) \leq 5$

  <u>**hence**</u>: $k = 7$ is enough

- but can we do better ?

Mixing of planar graphs

no :



frozen 6-colouring of the icosahedron

- for planar graphs, 7 colours are needed to guarantee mixing

Mixing of graphs on surfaces

- for all surfaces (orientable and non-orientable)
  - we know similar "mixing numbers"
  - except for the Klein bottle

- for all surfaces S
  - there is a sharp upper bound δ_S so that

    $\text{col}(G) \leq \delta_S$ for all G embeddable on S (Heawood)
  - **hence**: $k \geq \delta_S + 2$ is enough to guarantee mixing

- for all surfaces S, except plane and Klein bottle
  - we can't do better, because there exist

    complete graphs of order $\delta_S + 1$ (Ringel & Youngs, 1968)

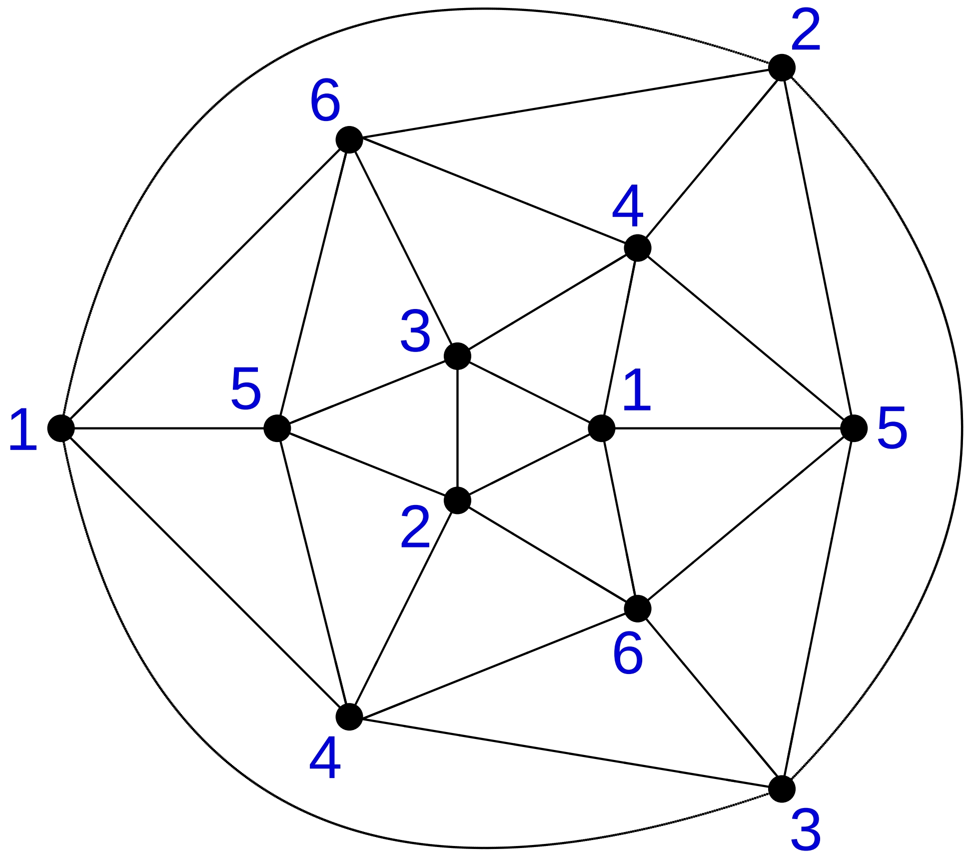
Mixing of graphs on surfaces

- for the plane : $\delta_{plane} + 2 = 7$
  - best possible because of
    5-regular icosahedron with a frozen 6-colouring

- for the Klein bottle : $\delta_{Klein} + 2 = 8$
  - but no graph embeddable on the Klein bottle can be
    6-regular and have a frozen 7-colouring

    (follows from result of Hliněný, 1999)

  - so are **7** or **8** colours needed for mixing on the Klein bottle ?



The end