Mixing Colour(ing)s in Graphs

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- reporting on research by: Paul Bonsma (Twente)
  Luis Cereceda (LSE)
  JvdH (LSE)
  and Matthew Johnson (Durham)
  - in several different combinations

- and it all started with a question of
  Hajo Broersma (Durham)

First definitions

- *graph* $G = (V, E)$: finite, simple, no loops, n vertices

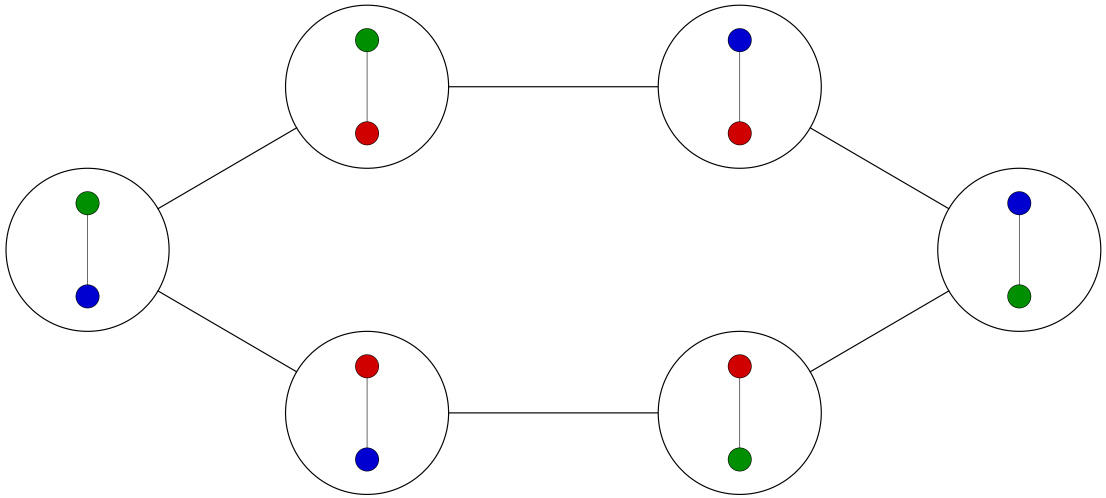
- *k-colouring* of G : proper vertex-colouring using colours from $\{1, 2, \ldots, k\}$
  - we always assume $k \geq \chi(G)$
  - we use $\alpha, \beta, \ldots$ to indicate k-colourings

- *k-colour graph* $\mathcal{C}(G; k)$
  - vertices are the k-colourings of G
  - two k-colourings are adjacent if they differ in the colour on exactly one vertex of G

Small example

- 2-colour graph for K_2:

- 3-colour graph for K_2:



Central question

General question

- Given G and k, what can we say about the colour graph $\mathcal{C}(G;k)$?

In particular

- is $\mathcal{C}(G;k)$ connected ?

  good way to think about it :

  - can we go from any k-colouring to any other k-colouring by recolouring one vertex at the time ?

Terminology : $\mathcal{C}(G;k)$ is connected $\iff$ G is k-mixing

Research on $\mathcal{C}(G;k)$

- little research in pure graph theory

- related to work in theoretical physics on
  Glauber dynamics of k-state anti-ferromagnetic Potts models at zero temperature

- related to work in theoretical computer science on
  Markov chain Monte Carlo methods for generating random k-colourings ("rapid mixing")

Some first results on mixing

- $k \geq \Delta(G) + 2 \quad \Longrightarrow \quad G$ is k-mixing ("Well-known")

- $\underline{D(G)} = \max\{\,\delta(H) \mid H \subseteq G\,\}$ (<u>degeneracy</u>)

<u>Property</u>

- $k \geq D(G) + 2 \quad \Longrightarrow \quad G$ is k-mixing (Dyer, et al., 2004)

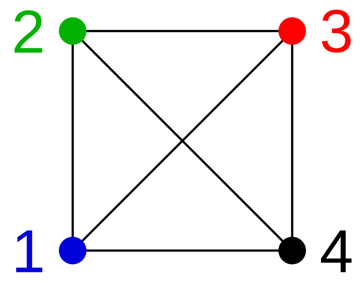
<u>Theorem</u>

- $k \geq D(G) + 3 \quad \Longrightarrow \quad \mathcal{C}(G;k)$ is hamiltonian

(Choo & MacGillivray, 2006)

Extremal graphs for the degree bounds

- “boring” extremal graph : complete graph K_m
  - $\Delta(K_m)+1 = D(K_m)+1 = m$
  - all m-colourings look the same :
  - no vertex can change colour

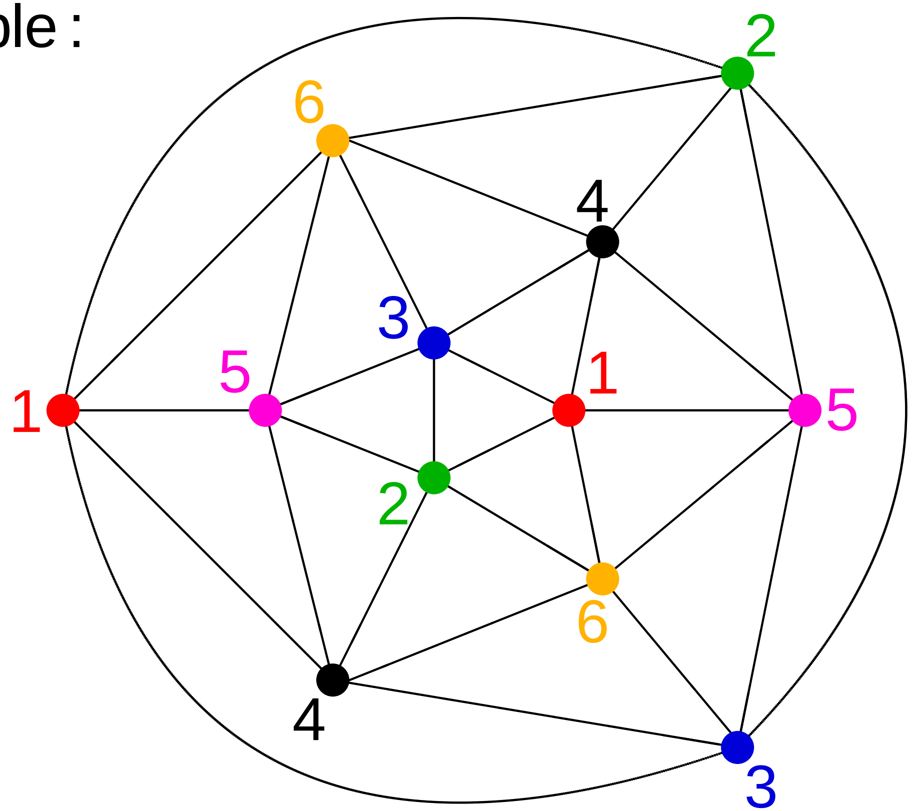


Terminology

- frozen k-colouring : colouring in which no vertex can change colour
  - frozen colourings form isolated nodes in $\mathcal{C}(G;k)$
  - immediately mean G is not k-mixing

The case for planar graphs

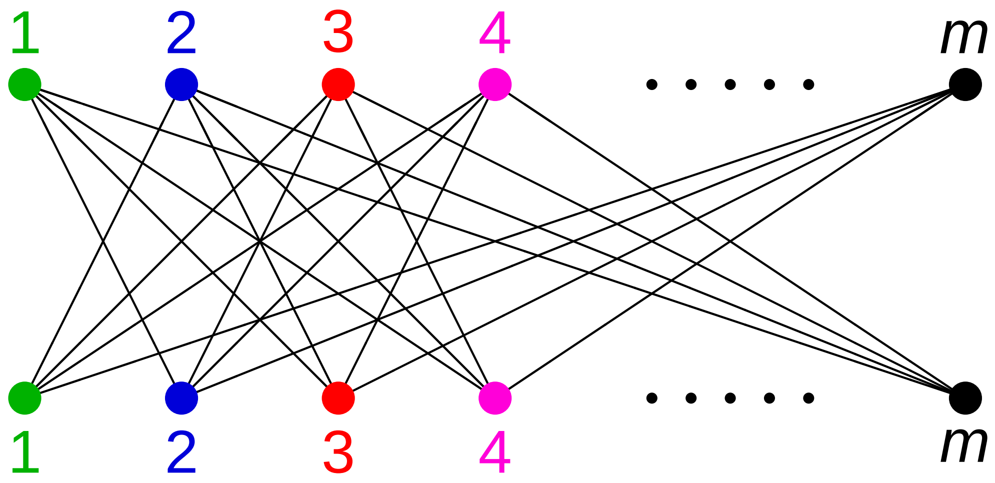
- G planar $\implies$ $D(G) \leq 5$ $\implies$ k-mixing for $k \geq 7$
- best possible :



frozen 6-colouring of the icosahedron

More interesting extremal graphs

- graph L_m : $K_{m,m}$ minus a perfect matching ($m \geq 3$)
  - $\Delta(L_m) + 1 = D(L_m) + 1 = m$

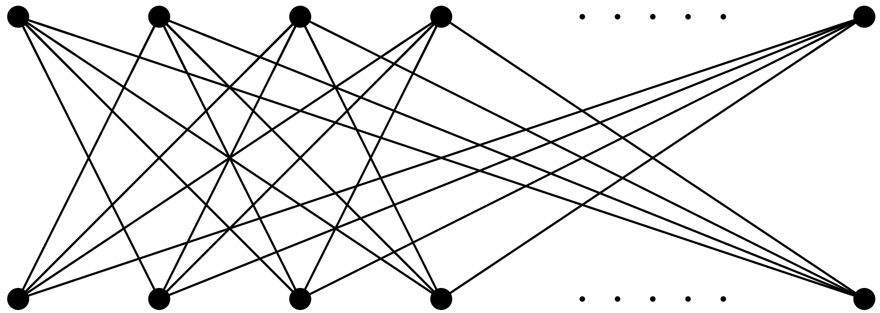


  - has frozen m -colourings – hence L_m is not m -mixing

- **so** :
  bipartite graphs can be non-k-mixing for arbitrarily large k

More interesting properties of L_m

- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m-1$
  - suppose L_m coloured with $k \leq m-1$ colours



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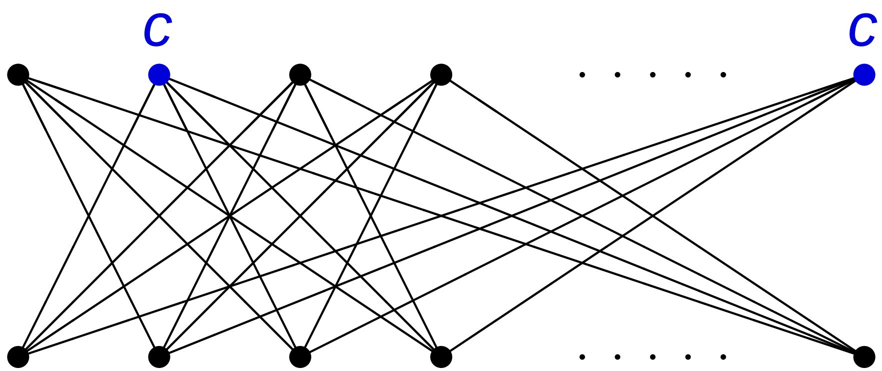

  - some colour c must appear more than once on the top

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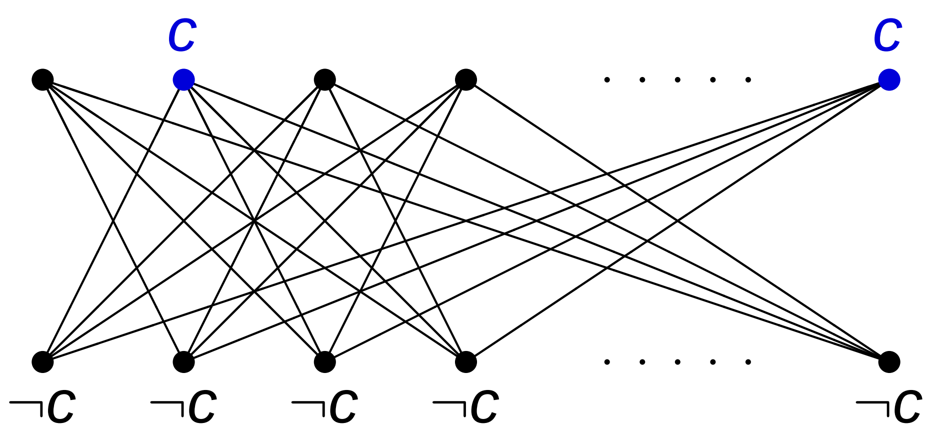

  - that colour c can't appear among the bottom vertices

More interesting properties of L_m

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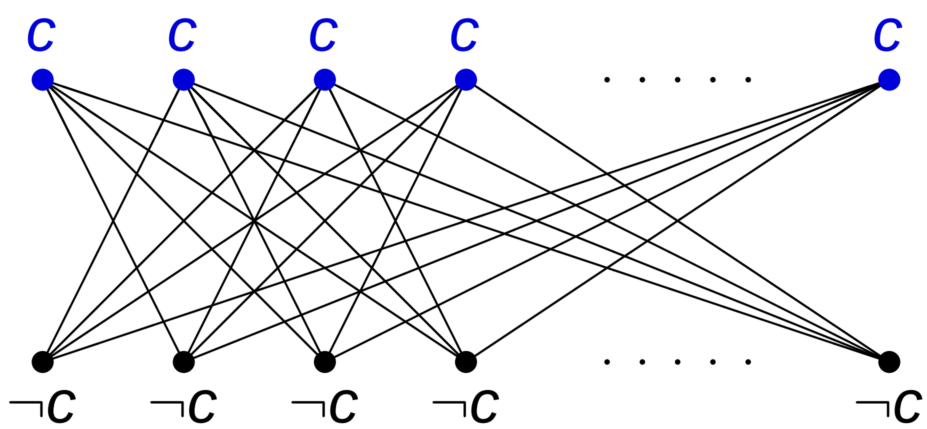
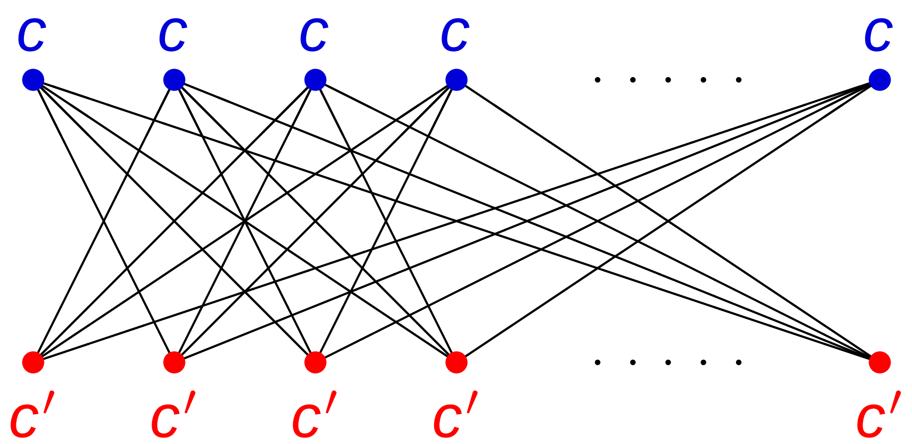
- but k-mixing for $3 \leq k \leq m-1$
  - suppose L_m coloured with $k \leq m-1$ colours



  - so all vertices on the top can be recoloured to c

More interesting properties of L_m

- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m - 1$
  - suppose L_m coloured with $k \leq m - 1$ colours



  - then the bottom can be recoloured to some $c' \neq c$

More interesting properties of L_m

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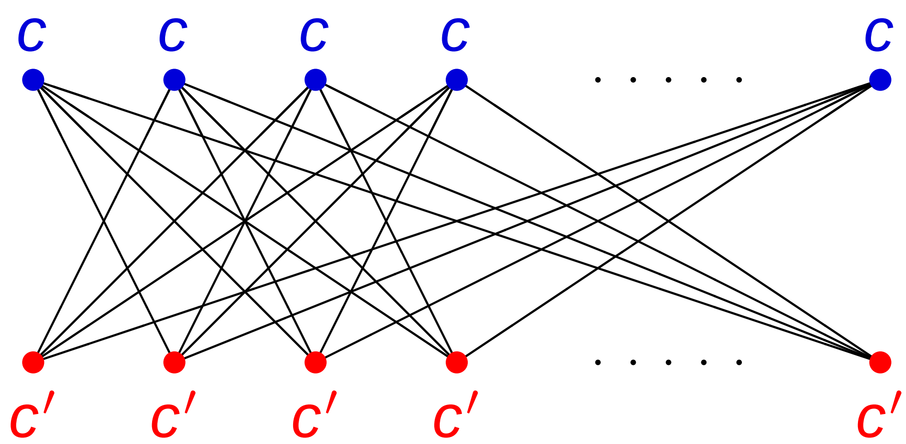
  - suppose L_m coloured with $k \leq m-1$ colours



  - hence any colouring is connected to a 2-colouring
  - easy to see that all these 2-colourings are connected

More interesting properties of L_m

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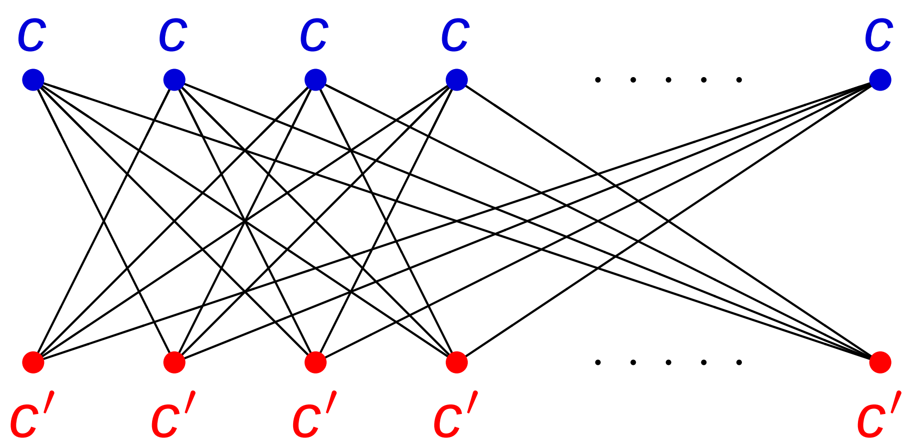
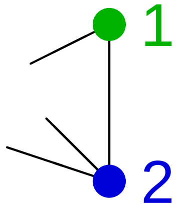


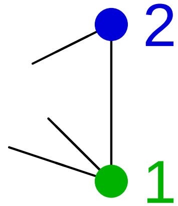
  - hence any colouring is connected to a 2-colouring
  - easy to see that all these 2-colourings are connected
- **so**: mixing is not a monotone property

Mixing for small values of k

- smallest possible is $k = \chi(G)$
- $\chi(G) = 1$: graph without edges – boring
- $\chi(G) = 2$: bipartite graph with at least one edge
  - not-mixing for $k = 2$:



can't become



The case $k = \chi = 3$

- $\chi(G) = 3$: 3-colourable graph with at least one odd cycle
- cycle C_3 has six 3-colourings, all frozen

  $\Longrightarrow$ C_3 is not 3-mixing
- cycle C_5 has 30 3-colourings, none of them frozen
  - the colour graph $\mathcal{C}(C_5; 3)$ is formed of two 15-cycles

    $\Longrightarrow$ C_5 is not 3-mixing

Theorem

- $\chi(G) = 3 \quad \Longrightarrow \quad G$ is not 3-mixing

Proof looks at 3-colourings of cycles

- suppose α is a 3-colouring of G
  and C is a cycle in G
  - choose an orientation $\overrightarrow{C}$ of the cycle
  - weight of an arc of $\overrightarrow{C}$:

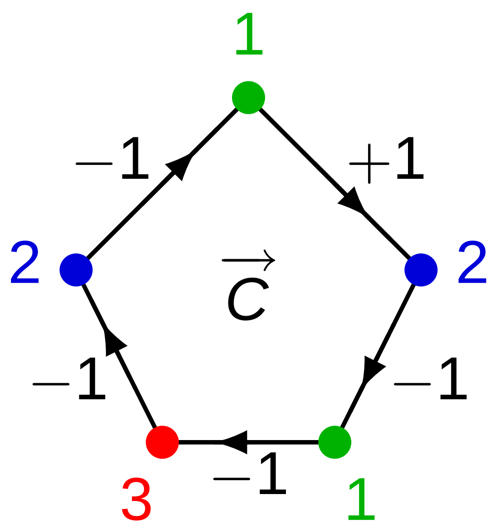
    $\boxed{+\mathbf{1}}$ if $1 \to 2$ or $2 \to 3$ or $3 \to 1$

    $\boxed{-\mathbf{1}}$ if $2 \to 1$ or $3 \to 2$ or $1 \to 3$
  - weight of the oriented cycle :

    $w(\overrightarrow{C};\alpha)$ $=$ sum of the weights of the arcs

Proof looks at 3-colourings of cycles

- Example :

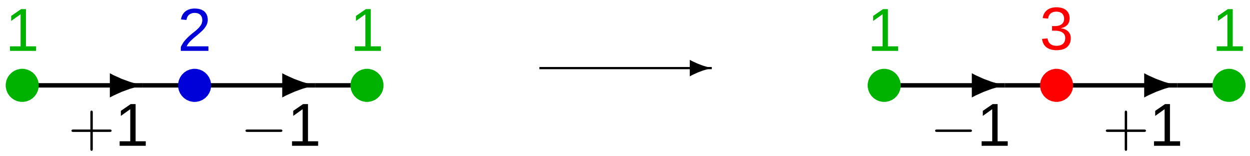


$$w(\overrightarrow{C};\alpha) = -3$$

Weights of 3-colourings of cycles

- recolour one vertex to go from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$

e.g.:



$$\Longrightarrow \quad w(\overrightarrow{C};\boldsymbol{\alpha}) = w(\overrightarrow{C};\boldsymbol{\beta})$$

Property

- $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ connected by a path in $\mathcal{C}(G;3)$

$\Longrightarrow$ for all cycles C in G : $w(\overrightarrow{C};\boldsymbol{\alpha}) = w(\overrightarrow{C};\boldsymbol{\beta})$

Weights of 3-colourings of cycles

- given 3-colouring α, form α^* by swapping colours 1 and 2

  $\Longrightarrow$ all arcs change sign

  $\Longrightarrow$ so for all C in G : $w(\overrightarrow{C};\alpha^*) = -w(\overrightarrow{C};\alpha)$

now: take 3-chromatic graph G with a 3-colouring α,
and take an odd cycle C in G

- $\Longrightarrow$ $w(\overrightarrow{C};\alpha) \neq 0$ (odd sum of $+1$s and -1s)

  $\Longrightarrow$ $w(\overrightarrow{C};\alpha^*) = -w(\overrightarrow{C};\alpha) \neq w(\overrightarrow{C};\alpha)$

  $\Longrightarrow$ α and α^* not connected in $\mathcal{C}(G;3)$

  $\Longrightarrow$ $\mathcal{C}(G;3)$ not connected

Mixing for larger values of $k = \chi$

- $\chi(G) = 2 \quad\Longrightarrow\quad G$ is not 2-mixing
- $\chi(G) = 3 \quad\Longrightarrow\quad G$ is not 3-mixing

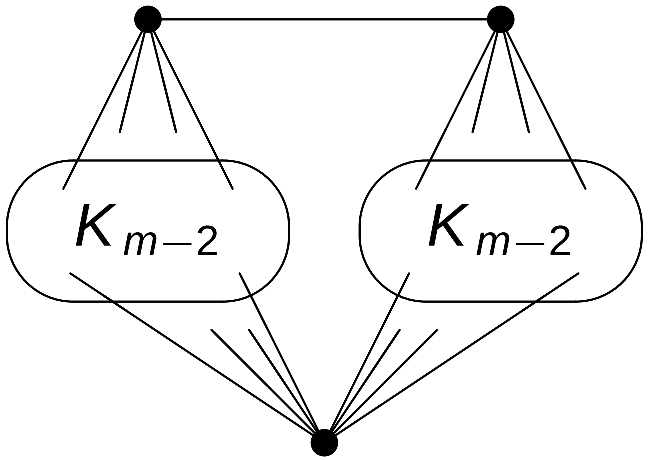
- What about $k \geq 4$?

- complete graph K_k has frozen k-colourings

  so: G has K_k as a subgraph $\quad\Longrightarrow\quad G$ not k-mixing

Mixing for larger values of $k = \chi$

- Hajos' graph H_m $(m \geq 3)$



- has $\chi(H_m) = m$
- and is m-mixing for $m \geq 4$
- **so** : graphs with $k = \chi(G) \geq 4$
  can be k-mixing or not k-mixing

Decision problems

k-MIXING

Input: graph G

Question: is G k-mixing ?

- probably very hard, since finding one k-colouring of a graph G is probably very hard, even if we know $k \geq \chi(G)$

Maybe easier:

BIPARTITE-k-MIXING

Input: bipartite graph G

Question: is G k-mixing ?

Is a given bipartite graph k-mixing ?

- trivial for $k = 2$ ("yes" if and only if G has no edges)

necessary for $k = 3$:

- for all 3-colourings α and cycles C in G : $w(\overrightarrow{C};\alpha) = 0$

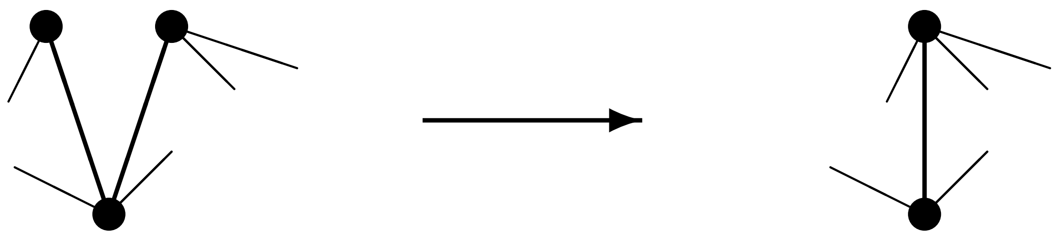
Theorem

- the condition is also **sufficient** for a graph to be 3-mixing

- **so** : BIPARTITE-3-MIXING is in coNP
- certificate for not 3-mixing :
  3-colouring α and cycle C in G with $w(\overrightarrow{C};\alpha) \neq 0$

A structural certificate for bipartite non-3-mixing

- pinch of two vertices at distance 2 :



- G pinchable to H : sequence of pinches changes G to H

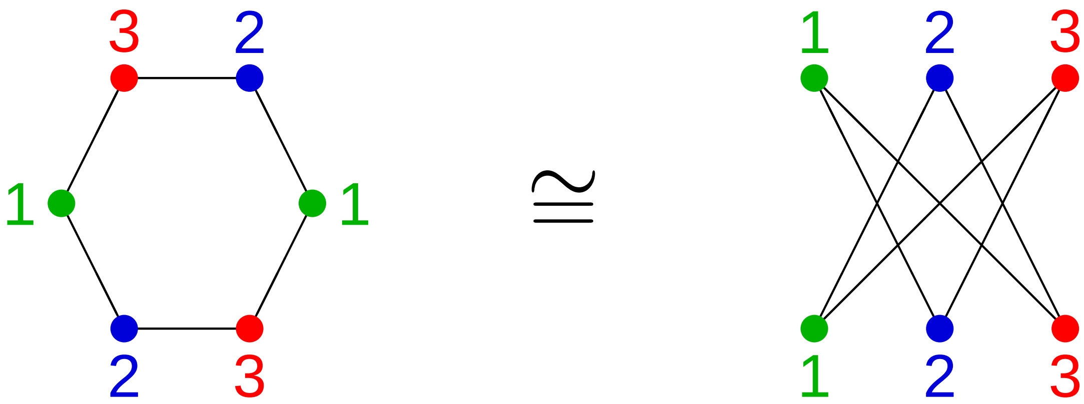
Theorem

- connected bipartite G is not 3-mixing

  $\iff$ G is pinchable to a chordless 6-cycle

Why the 6-cycle ?

- $C_6 \cong L_3$ – so C_6 is not 3-mixing



- **note** : C_4 is 3-mixing

Deciding bipartite mixing

- bipartite G not 3-mixing $\iff$ G pinchable to C_6

Theorem

- deciding pinchability to C_6 is NP-complete

hence

- BIPARTITE-3-MIXING is coNP-complete

Theorem

- BIPARTITE-3-MIXING is polynomial for planar graphs

open : what happens for $k \geq 4$?

A decision problem for general graphs

k-COLOUR-PATH

Input: graph G and two k-colourings α and β

Question: is there is a path in $\mathcal{C}(G;k)$ from α to β ?

or: "are α and β connected ?"

- this question might be doable for any k
- trivially decidable for $k = 2$

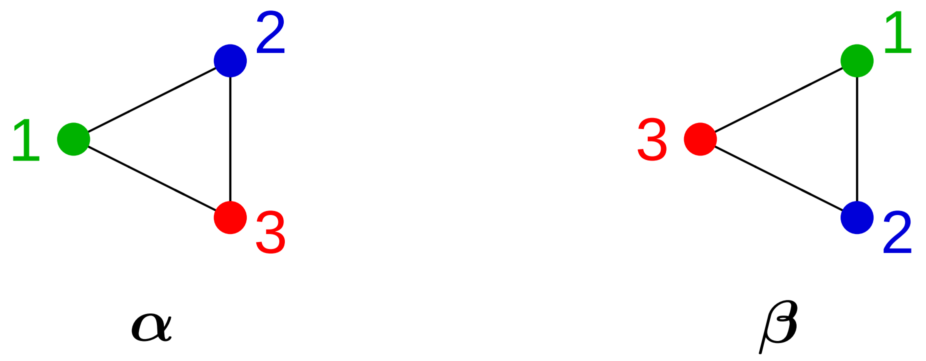
Connected 3-colourings

- **necessary condition 1**

  for two 3-colourings α and β to be connected :

  - for all cycles C in G : $w(\overrightarrow{C};\alpha) = w(\overrightarrow{C};\beta)$

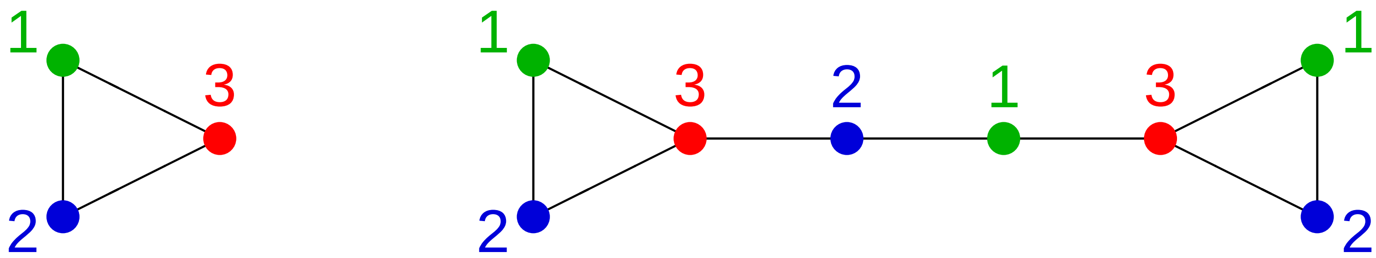
- but **not sufficient** :



Connected 3-colourings

- fixed vertex of a colouring : can never change colour

examples



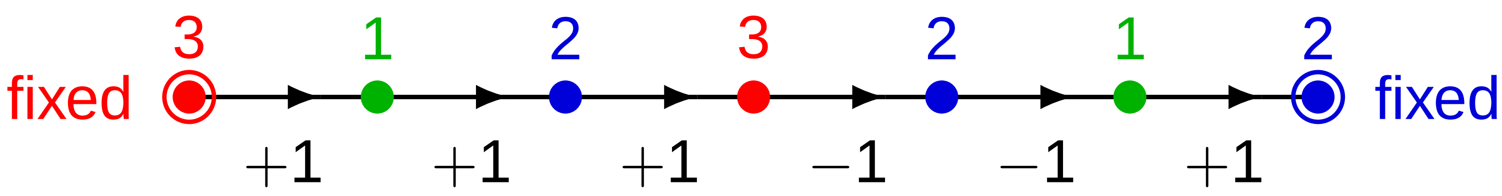
- **necessary condition 2**

  for two 3-colourings α and β to be connected :

  - all fixed vertices in α must be fixed in β as well and must have the same colour in both

Connected 3-colourings

- a path P with two fixed end vertices
  can also be given a weight $w(\overrightarrow{P};\alpha)$



- and this weight stays the same when recolouring

- **necessary condition 3**
  for two 3-colourings α and β to be connected :
  - for all fixed-ends paths P : $w(\overrightarrow{P};\alpha) = w(\overrightarrow{P};\beta)$

Connected 3-colourings

- two 3-colourings α and β can only be connected if :
  - for all cycles C : $w(\overrightarrow{C};\alpha) = w(\overrightarrow{C};\beta)$
  - for all fixed-ends paths P : $w(\overrightarrow{P};\alpha) = w(\overrightarrow{P};\beta)$
  - the sets of fixed vertices in α and β must be identical

Theorem

- the conditions above are also **sufficient**
- the conditions can be checked in polynomial time

and

- if connected, then there is a path of length $O(n^2)$

k-COLOUR-PATH for $k \geq 4$

Theorem

- for $k \geq 4$, k-COLOUR-PATH is PSPACE-complete

PSPACE

- decision problems that can be solved using a polynomial amount of memory (no restrictions on time)
- contains NP and coNP
- equal to its non-deterministic variant NPSPACE

k-COLOUR-PATH for $k \geq 4$

Theorem

k-COLOUR-PATH for bipartite, planar graphs:

- $k = 2$: trivially decidable
- $k = 3$: decidable in polynomial time
- $k = 4$: PSPACE-complete
- $k \geq 5$: always “YES”

Length of paths between connected colourings

Theorem

- for $k \geq 4$, k-COLOUR-PATH is PSPACE-complete
- if NP $\neq$ PSPACE (similar status as P $\neq$ NP), then no PSPACE-complete problem should have polynomial length certificates

so: for $k \geq 4$ path length between two connected k-colourings should not always be polynomial

Length of paths between connected colourings

Theorem

- for all $k \geq 4$, there exists graphs G with two k-colourings α and β so that
  - α and β are connected
  - the shortest path from α to β has exponential length

- the graphs can be bipartite
- and for $k = 4$ even bipartite and planar

Something different : using extra colours

- given a graph G and two k-colourings α and β
- suppose we can "buy" extra colours to go from α to β

how many extra colours do we need ?

Theorem

- $\chi(G) - 1$ extra colours is always enough

$\chi - 1$ *extra colours are always enough*

sketch of the proof

- take a χ-colouring using colours $-1, -2, \ldots, -\chi$

  say with colour-classes $V_{-1}, V_{-2}, \ldots, V_{-\chi}$
- starting with the k-colouring α (using colours $1, 2, \ldots, k$)
  - recolour vertices in V_{-1} with colour -1
  - recolour vertices in V_{-2} with colour -2
  - etc., until vertices in $V_{-(\chi-1)}$ with colour $-(\chi - 1)$
- the remaining vertices in $V_{-\chi}$ form an independent set
  - hence can be recoloured to their colours according to β
- now recolour vertices in $V_{-1} \cup V_{-2} \cup \cdots \cup V_{-(\chi-1)}$

  according to β as well

$\chi - 1$ *extra colours may be needed*

Theorem

- for all C, k with $k \geq C \geq 2$
  there exists graphs G with $\chi(G) = C$
  and two k-colourings α and β so that
  - to get from α to β requires $C - 1$ extra colours

$\chi - 1$ *extra colours may be needed*

Theorem

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  there exists graphs G with $\chi(G) = C$

  and two k-colourings α and β so that

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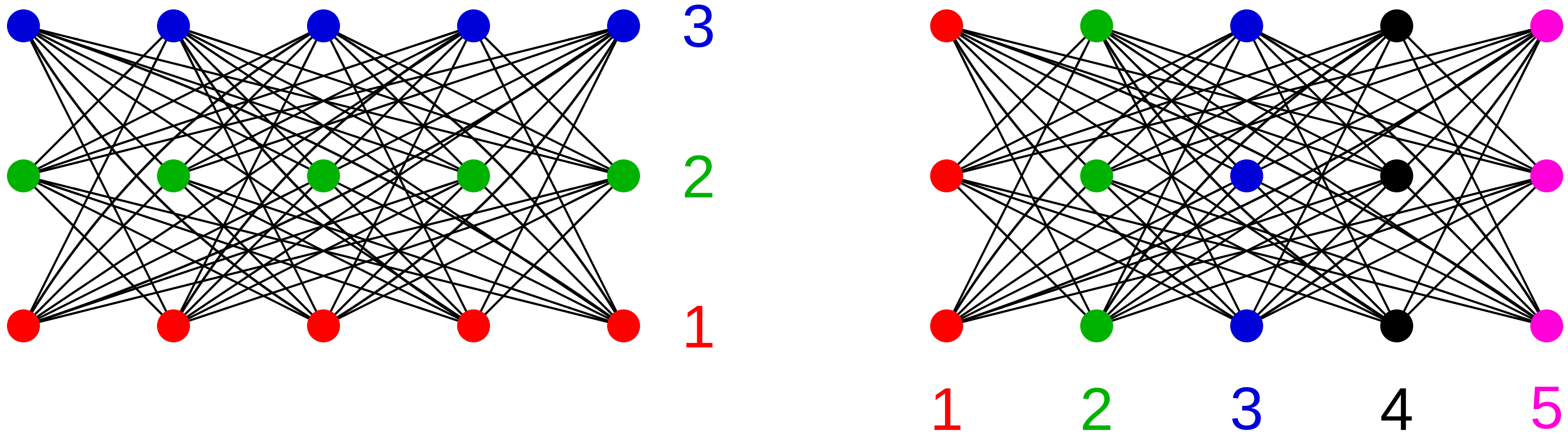

the graph for $C = 3$ and $k = 5$