Circular Arboricity of Graphs

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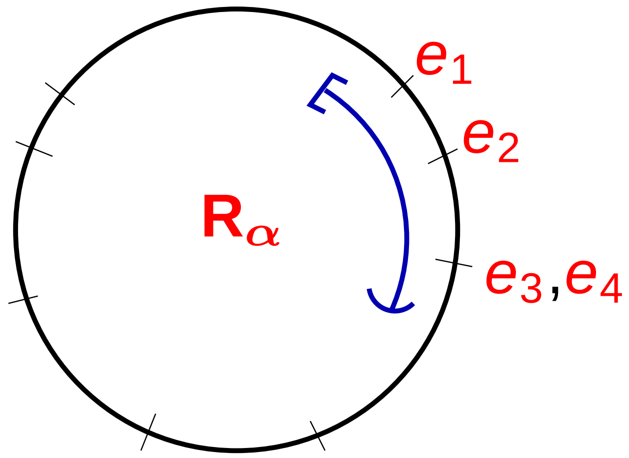
First definitions / notation

- $G = (V(G), E(G))$: finite graph,
  no loops, but multiple edges allowed
  - n : number of vertices
  - m : number of edges
- **forest** : subgraph of G without cycles
- $\mathbf{R}_\alpha$: circle with circumference α $(\alpha \in \mathbf{R},\ \alpha > 0)$
  think : interval $[0, \alpha)$ with a circular ordering
- $\mathbf{Z}_k$: integers modulo k $(k \in \mathbf{N})$
  think : numbers $1, 2, \ldots, k$ with a circular ordering

Circular arboricity

- we want to map the edges of G to $\mathbf{R}_\alpha$ so that:
  - for every unit interval $[\,a,\ a+1\,)$ of $\mathbf{R}_\alpha$:
    the edges mapped into that interval form a forest



- **circular arboricity** of G, $\Upsilon_C(G)$:
  minimum α for which this is possible

A bound on the circular arboricity

- for every subgraph $H \subseteq G$ we must have :
  - a forest can have at most $|V(H)| - 1$ edges from H
  - so every unit interval of $\mathbf{R}_\alpha$ can have at most $|V(H)| - 1$ edges from H
  - so we need $\alpha \geq \dfrac{|E(H)|}{|V(H)| - 1}$
- and hence $\Upsilon_C(G) \geq \max\limits_{H \subseteq G} \dfrac{|E(H)|}{|V(H)| - 1}$

<u>**Conjecture** (Goncalves)</u>: $\Upsilon_C(G) = \max\limits_{H \subseteq G} \dfrac{|E(H)|}{|V(H)| - 1}$

Integral arboricity

Theorem (Nash-Williams, 1964)

- If: $K \geq \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$, for some $K \in \mathbf{N}$

  Then: $E(G)$ can be partitioned into K disjoint forests

- this means: $\Upsilon_C(G) \leq \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$

Integral arboricity

Theorem (Nash-Williams, 1964)

- If: $K \geq \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$, for some $K \in \mathbf{N}$

  Then: $E(G)$ can be partitioned into K disjoint forests

- generalised to matroids by Edmonds (1964)

- in fact: everything in this talk can be (and has been) formulated / asked / proved for matroids as well

Fractional arboricities

- the circular arboricity can be considered as some kind of "**fractional**" **arboricity**
- a more natural fractional arboricity concept is the solution to the following LP-problem :
  - x_F : real-valued variable for a forest F
  - minimise : $\sum_F x_F$

    such that : $\forall e \in E(G) : \sum_{F \ni e} x_F \geq 1$

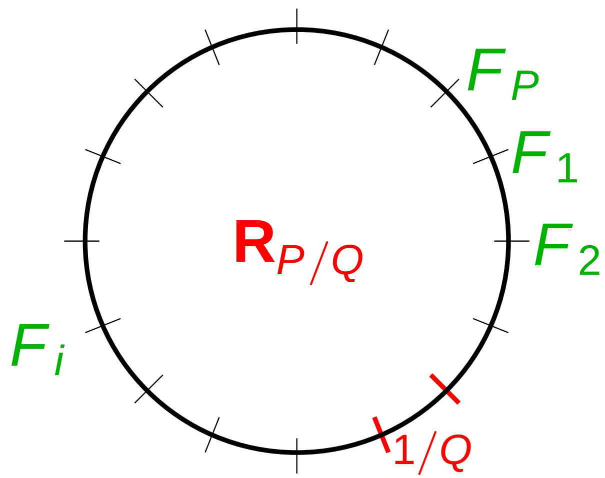
    $\forall F : x_F \geq 0$
- <u>**folklore**</u> : this minimum is equal to $\max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1}$

Quick proof of the fractional arboricity

- suppose $\max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1} = \frac{P}{Q}$
- form G^Q by replacing each edge by Q parallel edges
- then $\max_{H \subseteq G^Q} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil = \max_{H \subseteq G^Q} \frac{|E(H)|}{|V(H)|-1} = P$
- Nash-Williams : G^Q can be covered by P disjoint forests
- so G has P forests covering each edge Q times
- set $x_F = 1/Q$ for these forests $\square$

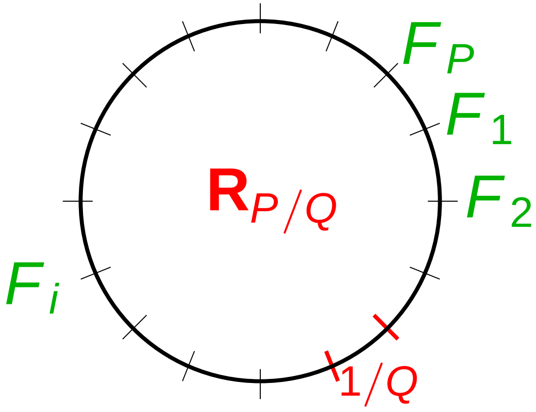
Forests in a circle

- **Conjecture** : $\Upsilon_C(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1} = \frac{P}{Q}$
- and we know : there is a collection $\mathcal{F} = \{F_1, \ldots, F_P\}$ of P forests covering each edge Q times
- if we give each forest in $\mathcal{F}$ weight $1/Q$ we can put them around $\mathbf{R}_{P/Q}$:



From forests in a circle to circular arboricity

- P forests with each edge appearing in Q of them



- we would be done if we can make sure that

  every edge occurs in Q consecutive forests

  - then : map each edge to the first forest it appears in
  - which would mean :

    set of edges in a unit interval

    $=$ edges of the last forest in that interval

A possible proof of Goncalves' Conjecture

- **Conjecture** : $\Upsilon_C(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1} = \frac{P}{Q}$

- we are done if we can prove :

  there exists a "cyclic" list of P forests so that

  - each edge appears in Q consecutive forests

- equivalent to :

  there exists a multimap $E(G) \lessdot \mathbf{Z}_P$ so that

  - each edge is mapped to Q consecutive numbers
  - for all $x \in \mathbf{Z}_P$: edges mapped to x form a forest

A general theorem

Theorem 1

- Given: $K \in \mathbf{N}$, edge weights $w : E(G) \longrightarrow \mathbf{N}$
- If: $\forall H \subseteq G$: $K \geq \dfrac{\sum_{e \in E(H)} w(e)}{|V(H)| - 1}$
- Then: there exists a multimap $E(G) \lessdot\!\!\!\!\!\!\!\!\equiv \mathbf{Z}_K$ so that
  - each edge e is mapped to $w(e)$ consecutive numbers
  - for all $x \in \mathbf{Z}_K$: edges mapped to x form a forest

Corollary: by taking $K = P$ and $\forall e$: $w(e) = Q$ we get:

$$\Upsilon_C(G) = \frac{P}{Q} = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1}$$

Some ideas from the the proof

- proof by induction on $\sum w(e)$
- choose an e_1 and replace $w(e_1)$ by $w(e_1) - 1$
  - find a multimap to $\mathbf{Z}_K$ with this reduced weight
- say e_1 gets mapped to the interval $x_0, \ldots, x_1 - 1$
  - map an extra copy of e_1 to position x_1
  - this may introduce a cycle at position x_1
- there is an edge e_2 in this cycle not mapped to $x_1 - 1$
- say e_2 gets mapped to the interval $x_1, \ldots, x_2 - 1$
  - remove the map from e_2 to x_1
  - map a new copy of e_2 to position x_2
  - this may introduce a cycle at position x_2

Some ideas from the the proof

-
  - map a new copy of e_2 to position x_2
  - this may introduce a cycle at position x_2
- there is an edge e_3 in this cycle not mapped to $x_2 - 1$
- say e_3 gets mapped to the interval $x_2, \ldots, x_3 - 1$
  - remove the map from e_3 to x_2
  - map a new copy of e_3 to position x_3
  - this may introduce a cycle at position x_3
- ad infinitum . . . .

NOT !

Disjoint spanning trees

- $\boldsymbol{\omega(G)}$: number of components of a graph G

Theorem (Nash-Williams, Tutte, 1961)

- If : $K \leq \min_{A \subseteq E(G)} \left\lfloor \frac{|A|}{\omega(G-A)-1} \right\rfloor$, for some $K \in \mathbf{N}$

  Then : G contains K disjoint spanning trees

- matroid version by Edmonds (1964)

The dual of Theorem 1

Theorem 1, using matroid duality, can be used to prove :

Theorem 2

- Given : $K \in \mathbf{N}$, edge weights $w : E(G) \longrightarrow \mathbf{N}$
- If : $\forall A \subseteq E(G) :\ K \leq \dfrac{\sum_{e \in A} w(e)}{\omega(G - A) - 1}$
- Then : there exists a multimap $E(G) \lessdot\!\!\!\!\!\!\to \mathbf{Z}_K$ so that
  - each edge e is mapped to $w(e)$ consecutive numbers
  - for all $x \in \mathbf{Z}_K$: edges mapped to x form a connected spanning subgraph

The dual of circular arboricity

Theorem 2 gives the following **circular version** of the Nash-Williams / Tutte Theorem :

Corollary

- If : $\alpha \le \min_{A \subseteq E(G)} \dfrac{|A|}{\omega(G-A)-1}$, for some $\alpha \in \mathbf{R}$, $\alpha > 0$
- Then : there exists a map $E(G) \longrightarrow \mathbf{R}_\alpha$ so that :
  - for every unit interval $[\,a,\ a+1\,)$ of $\mathbf{R}_\alpha$:
    the edges mapped into that interval
    form a connected spanning subgraph
- this upper bound on α is best possible

More on circular mappings of edges

- condition from **Theorem 1**:

$$\forall H \subseteq G: \quad K \geq \frac{\sum_{e \in E(H)} w(e)}{|V(H)| - 1}$$

- suppose we take

$$\forall e: \ w(e) = |V(G)| - 1 = n - 1 \text{ and } K = |E(G)| = m$$

Corollary

- If: $\forall H \subseteq G: \ \dfrac{m}{n-1} \geq \dfrac{|E(H)|}{|V(H)| - 1}$
- Then: there exists a multimap $E(G) \lessdot\!\!\!\lessdot \mathbf{Z}_m$ so that
  - each edge is mapped to $n - 1$ consecutive numbers
  - for all $x \in \mathbf{Z}_m$: edges mapped to x form a forest

More on circular mappings of edges

- there exists a multimap $E(G) \Lleftarrow \mathbf{Z}_m$ so that
  - each edge is mapped to $n-1$ consecutive numbers
  - for all $x \in \mathbf{Z}_m$: edges mapped to x form a forest

Corrolary is equivalent to

- If : $\forall H \subseteq G : \quad \dfrac{m}{n-1} \geq \dfrac{|E(H)|}{|V(H)|-1}$
- Then : there exists a **function** $E(G) \longrightarrow \mathbf{Z}_{|E(G)|}$ so that
  - for all intervals of $n-1$ consecutive numbers :
    edges mapped to that interval form a spanning tree

Question : can we make this function a bijection ?

Circular orderings of edges

Conjecture (Kajitani, Ueno & Miyano, 1988)

- If: $\forall H \subseteq G: \quad \dfrac{m}{n-1} \geq \dfrac{|E(H)|}{|V(H)|-1}$
- Then: there exists a **circular ordering** of $E(G)$ so that
  - each $n-1$ consecutive edges form a spanning tree

- they posed the same conjecture for matroids
- known to be true for
  - a few special classes of graphs
  - graphs consisting of **two** edge-disjoint spanning trees

    (but even that case is open for matroids)

A result on circular orderings

Theorem 3

- If: $\forall H \subseteq G: \quad \dfrac{m}{n-1} \geq \dfrac{|E(H)|}{|V(H)|-1}$

  and: $n-1$ and m are co-prime

- Then: there exists a circular ordering of $E(G)$ so that
  - each $n-1$ consecutive edges form a spanning tree

- holds for matroids as well