The Hirsch Conjecture and the Transportation Polytope

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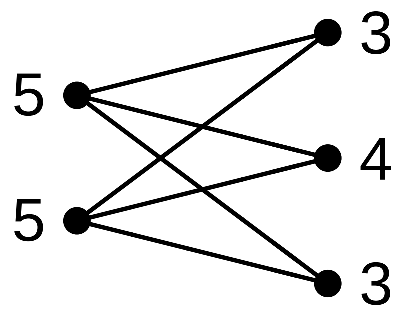
The Transportation Problem

- ***m* supply points**, each holding quantity $r_i > 0$
- ***m* demand points**, each wanting quantity $c_j > 0$

- total supply $=$ total demand : $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$

supply points



demand points

The Transportation Polytope

- x_{ij} : amount transported from i to j
- a **feasible solution** X is an $m \cdot n$ vector $X = (x_{ij})$ so that

$$\sum_{j=1}^{n} x_{ij} = r_i, \qquad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = c_j, \qquad j = 1, \ldots, n$$

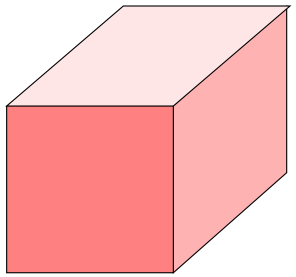
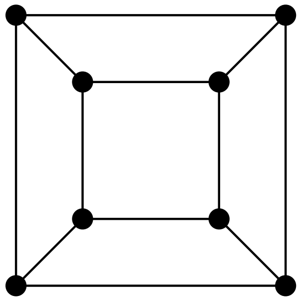
$$x_{ij} \geq 0, \qquad i = 1, \ldots, m, \; j = 1, \ldots, n$$

- **transportation polytope** $\mathcal{T}$: convex polytope formed by the set of all feasible solutions in $\mathbb{R}^{mn}$

The diameter of $\mathcal{T}$

- **1-skeleton** or **edge graph** of $\mathcal{T}$:

  graph formed by 0-faces as vertices and 1-faces as edges

polytope   its 1-skeleton

- **diam($\mathcal{T}$)** : graph diameter of the **1-skeleton** of $\mathcal{T}$

question of the day : what can we say about diam($\mathcal{T}$) ?

Why ?

- **Conjecture** (Hirsch, 1957)
  - $\mathcal{P}$ a polytope with ***f*** **facets** and **dimension** ***d***

$$\Longrightarrow \qquad \mathrm{diam}(\mathcal{P}) \leq f - d$$

- Kalai & Kleitman, 1992
  - $$\mathrm{diam}(\mathcal{P}) \leq f^{\log_2(d)+2}$$

The Hirsch Conjecture for $\mathcal{T}$

- $\mathcal{T}$ contains points $X = (x_{ij})$ from $\mathbb{R}^{mn}$

  but there are $m + n - 1$ independent equalities of type

$$\sum_{j=1}^{n} x_{ij} = r_i \qquad \text{and} \qquad \sum_{i=1}^{m} x_{ij} = c_j$$

  - so: $\dim(\mathcal{T}) = mn - m - n + 1$

- each inequality $x_{ij} \geq 0$ gives a facet

  - so: $\#\text{facets} = mn$

- Hirsch Conjecture true $\Longrightarrow$ $\operatorname{diam}(\mathcal{T}) \leq m + n - 1$

Bounds on diam($\mathcal{T}$)

- Hirsch Conjecture true $\Longrightarrow$ $\mathrm{diam}(\mathcal{T}) \leq m + n - 1$
  - also best possible

- Dyer & Frieze, 1994 $\mathrm{diam}(\mathcal{T}) \leq O(m^{16} n^3 \log^3 n)$
  (corollary of much more general result)

- Stougie, Oct 2002 $\mathrm{diam}(\mathcal{T}) \leq m^2 n$

- vdH & Stougie, Nov 2002 $\mathrm{diam}(\mathcal{T}) \leq \frac{1}{2}(m + n - 1)^2$

- Brightwell, vdH & S, Dec 2002
  $\mathrm{diam}(\mathcal{T}) \leq 8(m + n - 2)$

The structure of the skeleton of $\mathcal{T}$

- for the remainder, assume the problem is **non-degenerate** :
  - $\forall I \subsetneq \{1, \ldots, m\},\ J \subsetneq \{1, \ldots, n\} : \ \sum_{i \in I} r_i \neq \sum_{j \in J} c_j$

(if $\mathcal{T}$ degenerate, then a small perturbation of r_i and c_j gives a non-degenerate $\mathcal{T}^*$ with a larger diameter)

The vertices of $\mathcal{T}$

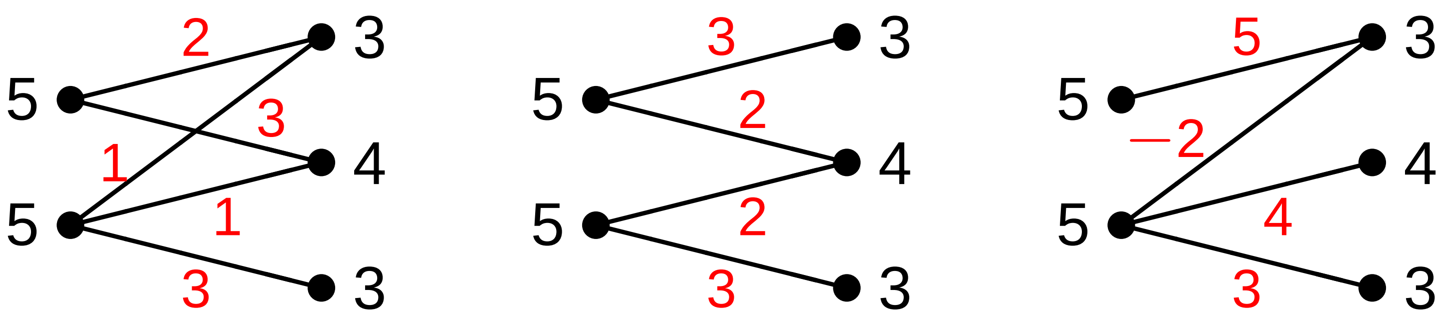
- for $X \in \mathcal{T}$, let $\mathbf{G(X)}$ be the subgraph of $K_{m,n}$ with edges

$$(i,j) \in E(X) \quad \Longleftrightarrow \quad x_{ij} > 0$$

- <u>Klee & Witzgall, 1968</u>

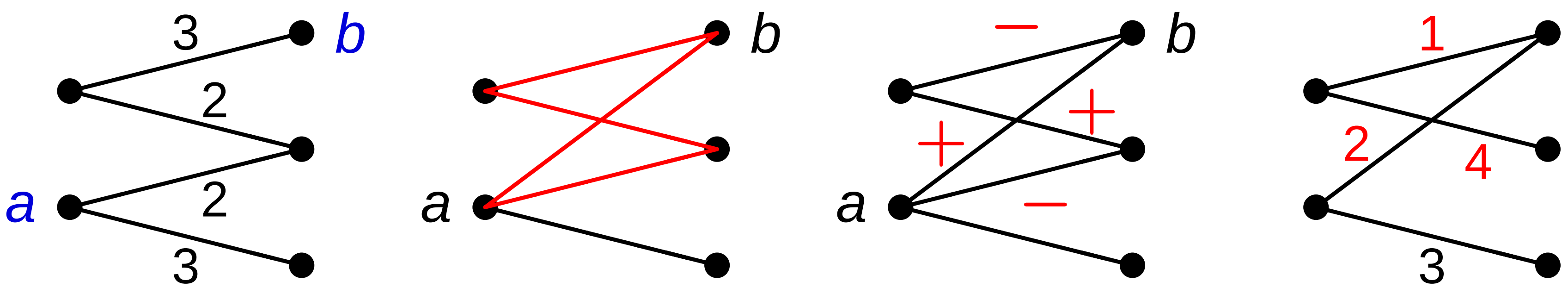
  X is a vertex of $\mathcal{T}$ $\quad \Longleftrightarrow \quad$ $G(X)$ is a (spanning) tree

- note: not every tree in $K_{m,n}$ can appear as a $G(X)$



The edges of $\mathcal{T}$

- vertex X with $x_{ab} = 0$, i.e., tree $G(X)$ and $(a,b) \notin E(X)$
- then a **pivot on (a,b)** is:
  - add edge (a,b): gives a unique cycle C of even length
  - label edges of C alternating $+/-$; giving (a,b) a $+$
  - remove $-$-edge with minimal value
  - change value other edges by $+/-$ the removed value



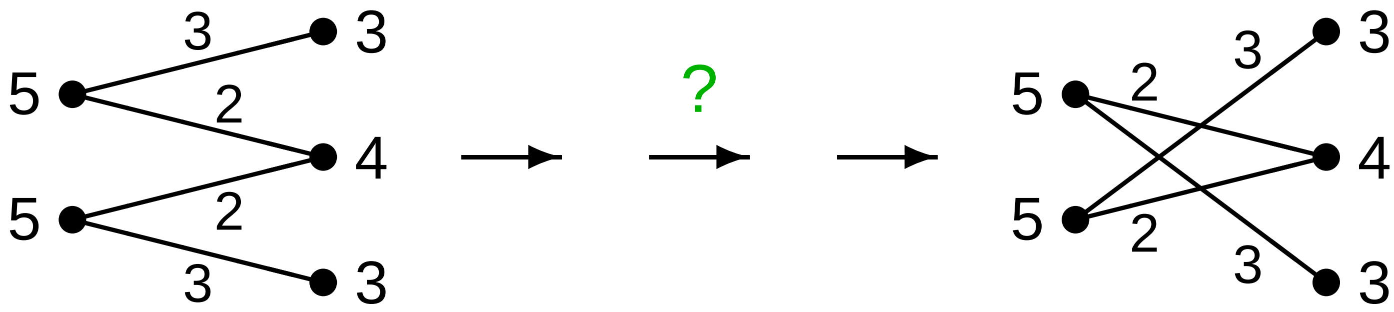
- pivot changing $G(X)$ to $G(Y)$ $\equiv$ edge from X to Y

The problem reformulated

- given m, n, $r_1, \ldots, r_m$, $c_1, \ldots, c_n$

  and a pair $X, Y \in \mathcal{T}$ so that $G(X), G(Y)$ are trees

- how many pivots are needed to get from $G(X)$ to $G(Y)$?



- easy to add a new edge (a, b) to a tree $G(X)$
  - but can we control the edge that gets removed ?

Some stronger conjectures

- $G(X), G(Y)$ trees corresponding to vertices $X, Y \in \mathcal{T}$
- a pivot in a tree adds one edge and removes one edge

Stronger Conjecture 1

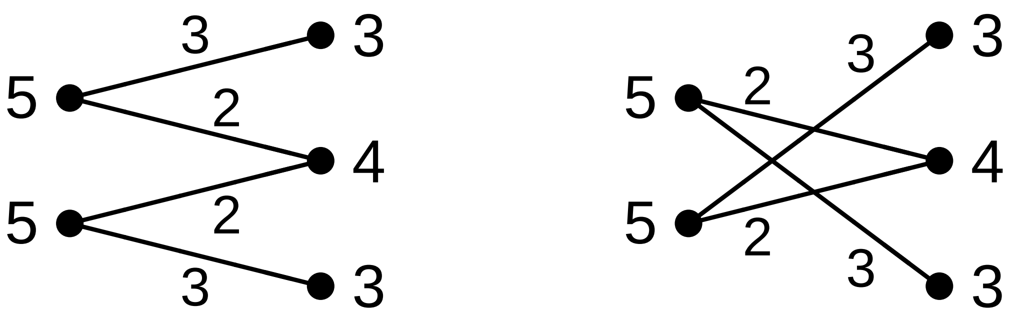
- there exists a pivot in $G(X)$ removing an edge from $G(X)$ and adding an edge from $G(Y)$

Stronger Conjecture 2

- the number of pivots needed to get from $G(X)$ to $G(Y)$ is $|E(X) \setminus E(Y)|$ $(= |E(Y) \setminus E(X)|)$

The stronger conjectures

- if $E(X) \cap E(Y) = \varnothing$:
  - Conjecture 1 trivially holds
  - Hirsch Conjecture $\Longrightarrow$ Conjecture 2 holds
    (since $|E(X) \setminus E(Y)| = |E(X)| = m + n - 1$)
- if $|E(X) \setminus E(Y)| = 1$: both conjectures hold
- but both conjectures are false in general :



Main ideas of our proofs : Leafs in trees

- **pendant edge** : edge incident with a leaf

- (a, b) a pendant edge in both $G(X)$ and $G(Y)$

  $\Longrightarrow$ either a is a leaf in both (if $r_a < c_b$),
  or b is a leaf in both

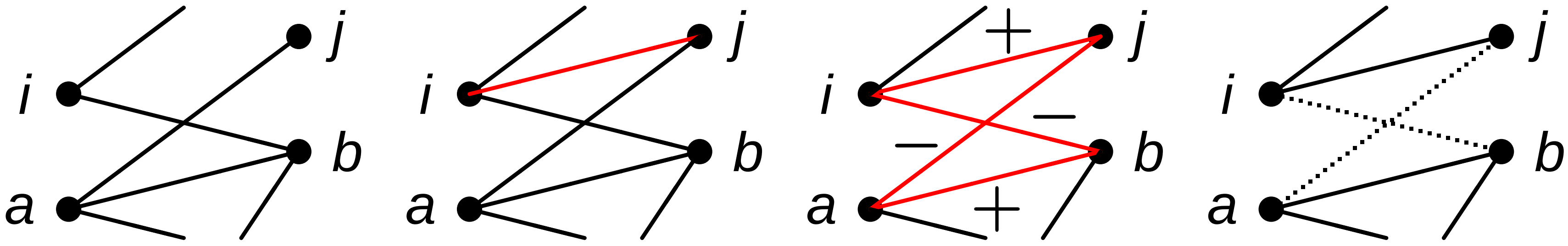
- a a leaf and (a, b) a pendant edge in both $G(X), G(Y)$

  $\Longrightarrow$ any pivot not involving a,
  leaves (a, b) a pendant edge
  and $x_{ab} = y_{ab}$ ($= r_a$)

  $\Longrightarrow$ $\text{dist}(G(X), G(Y)) = \text{dist}(G(X) - a, G(Y) - a)$

Making (a,b) *a pendant edge in a tree* $G(X)$

- if $(a,b) \notin E(X)$, insert it in one pivot step
- as long as (a,b) not a pendant edge
  (both $d_{G(X)}(a), d_{G(X)}(b) > 1$)
  - find $i \neq a$ and $j \neq b$ with $(i,b),(a,j) \in E(X)$
  - do a pivot inserting (i,j)
  - this removes one of $(i,b),(a,j)$,
    i.e., reduces $d_{G(X)}(a) + d_{G(X)}(b)$ by one



Making (a,b) a pendant edge in a tree $G(X)$

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  - (a,b) becomes a pendant edge

    when one of $d_{G(X)}(a), d_{G(X)}(b)$ becomes 1

    which happens after at most

    $$d_{G(X)}(a) + d_{G(X)}(b) - 3 \leq n + m - 3 \text{ pivots}$$

A quadratic bound on the diameter

Input: two trees $G(X), G(Y)$

- choose a pendant edge (a, b) in $G(Y)$
- transform $G(X)$ to $G(X^*)$,
  with (a, b) a pendant edge in $G(X^*)$
  - requires at most
    $1 + d_{G(X)}(a) + d_{G(X)}(b) - 3 \leq n + m - 2$ pivots
- now (a, b) is a pendant edge in both $G(X^*)$ and $G(Y)$
  $\Longrightarrow$ same end vertex of (a, b) is leaf in both
- remove common leaf from both $G(X^*)$ and $G(Y)$
- proceed by induction

Towards a linear bound

Main extra idea

- not : transform $G(X)$ to $G(X^*)$ to get closer to $G(Y)$
- but : transform $G(X)$ to $G(X^*)$ **and** $G(Y)$ to $G(Y^*)$ such that $G(X^*)$ and $G(Y^*)$ have common pendant edge
- remove the common leaf from $G(X^*)$ and $G(Y^*)$
- continue by induction

Claim : by choosing the edge (a, b) to be inserted carefully, one iteration of the above can be done in at most 8 pivots

- **uses** : trees have low average degree

Using average degree of trees

two very different cases

- $\frac{1}{2}n \leq m \leq 2n \quad \Longrightarrow \quad$ there exist a, b with

  $$d_{G(X)}(a) + d_{G(X)}(b) + d_{G(Y)}(a) + d_{G(Y)}(b) \leq 8$$

- $m > 2n$

  $\Longrightarrow$ every tree in $K_{m,n}$ has at least $\frac{1}{2}(m+1)$ leafs among the sources

  $\Longrightarrow$ there is a source that is leaf in both $G(X)$ and $G(Y)$

  - some further analysis . . .