List Colouring Squares of Planar Graphs

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The start of it all

- **chromatic number** $\chi(G)$:
  minimum number of colours needed for a vertex-colouring
- **vertex-colouring** :
  vertices at distance one must receive different colours

- suppose we require vertices at larger distances
  to receive different colours as well

- describe using vertex-colouring of powers of graphs
- G^p, p**-th power of** G :
  - same vertex set as G
  - edges between vertices with distance at most p in G

Bounds on the chromatic number of powers

- Δ : maximum degree of G

Easy facts

- any G $\Longrightarrow$ $\chi(G^p) \leq \Delta^p + O(\Delta^{p-1})$
- G contains a **regular Δ-tree** (all internal vertices have degree Δ) of height $\lfloor p/2 \rfloor$

$$\Longrightarrow \quad \chi(G^p) \geq \Delta^{\lfloor p/2 \rfloor} + O(\Delta^{\lfloor p/2 \rfloor - 1})$$

Theorem (Agnarsson & Halldórsson, 2003)

- G planar $\Longrightarrow$ $\chi(G^p) \leq O(\Delta^{\lfloor p/2 \rfloor})$

The square of planar graphs

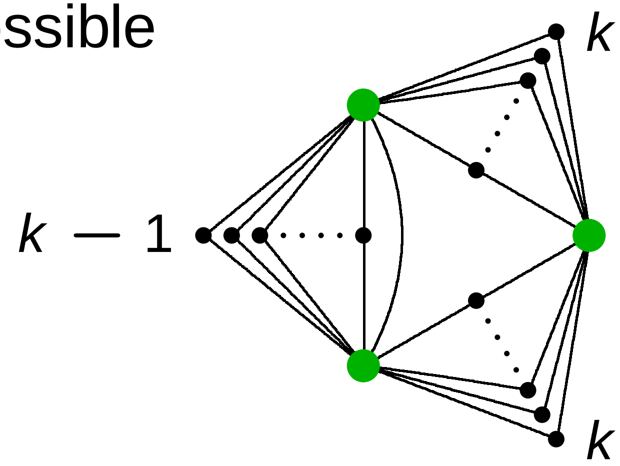
Conjecture (Wegner, 1977)

- G planar $\Longrightarrow$

$$\chi(G^2) \le \begin{cases} 7, & \text{if } \Delta = 3 \\ \Delta + 5, & \text{if } 4 \le \Delta \le 7 \\ \lfloor \frac{3}{2}\Delta \rfloor + 1, & \text{if } \Delta \ge 8 \end{cases}$$

- bounds would be best possible

case $\Delta = 2k \ge 8$:



What is known for large Δ

G planar $\Longrightarrow$

- $\chi(G^2) \leq 8\,\Delta - 22$ (Jonas, PhD, 1993)
- $\chi(G^2) \leq 3\,\Delta + 5$ (Wong, MSc, 1996)
- $\chi(G^2) \leq 2\,\Delta + 25$ (vdH & McGuinness, 2003)
- $\chi(G^2) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$ (for $\Delta \geq 47$)
  (Borodin, Broersma, Glebov & vdH, 2001)
- $\chi(G^2) \leq \lceil \frac{5}{3}\,\Delta \rceil + 24$ (for $\Delta \geq 241$)
  (Molloy & Salavatipour, 2005)

The new results

Theorem

- G planar $\Longrightarrow$ $\chi(G^2) \leq \frac{3}{2}\boldsymbol{\Delta} + o(\boldsymbol{\Delta}) \quad (\boldsymbol{\Delta} \to \infty)$

or, more precisely, a "list version with extras" :

- for all $\varepsilon > 0$ there exists a $\boldsymbol{\Delta}_\varepsilon$ so that if $\boldsymbol{\Delta} \geq \boldsymbol{\Delta}_\varepsilon$:
  - G a planar graph with maximum degree $\boldsymbol{\Delta}$
  - each vertex v has a list $L(v)$ of at least $(\frac{3}{2} + \varepsilon)\,\boldsymbol{\Delta}$ colours from $\mathbf{Z}$
- then there exists a proper colouring of G^2, such that
  - each vertex gets a colour from its own list
  - neighbours in G get colours at least $\boldsymbol{\Delta}^{1/4}$ apart

Basis of the proof : induction on the number of vertices

- **2-neighbour** : vertex at distance one or two
- $\boldsymbol{d^2(v)}$: number of 2-neighbours of v

  $=$ number of neighbours of v in G^2

- we would like to remove a vertex v with $d^2(v) \leq \frac{3}{2}\Delta$
  - but that can change distances in $G - v$
- contraction to a neighbour u will solve the distance problem
  - but may increase maximum degree if $d(u) + d(v) > \Delta$
- easy induction possible if there is an edge uv

  with $d(u) + d(v) \leq \Delta$ and $d^2(v) \leq \frac{3}{2}\Delta$

When easy induction is not possible

S, **small** vertices : degree at most $\Delta^{1/4}$

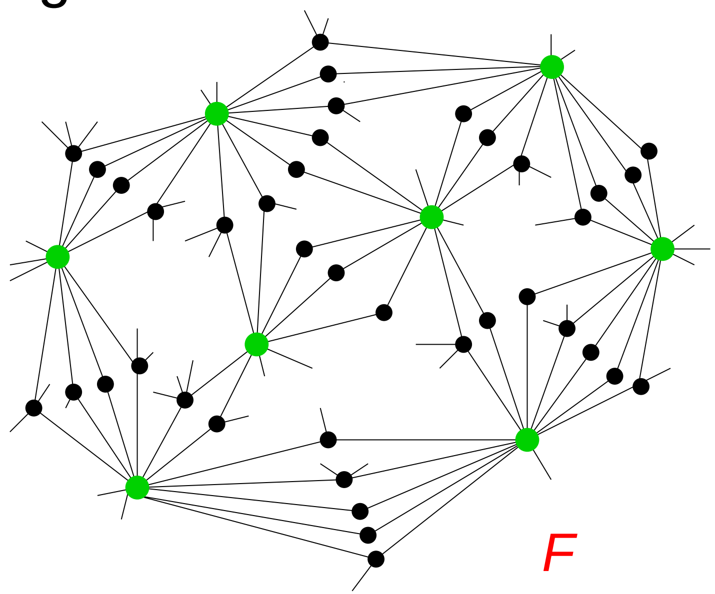
B, **big** vertices : degree more than $\Delta^{1/4}$

H, **huge** vertices : degree at least $\frac{1}{2}\Delta$

- small vertices have a least two big neighbours
- a planar graph on n vertices has fewer than $3n$ edges and fewer than $2n$ edges if it is bipartite
  - $\Longrightarrow$ all but $O(|V|/\Delta^{1/4})$ vertices are small
  - $\Longrightarrow$ fewer than $2|B|$ vertices in $V \setminus B$ have more than two neighbours in B
- "most" vertices are small
- and these have exactly two big neighbours — in fact huge

The structure so far

- there is a subgraph $\boldsymbol{F}$ of G looking like :



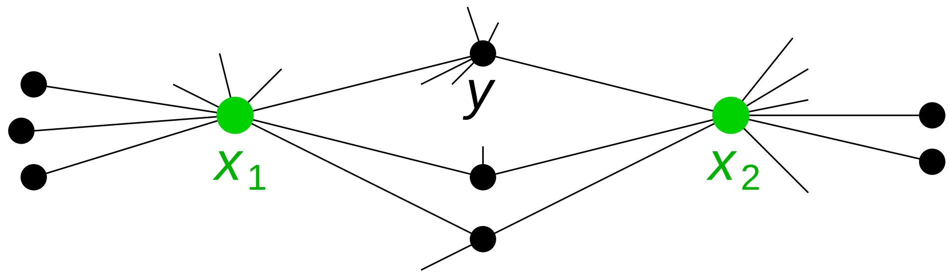
- green vertices $\boldsymbol{X}$
  have degree at least $\frac{1}{2}\,\Delta$
- black vertices $\boldsymbol{Y}$
  have degree at most $\Delta^{1/4}$
- all other neighbours of Y-vertices are also small

we can guarantee additionally :

- only "few" edges from X to rest of G
- F satisfies "some edge density condition"

The other induction step

- remove the vertices from Y (using contraction)
- colour the smaller graph (which is possible by induction)
- what to do with the uncoloured Y-vertices?



- y has
  - a "lot" of 2-neighbours in Y via x_1, x_2
  - at most $O(\Delta^{1/4} \cdot \Delta^{1/4})$ other 2-neighbours in Y
  - at most $(d_G(x_1) - d_F(x_1)) + (d_G(x_2) - d_F(x_2))$ 2-neighbours outside Y via x_1, x_2
  - at most $O(\Delta^{1/4} \cdot \Delta^{1/4})$ other 2-neighbours outside Y

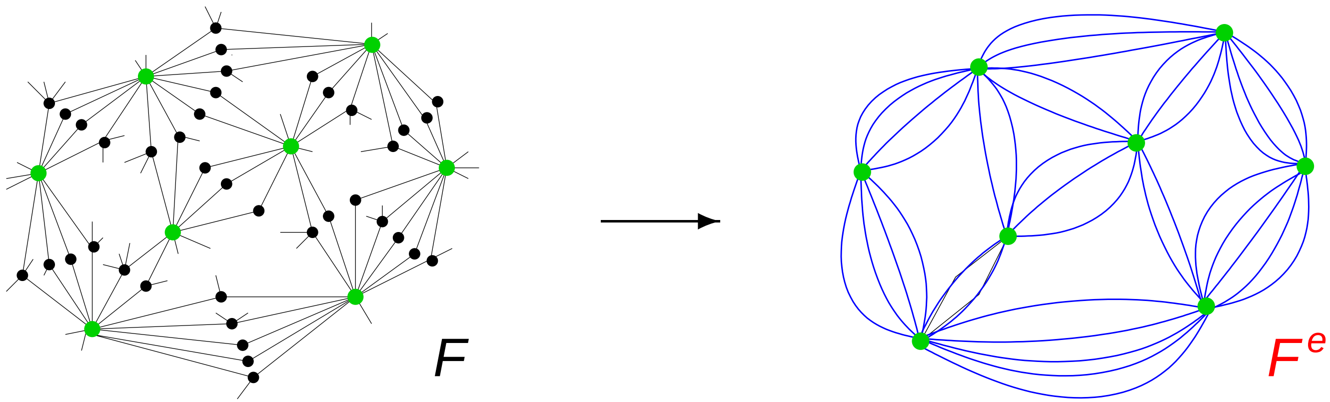
Transferring to edge-colouring

- so a vertex y from Y has at least

$$\left(\tfrac{3}{2}+\varepsilon\right)\Delta-(d_G(x_1)-d_F(x_1))-(d_G(x_2)-d_F(x_2))-O(\Delta^{1/2})$$

colours still available

- colouring Y is like colouring edges of the multigraph $\boldsymbol{F^e}$:



- but ... there may be up to $O(\Delta^{1/2})$ extra connections from an edge in F^e to other edges in F^e

Edge-colouring multigraphs

- $\chi'(G)$: chromatic index of multigraph G
- $\chi'_L(G)$: list chromatic index of multigraph G
- $\chi'_F(G)$: fractional chromatic index of multigraph G

Theorem (Kahn, 1996, 2000)

- G multigraph $\Longrightarrow$ $\chi'_L(G) \approx \chi'(G) \approx \chi'_F(G)$

(where "$\approx$" means "$=\ (1+o(1))\times$")

- in fact, Kahn's proofs provide something much more general

Kahn's result

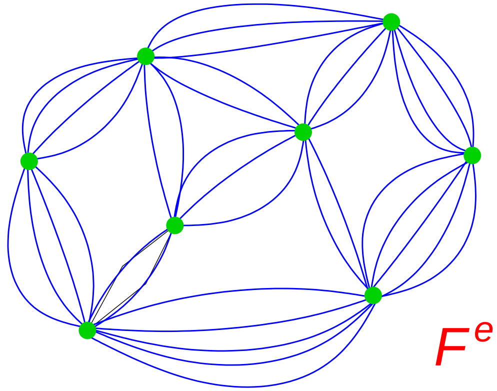
Theorem (Kahn, 2000)

for $0 < \delta < 1$, $C > 0$ there exists $\boldsymbol{\Delta}_{\delta,C}$ so that if $\boldsymbol{\Delta} \geq \boldsymbol{\Delta}_{\delta,C}$:

- G a multigraph with maximum degree $\boldsymbol{\Delta}$
- each edge e has a list $L(e)$ of colours so that
  - for all vertices v : $$\sum_{e \ni v} |L(e)|^{-1} \leq 1 \cdot (1 - \delta)$$
  - for all $K \subseteq G$ with $|V(K)| \geq 3$ odd : $$\sum_{e \in E(K)} |L(e)|^{-1} \leq \tfrac{1}{2}\,(|V(K)| - 1) \cdot (1 - \delta)$$
- then there exists a proper colouring of the edges of G so that each edge gets colours from its own list

Kahn's approach for our case

- we have a multigraph F^e :



- so that each edge $e = x_1 x_2$ has a list $L(e)$ of at least

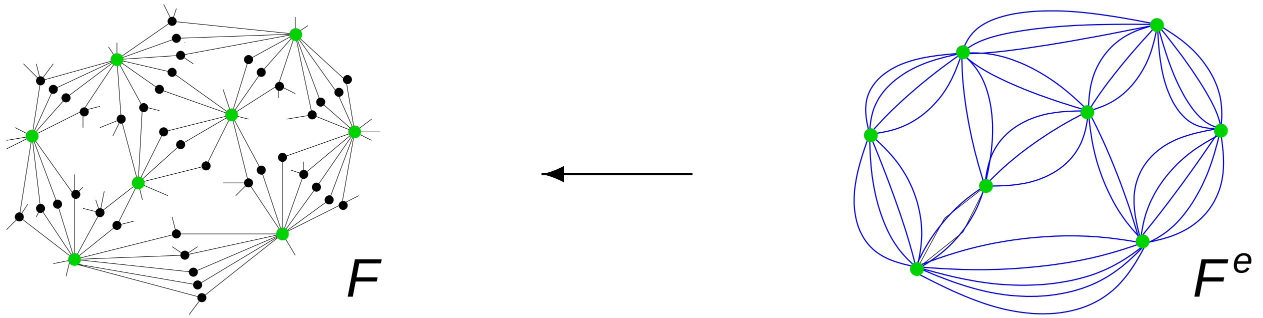
$$\left(\tfrac{3}{2}+\varepsilon\right)\Delta-(d_G(x_1)-d_F(x_1))-(d_G(x_2)-d_F(x_2))-O(\Delta^{1/2})$$

colours

- and F^e satisfies "some edge density condition"

Extending Kahn's approach

- these conditions guarantee that Kahn's conditions are satisfied for F^e
- $\Longrightarrow$ we can edge-colour F^e



- $\Longrightarrow$ we can colour the Y-vertices in F
  choosing from the left-over colours for each
- but ... there may be up to $O(\Delta^{1/2})$ extra connections
  from an edge in F^e to other edges in F^e
- redo Kahn's proof to take this "noise" into account

The next steps

- next to prove :

  G planar $\Longrightarrow$ $\chi(G^2) \leq \frac{3}{2}\Delta + O(1)$

- structural analysis of planar graphs for this can be done

- but would need a stronger version of Kahn's result :

  G multigraph $\Longrightarrow$ $\chi'_L(G) \leq \chi'_F(G) + O(1)$

Conjecture

(Vizing, 1975) and (Goldberg, 1973; Andersen, 1977; Seymour, 1979)

- G multigraph $\Longrightarrow$

  $\chi'_L(G) = \chi'(G)$ and $\chi'(G) \leq \chi'_F(G) + 1$