# Circular Orderings of Matroids

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## *Matroids – the basics*

- **matroid** $\mathcal{M}$

  with **ground set** $E$ and **independent subsets** $\mathcal{I}$

  - empty set is independent
  - subsets of independent sets are again independent
  - "exchange property"

**best example for today**

- start with a connected graph $G$
  - $E$ : edge set of $G$
  - $\mathcal{I}$ : subsets of edges without a cycle

## *Matroids – the basics*

- **rank** $\boldsymbol{r(A)}$: size of largest independent subset of $A$
  - $A$ dependent $\Leftrightarrow$ $r(A) < |A|$
- **base**: maximal independent set
  - all bases have the same size

    **rank of matroid** $\boldsymbol{r} = r(E)$
- **circuit**: minimal dependent set

**for the graphical matroid**

- **base**: spanning tree
- **rank** $= |V(G)| - 1$
- **circuit**: cycle

## *The main property we need*

- take $A$ independent and $x \in E \setminus A$
- consider $A + x$

**then either**

- $A + x$ is still independent

**or**

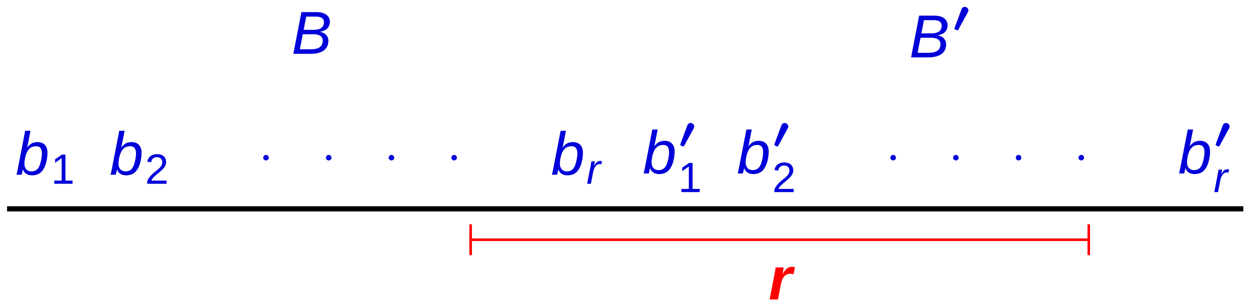
- $A + x$ is dependent, and then
  - there is a unique circuit $C \subseteq A + x$
  - for all $c \in C$: $(A + x) - c$ is independent

## *Linear ordering of two bases*

- take two disjoint bases $B$ and $B'$

<u>**then**</u>

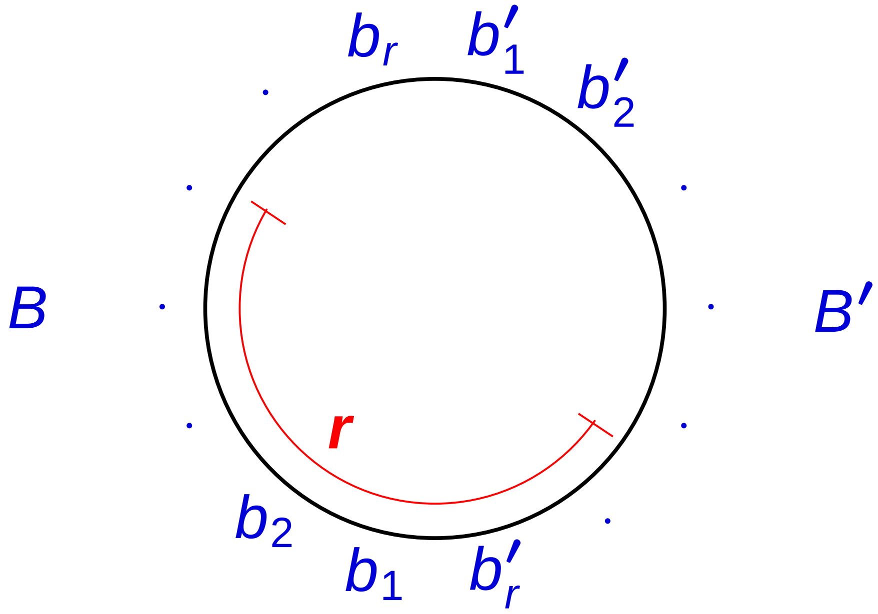
- we can order the elements of $B$ as $b_1, b_2, \ldots, b_r$
  and the elements of $B'$ as $b'_1, b'_2, \ldots, b'_r$
- so that in the total sequence $b_1, b_2, \ldots, b_r, b'_1, b'_2, \ldots, b'_r$ :
  each $\boldsymbol{r}$ **consecutive elements** are **independent**



# *Circular ordering of two bases*

- **there exists** : a sequence $b_1, b_2, \ldots, b_r, b'_1, b'_2, \ldots, b'_r$ in which each $r$ **consecutive elements** are **independent**

- **question** ( Gabov, 1976 ) can we form a **circular ordering** with the same property ?



# *Circular ordering of bases*

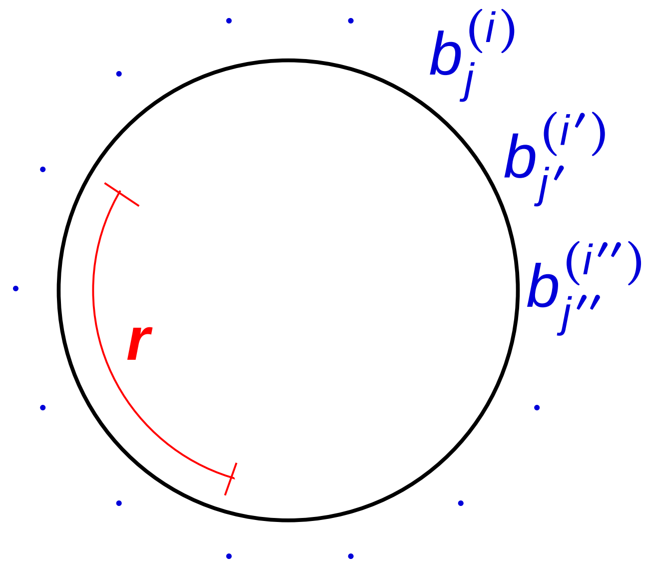
- **there exists** : a sequence $b_1, b_2, \ldots, b_r, b'_1, b'_2, \ldots, b'_r$ in which each $r$ **consecutive elements** are **independent**
- **question** ( Gabov, 1976 )
  can we form a **circular ordering** with the same property ?
  - **yes** for two spanning trees in a graph
  - **open** for matroids in general
- linear ordering can easily be extended to more bases
- what about a **circular ordering** for more than two bases ?
  - **open** for all cases

## *An easier (?) conjecture*

**Conjecture**

- given: $k$ disjoint bases $B^{(1)}, B^{(2)}, \ldots, B^{(k)}$
  with $B^{(i)} = \{\, b_1^{(i)}, b_2^{(i)}, \ldots, b_r^{(i)} \,\}$,
- then: there is a **circular ordering of all $k \cdot r$ elements**

  so that: each $r$ **consecutive elements** are **independent**



## *An easier (?) conjecture – cont*

- we can assume $E = B^{(1)} \cup B^{(2)} \cup \cdots \cup B^{(k)}$

  (remove all elements not in one of the bases)

**Conjecture** again

- a matroid with $E$ the disjoint union of $k$ of its bases

  has a "good" circular ordering

- conjectured by Wiedemann, 1984 (for union of two bases)
  - **yes** for graph formed by two disjoint spanning trees
  - **open** for all other cases

## *Towards a more general conjecture*

- does **every** matroid have a "good" circular ordering ?
- **no**, things go wrong if

  there is a subset $A \subseteq E$ so that $\dfrac{|A|}{r(A)} > \dfrac{|E|}{r}$

  then : for each circular ordering of $E$

  - there are $\boldsymbol{r}$ consecutive elements that contain more than $r(A)$ elements from $A$
  - those $\boldsymbol{r}$ consecutive elements contain a dependent subset
  - and hence those $\boldsymbol{r}$ consecutive elements are dependent themselves

## *The general conjecture*

**Conjecture** ( Kajitani *et al*, 1988 )

- given : matroid $\mathcal{M}$ with ground set $E$ and rank $r$
- then : there is a **circular ordering of $E$** so that each $r$ **consecutive elements** are **independent**

$\Leftrightarrow$

for all $A \subseteq E$ : $\dfrac{|A|}{r(A)} \leq \dfrac{|E|}{r}$

**Theorem**

- this is true if $|E|$ and $r$ are co-prime

## *The general conjecture – revisited*

- for all $A \subseteq E$ : $\dfrac{|A|}{r(A)} \leq \dfrac{|E|}{r}$

  $\Leftrightarrow$ ? there is a "good" circular ordering

- a "good" circular ordering is equivalent to
  - there exists a circular ordering of $|E|$ bases such that
    - two consecutive bases **differ in one element**
    - each element appears in a list of **consecutive bases**
    - each element appears in $\boldsymbol{r}$ bases

**<u>Theorem</u>**

- we can guarantee this for any two of the three properties

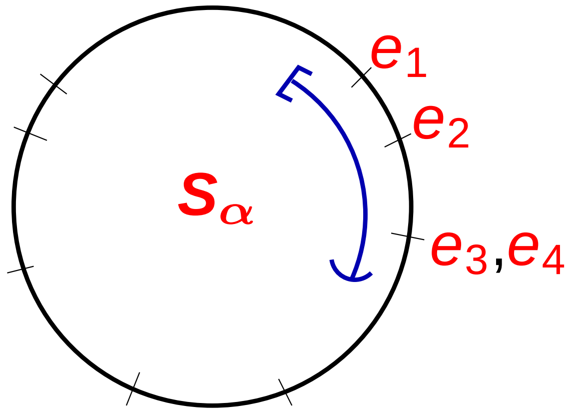
## *A different kind of circular ordering*

- $\boldsymbol{S}_\alpha$ : circle with circumference $\alpha$ $(\alpha \in \boldsymbol{S},\ \alpha > 0)$

  think : interval $[\,0\,,\,\alpha\,)$ with a circular ordering

- we want to map the elements $E$ of $\mathcal{M}$ to $\boldsymbol{S}_\alpha$ so that :
  - for every unit interval $[\,a,\ a+1\,)$ of $\boldsymbol{S}_\alpha$ :

    the elements mapped into that interval are independent




- **circular arboricity** of $\mathcal{M}$, $\boldsymbol{\Upsilon_C(\mathcal{M})}$ :

  minimum $\alpha$ for which this is possible

## *A bound on the circular arboricity*

- we must have for every subset $A \subseteq E$ :
  - an independent set has at most $r(A)$ elements from $A$
  - so every unit interval of $\boldsymbol{S}_\alpha$ can have at most $r(A)$ elements from $A$
  - so we need $\alpha \geq \dfrac{|A|}{r(A)}$
- and hence $\Upsilon_C(\mathcal{M}) \geq \max\limits_{A \subseteq E} \dfrac{|A|}{r(A)}$

**Conjecture** (Gonçalves): $\Upsilon_C(\mathcal{M}) = \max\limits_{A \subseteq E} \dfrac{|A|}{r(A)}$

**It's a Theorem !!**

## *Integral arboricity*

**Theorem** ( Nash-Williams, 1964; Edmonds, 1964 )

- if : $K \geq \left\lceil \max_{A \subseteq E} \frac{|A|}{r(A)} \right\rceil$, for some $K \in \mathbf{N}$

  then : $E$ can be partitioned into $K$ disjoint independent sets

- this means : $\Upsilon_C(\mathcal{M}) \leq \left\lceil \max_{A \subseteq E} \frac{|A|}{r(A)} \right\rceil$

- also provides polynomial time algorithm to find $\max_{A \subseteq E} \frac{|A|}{r(A)}$

# *Fractional arboricity*

- circular arboricity can be considered as some kind of "**fractional**" **arboricity**
- a more natural fractional arboricity concept is the solution to the following LP-problem :
  - $x_J$ : real-valued variable for an independent set $J$
  - minimise : $\sum_{J \in \mathcal{I}} x_J$

    such that : $\forall e \in E : \sum_{J \ni e} x_J \geq 1$

    $\forall J : x_J \geq 0$
- <u>**folklore**</u> : this minimum is equal to $\max_{A \subseteq E} \frac{|A|}{r(A)}$

# *Quick proof of the fractional arboricity*

- suppose $\max_{A \subseteq E} \frac{|A|}{r(A)} = \frac{P}{Q}$
- form $\mathcal{M}^Q$ : replace each element by $Q$ parallel elements
- then $\left\lceil \max_{A \subseteq E^Q} \frac{|A|}{r(A)} \right\rceil = \max_{A \subseteq E^Q} \frac{|A|}{r(A)} = P$
- Edmonds :

  $\mathcal{M}^Q$ can be covered with $P$ disjoint independent sets
- so $P$ independent sets covering each element $Q$ times
- set $x_J = 1/Q$ for these independent sets □

## *Points and intervals on a circle*

- $\forall\, \alpha \geq \max\limits_{A \subseteq E} \dfrac{|A|}{r(A)} \quad \Rightarrow$
  - there is a mapping $\varphi : E \longrightarrow \boldsymbol{S}_\alpha$
  - so that for every unit interval $[\, a,\, a+1\,)$ of $\boldsymbol{S}_\alpha$ :

    the elements mapped into that interval are independent

- **equivalent conclusion** :
  - there is a mapping $\varphi^* : E \longrightarrow \boldsymbol{S}_\alpha$
  - so that for every element $x \in \boldsymbol{S}_\alpha$ :

    the elements $e$ whose unit intervals

    $[\, \varphi^*(e)\,,\ \varphi^*(e)+1\,)$ contain $x$ are independent

## *A weighted variant*

- suppose we are given non-negative weights $\omega(e)$
- we want to map the elements $E$ to intervals $[\,\varphi^*(e)\,,\,\varphi^*(e)+\omega(e)\,)$ on $\boldsymbol{S}_\alpha$ so that :
  - for every point $x \in \boldsymbol{S}_\alpha$ :
    the elements whose intervals contain $x$ are independent
- **weighted circular arboricity** of $\mathcal{M}$, $\boldsymbol{\Upsilon_C(\mathcal{M},\omega)}$ :
  minimum $\alpha$ for which this is possible
- obvious lower bound again : $\Upsilon_C(\mathcal{M},\omega) \geq \max\limits_{A\subseteq E} \dfrac{\sum_{a\in A}\omega(a)}{r(A)}$

## *Weighted arboricity*

**Theorem**

- any matroid $\mathcal{M}$ and non-negative weight $\omega : E \longrightarrow \mathbf{R}^+$ :

$$\Upsilon_C(\mathcal{M}, \omega) = \max_{A \subseteq E} \frac{\sum_{a \in A} \omega(a)}{r(A)}$$

- all other results so far follow by taking appropriate weights

## *A little about the proof of the weighted case*

- **to prove**: $\alpha \geq \max_{A \subseteq E} \left[ \sum_{a \in A} \omega(a) \,/\, r(A) \right] \qquad \Rightarrow$
  - there is a mapping $\varphi^* : E \longrightarrow \boldsymbol{S}_\alpha$
  - so that for every point $x \in \boldsymbol{S}_\alpha$ :
    the elements $e$ whose intervals
    $[\,\varphi^*(e)\,,\,\varphi^*(e) + \omega(e)\,)$ contain $x$ are independent

- start of proof
  - we can assume $\alpha$ and weights are integers
  - can be seen as a problem about
    $\omega(e)$ consecutive positions on the "discrete" cycle $\boldsymbol{Z}_\alpha$
  - use induction on $\sum_{e \in E} \omega(e)$