A General Approach to Distance-Two Colouring of Graphs

Jan van den Heuvel

based on joint work with Omid Amini & Louis Esperet

and Frédéric Havet, Colin McDiarmid & Bruce Reed

Department of Mathematics
London School of Economics and Political Science

The basics of graph colouring

- **graph** $G = (V, E)$: finite, simple, undirected
- **vertex-colouring** with k colours :
  adjacent vertices receive different colours
- **chromatic number** $\chi(G)$:
  minimum k so that a vertex-colouring exists
- **list vertex-colouring** : as vertex-colouring,
  but each vertex v has its own list $L(v)$ of colours
- **choice number** $ch(G)$: minimum k so that if $|L(v)| \geq k$,
  then a proper list vertex-colouring exists

Easy and not-so-easy facts

- **vertex degree** $d_G(v)$: number of vertices adjacent to v
- **maximum degree** $\Delta(G)$, or just Δ

Easy fact

- $1 \le \chi(G) \le ch(G) \le \Delta(G)+1$

(and all inequalities can be arbitrarily far off)

Far from easy facts (Appel & Haken, 1977; Thomassen, 1994)

- G planar $\implies$ $\chi(G) \le 4$, $ch(G) \le 5$

Another way to look at vertex-colouring

- vertex-colouring :

  vertices at distance one must receive different colours

- suppose we require vertices at larger distances

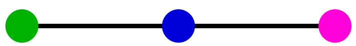
  to receive different colours as well

- for today we only look at distance two

- yet another way to look at this :
  - vertex-colouring :

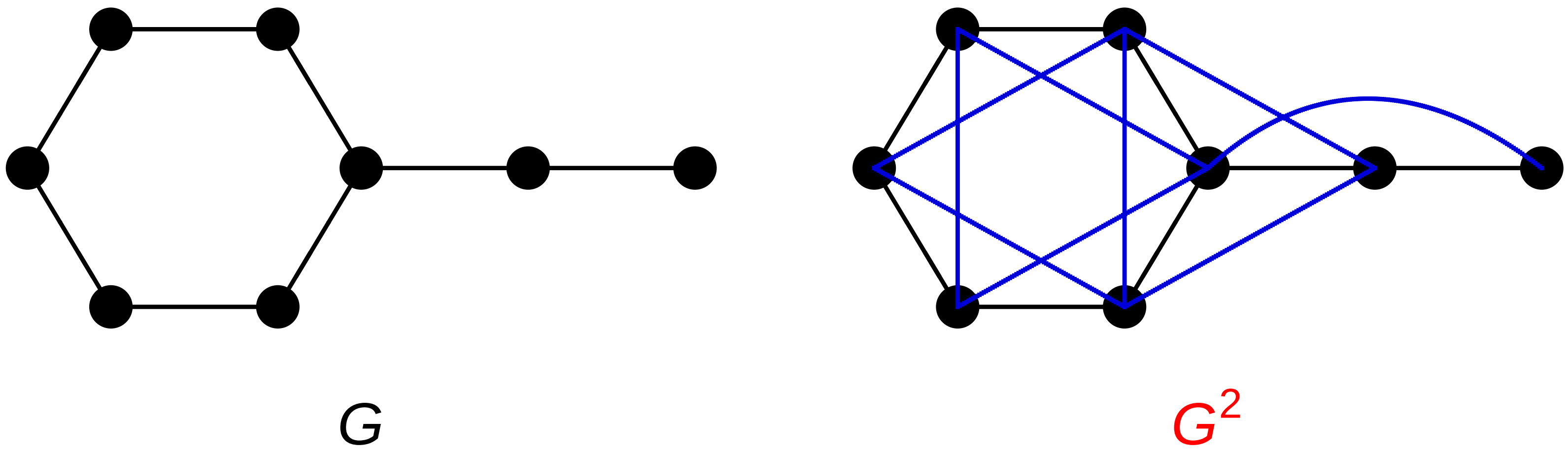
    vertices in all P_2  must get two colours

  - colouring at distance two :

    vertices in all P_3  must get three colours

The square of a graph

- distance-two colouring can be modelled using the **square G^2 of a graph**:
  - same vertex set as G
  - edges between vertices with distance at most 2 in G
    ($=$ are adjacent or have a common neighbour)



Colouring the square of a graph

Easy facts

- $\chi(G^2) \geq \Delta(G)+1$

and

- $\Delta(G^2) \leq \Delta(G)^2$, **so** $\chi(G^2) \leq \Delta(G)^2+1$

Question

- is the upper bound relevant ?

- there are at most 4 graphs with $\chi(G^2) = \Delta(G)^2+1$
- and infinitely many graphs with
$$\chi(G^2) = \Delta(G)^2 - \Delta(G) + 1$$

The square of planar graphs

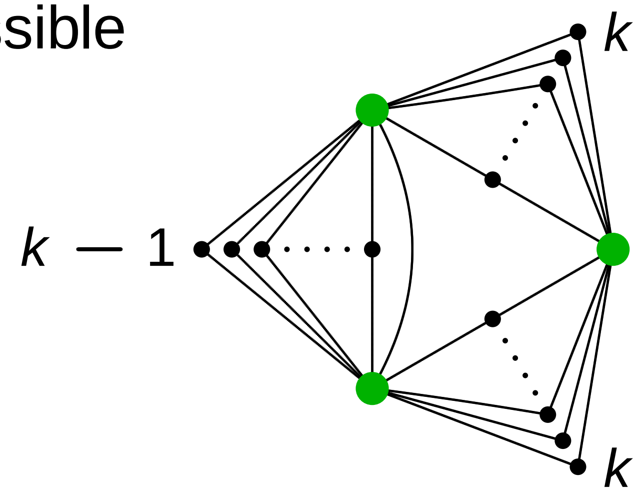
Conjecture (Wegner, 1977)

- G planar

$$\Longrightarrow \quad \chi(G^2) \le \begin{cases} 7, & \text{if } \boldsymbol{\Delta} = 3 \\ \boldsymbol{\Delta} + 5, & \text{if } 4 \le \boldsymbol{\Delta} \le 7 \\ \lfloor \frac{3}{2}\boldsymbol{\Delta} \rfloor + 1, & \text{if } \boldsymbol{\Delta} \ge 8 \end{cases}$$

- bounds would be best possible

case $\boldsymbol{\Delta} = 2k \ge 8$:



What is known for large Δ

G planar $\Longrightarrow$

- $\chi(G^2) \leq 8\,\Delta - 22$ (Jonas, PhD, 1993)
- $\chi(G^2) \leq 3\,\Delta + 5$ (Wong, MSc, 1996)
- $\chi(G^2) \leq 2\,\Delta + 25$ (vdH & McGuinness, 2003)
- $\chi(G^2) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$ (for $\Delta \geq 47$)
  (Borodin, Broersma, Glebov & vdH, 2001)
- $\chi(G^2) \leq \lceil \frac{5}{3}\,\Delta \rceil + 24$ (for $\Delta \geq 241$)
  (Molloy & Salavatipour, 2005)

First new results

Theorem

- G planar $\Longrightarrow$ $\chi(G^2) \leq \left(\frac{3}{2} + o(1)\right)\Delta$ $(\Delta \to \infty)$

we actually prove the list-colouring version :

- for all $\varepsilon > 0$ there exists a Δ_ε so that if $\Delta \geq \Delta_\varepsilon$:
  - G a planar graph with maximum degree Δ
  - all lists $L(v)$ satisfy $|L(v)| \geq \left(\frac{3}{2} + \varepsilon\right)\Delta$
- $\Longrightarrow$ there exists a proper colouring of G^2, so that
  - each vertex v gets a colour from its own list $L(v)$

- the proof does not give an algorithm

Beyond planar ?

- a graph H is a **minor** of a graph G :

  H can be obtained from G by sequence of
  - vertex removals
  - edge removals
  - edge contractions

- otherwise : G is $\boldsymbol{H}$**-minor free**

<u>**Well-known**</u> (Kuratowski, Wagner)

- G planar $\iff$ G is K_5-minor free and $K_{3,3}$-minor free

Beyond planar!

Theorem

- graph G $K_{3,k}$-minor free for some fixed k

$$\Longrightarrow \quad ch(G^2) \;\leq\; \left(\tfrac{3}{2}+o(1)\right)\Delta$$

Question

- what is the best upper bound for $ch(G^2)$

  if G is H-minor free for some fixed graph H ?

  (in the talk I conjectured that always $ch(G^2) \leq \frac{3}{2}\,\Delta + c_H$, but that is false)

Property

- graph G H-minor free for some fixed graph H

$$\Longrightarrow \quad ch(G^2) \;\leq\; C_H\,\Delta \quad \text{for some fixed constant } C_H$$

The clique number

- **clique number** $\omega(G)$: maximum size of a **clique** in G
- obvious that $\omega(G) \leq \chi(G)$

Corollary

- graph G $K_{3,k}$-minor free for some fixed k

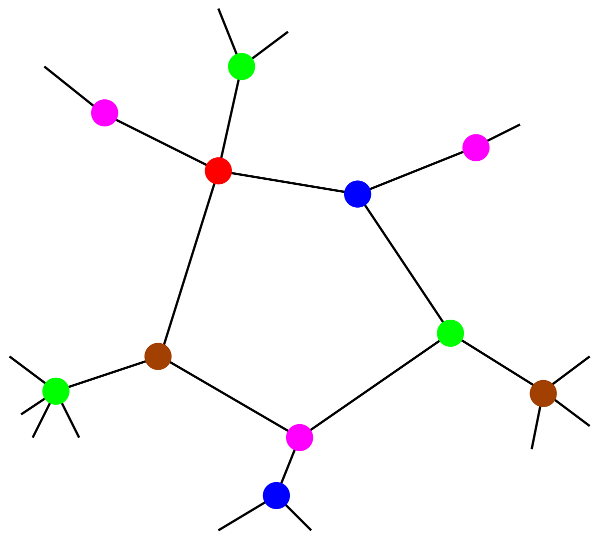
$$\Longrightarrow \quad \omega(G^2) \leq \left(\tfrac{3}{2} + o(1)\right)\Delta$$

Theorem

- G planar $\Longrightarrow \quad \omega(G^2) \leq \frac{3}{2}\Delta + O(1)$

A related (?) problem

- **plane graph** : planar graph with a given embedding
- **cyclic colouring** of a plane graph :
  - vertex-colouring so that
  - vertices incident to the same face get a different colour



A related (?) problem

- **plane graph** : planar graph with a given embedding
- **cyclic colouring** of a plane graph :
  - vertex-colouring so that
  - vertices incident to the same face get a different colour
- **cyclic chromatic number** $\chi^*(G)$:
  minimum number of colours needed for a cyclic colouring
- $\Delta^*(G)$: size of largest face of G

Easy fact

- G plane graph $\Longrightarrow$ $\chi^*(G) \geq \Delta^*(G)$

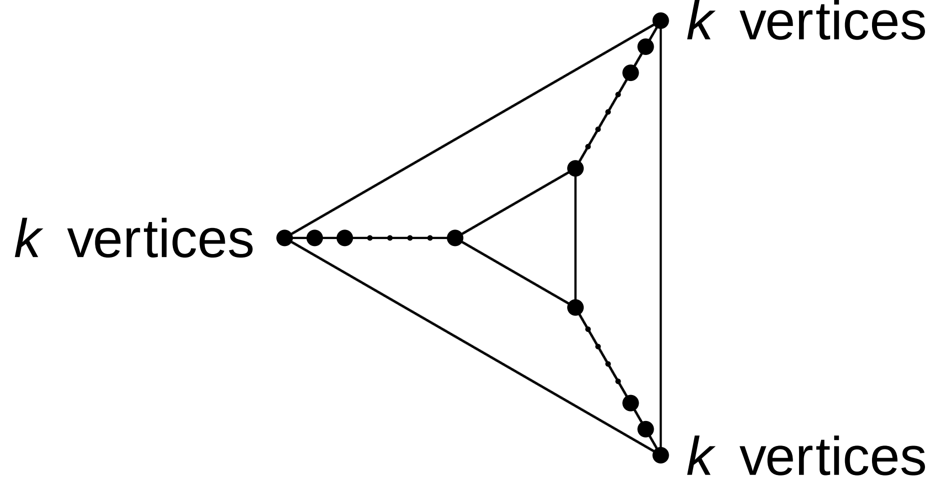
Conjecture for the related(?) problem

Conjecture (Borodin, 1984)

- G plane graph $\Longrightarrow$ $\chi^*(G) \le \frac{3}{2}\,\Delta^*(G)$

- bound would be best possible

case $\Delta^* = 2\,k$:



Bounds on the cyclic chromatic number

G plane graph $\Longrightarrow$

- $\chi^*(G) \le 2\,\Delta^*(G)$ (Ore & Plummer, 1969)
- $\chi^*(G) \le \frac{9}{5}\,\Delta^*(G)$ (Borodin, Sanders & Zhao, 1999)
- $\chi^*(G) \le \frac{5}{3}\,\Delta^*(G)$ (Sanders & Zhao, 2001)

Theorem

- G plane graph

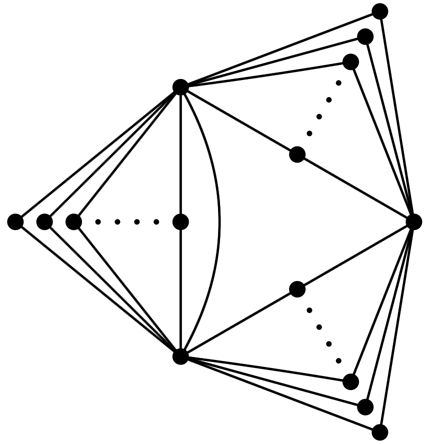
$$\Longrightarrow \quad \chi^*(G) \le \left(\tfrac{3}{2} + o(1)\right)\Delta^* \quad (\Delta^* \to \infty)$$

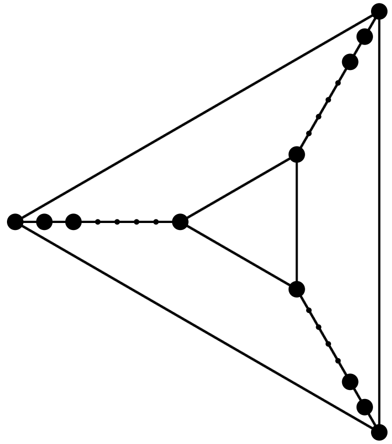
(in fact, we again proved the list-colouring version)

Are the problems the same ?

colouring square of planar graphs
vs
cyclic colouring of plane graphs

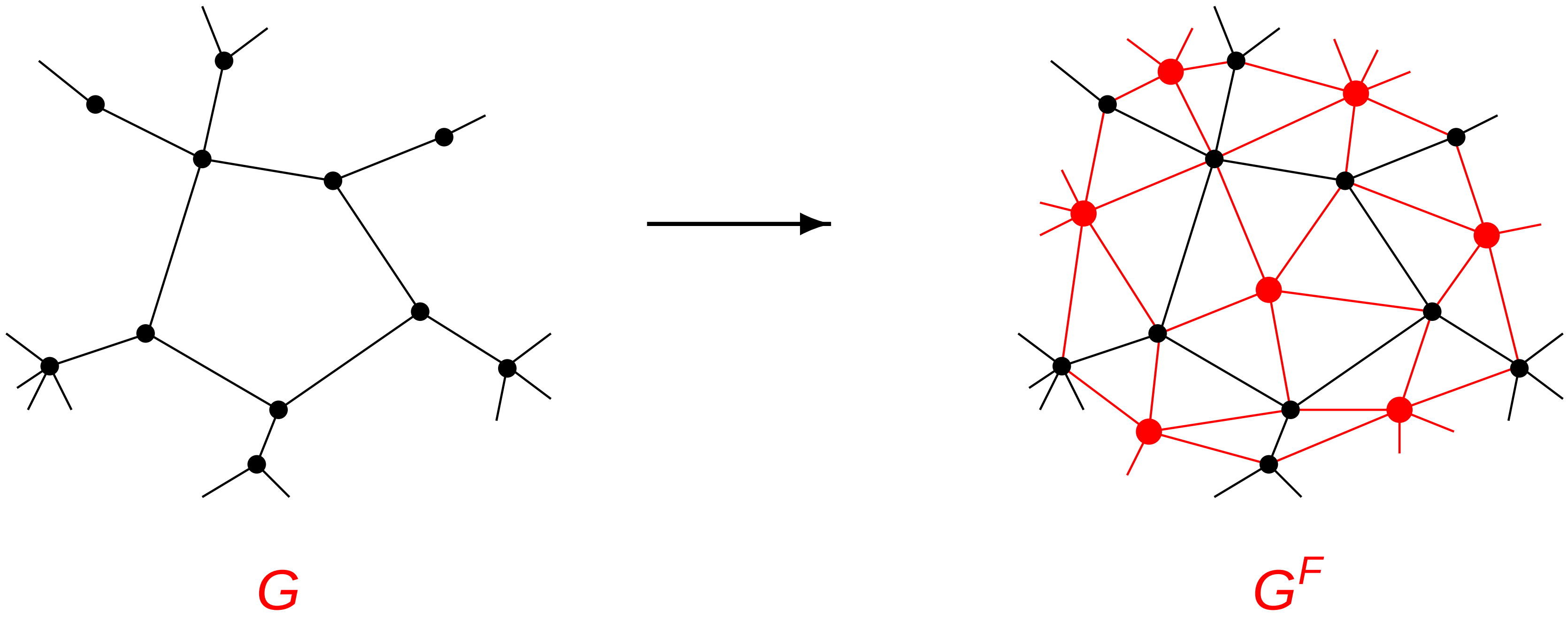
- similar conjectured upper bounds : $\frac{3}{2}\,\Delta + 1$ vs $\frac{3}{2}\,\Delta^*$
- similar known bounds, like : $\frac{5}{3}\,\Delta + 24$ vs $\frac{5}{3}\,\Delta^*$
- similar ("dual") extremal graphs :



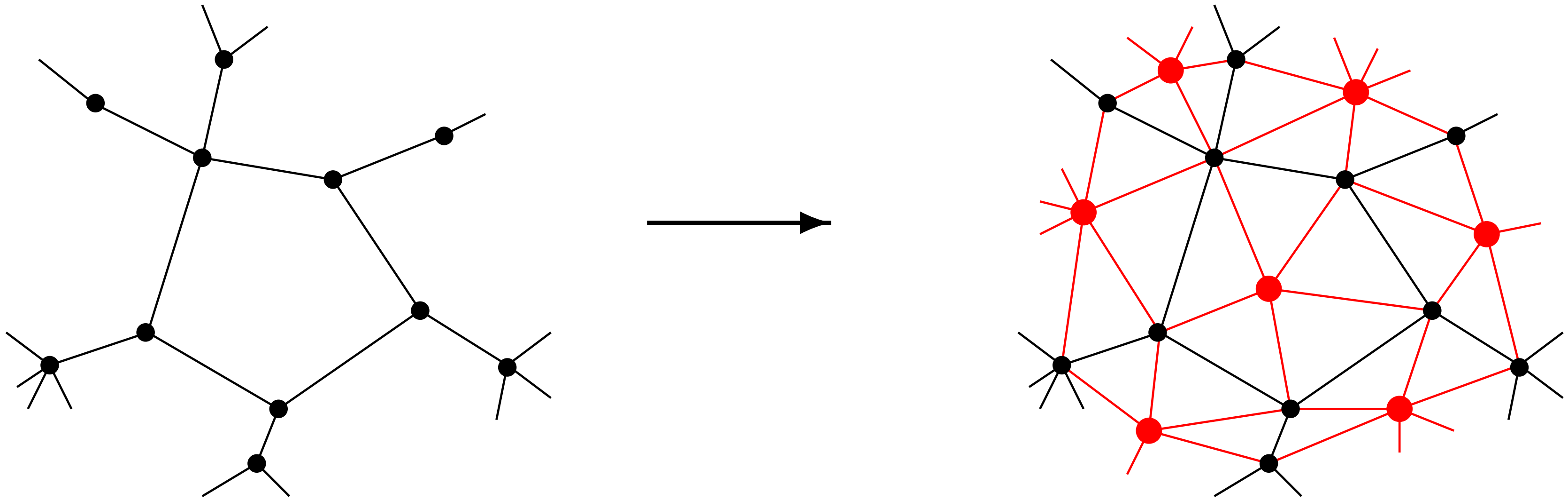


From cyclic colouring to “square” colouring

- given a plane graph G, form new graph G^F by
  - adding a vertex in each face
  - adding edges from new vertex to all vertices of the face



From cyclic colouring to "square" colouring



- colouring the square of the new graph will give cyclic colouring of original graph
  - but also colours the faces (not asked for, but o.k.)
  - and degrees of old vertices may be more than $\Delta^*(G)$

    (serious problem)

So close, yet not the same

how to combine the two concepts ?

idea : do not treat all vertices equal :

- some vertices need to be coloured
- some vertices determine distance two

- vertices can be of both types
  - as are all vertices when colouring the square
- or the types can be disjoint
  - as for “faces” and “vertices” for cyclic colouring
- or anything in between

Formalising our clever little idea

- **given**: graph G, subsets $A, B \subseteq V(G)$
- $(\boldsymbol{A}, \boldsymbol{B})$**-colouring of** $\boldsymbol{G}$: colouring of vertices in B so that
  - adjacent vertices get different colours
  - vertices with a common neighbour in A
    get different colours
- **list** $(\boldsymbol{A}, \boldsymbol{B})$**-colouring of** $\boldsymbol{G}$:
  - similar, but each vertex in B has its own list
- $\chi(G; A, B)$ / $ch(G; A, B)$:
  minimum number of colours needed /
  minimum size of each list needed

The first baby steps

- $\chi(G) \;=\; \chi(G;\varnothing,V)$
- $\chi(G^2) \;=\; \chi(G;V,V)$
- $\chi^*(G) \;=\; \chi(G^F;\text{“faces”},\text{“vertices”})$

- relevant “**degree**” : $d_B(v) =$ number of neighbours in B
- “**maximum degree**” :

$$\boldsymbol{\Delta}(G;A,B) = \text{ maximum of } d_B(v) \text{ over all } v \in A$$

Easy fact

- $\boldsymbol{\Delta}(G;A,B) \;\le\; \chi(G;A,B) \;\le\; ch(G;A,B)$

$$\le\; \boldsymbol{\Delta}(G) + \boldsymbol{\Delta}(G;A,B)\cdot(\boldsymbol{\Delta}(G;A,B)-1)+1$$

The obvious(?) conjecture & results

Conjecture

- G planar, $A, B \subseteq V(G)$, $\mathbf{\Delta}(G; A, B)$ large enough

$$\Longrightarrow \quad ch(G; A, B) \leq \tfrac{3}{2}\,\mathbf{\Delta}(G; A, B) + 1$$

Theorem

- G planar, $A, B \subseteq V(G)$

$$\Longrightarrow \quad ch(G; A, B) \leq \left(\tfrac{3}{2} + o(1)\right)\mathbf{\Delta}(G; A, B)$$

Corollary

(asymptotic list version of Wegner's and Borodin's Conjecture)

- G planar $\quad \Longrightarrow \quad ch(G^2) \leq \left(\tfrac{3}{2} + o(1)\right)\mathbf{\Delta}(G)$
- G plane graph $\quad \Longrightarrow \quad ch^*(G) \leq \left(\tfrac{3}{2} + o(1)\right)\mathbf{\Delta}^*(G)$

Sketch of the proof of square of planar graph

uses induction on the number of vertices

- **2-neighbour** : vertex at distance one or two
- $\boldsymbol{d^2(v)}$: number of 2-neighbours of v
  $=$ number of neighbours of v in G^2
- we would like to remove a vertex v with $d^2(v) \leq \frac{3}{2}\Delta$
  - but that can change distances in $G - v$
- contraction to a neighbour u will solve the distance problem
  - but may increase maximum degree if $d(u) + d(v) > \Delta$
- easy induction possible if there is an edge uv
  with $d(u) + d(v) \leq \Delta$ and $d^2(v) \leq \frac{3}{2}\Delta$

When easy induction is not possible

S, **small** vertices : degree at most some constant C

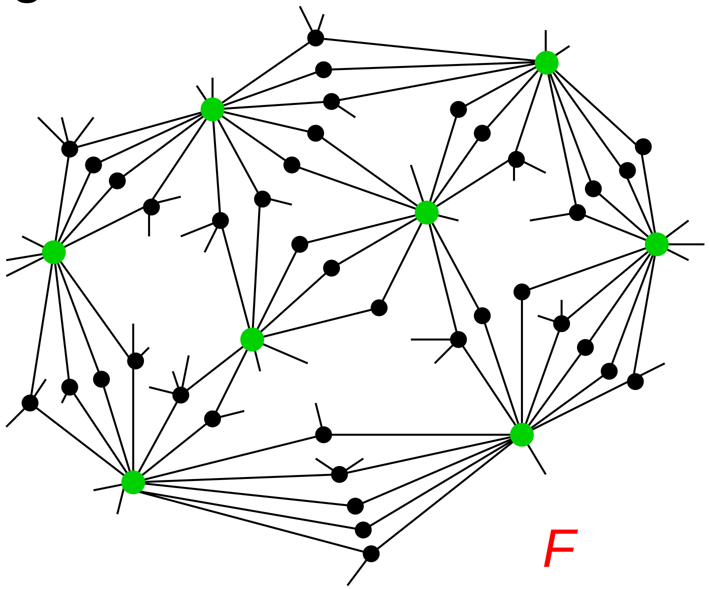
B, **big** vertices : degree more than C

H, **huge** vertices : degree at least $\frac{1}{2}\,\Delta$

- small vertices have a least two big neighbours
- a planar graph has fewer than $3\,|V|$ edges
  and fewer than $2\,|V|$ edges if it is bipartite
  - $\Longrightarrow$ all but $O(|V|/C)$ vertices are small
  - $\Longrightarrow$ fewer than $2\,|B|$ vertices in $V \setminus B$ have more than two neighbours in B
- **so** : "most" vertices are small
- and these have exactly two big neighbours — in fact huge

The structure so far

- there is a subgraph ***F*** of G looking like :



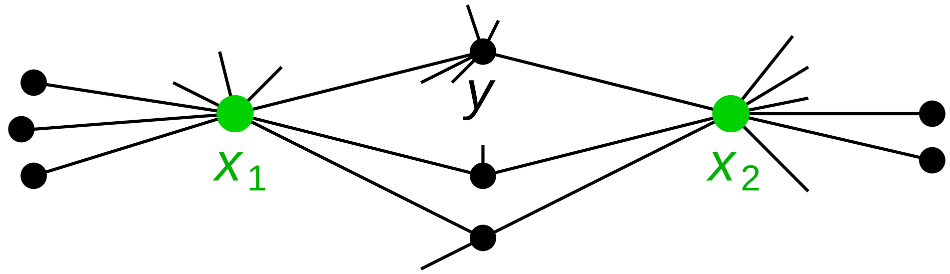
- green vertices ***X***
  have degree at least $\frac{1}{2}\,\Delta$
- black vertices ***Y***
  have degree at most C
- all other neighbours of Y -vertices are also small

we can guarantee additionally :

- only “few” edges from X to rest of G
- F satisfies “some edge density condition”

The other induction step

- remove the vertices from Y (using contraction)
- colour the smaller graph (which is possible by induction)
- what to do with the uncoloured Y-vertices?



- y has
  - a "lot" of 2-neighbours in Y via x_1, x_2
  - at most C^2 other 2-neighbours in Y
  - at most $(d_G(x_1) - d_F(x_1)) + (d_G(x_2) - d_F(x_2))$ 2-neighbours outside Y via x_1, x_2
  - at most C^2 other 2-neighbours outside Y

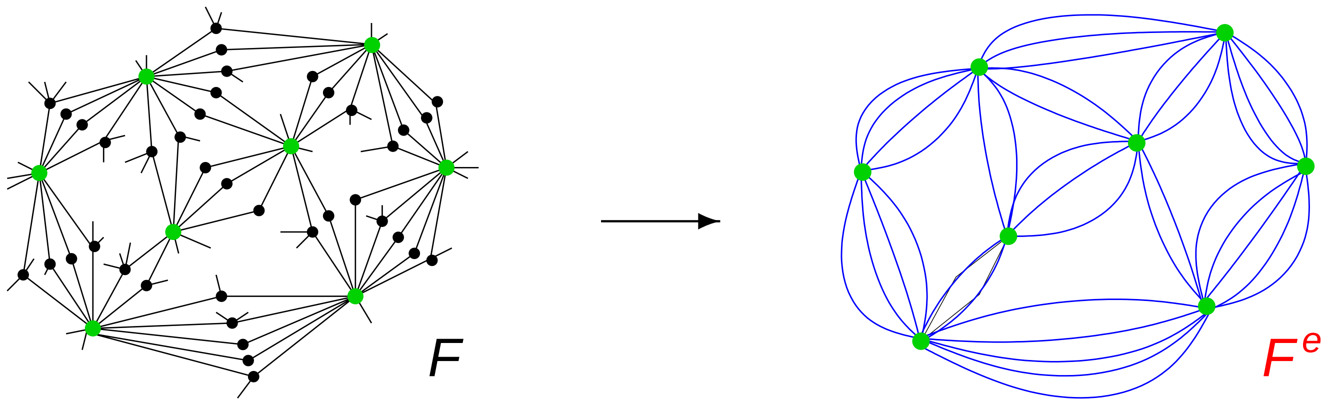
Transferring to edge-colouring

- so a vertex y from Y has at least

  $$\left(\tfrac{3}{2}+\varepsilon\right)\boldsymbol{\Delta} - (d_G(x_1) - d_F(x_1)) - (d_G(x_2) - d_F(x_2)) - C^2$$

  colours still available

- colouring Y is like colouring edges of the multigraph $\boldsymbol{F^e}$:



- but . . . there may be up to C^2 extra connections

  from an edge in F^e to other edges in F^e

Edge-colouring multigraphs

- $\chi'(G)$: chromatic index of multigraph G
- $ch'(G)$: list chromatic index of multigraph G
- $\chi'_F(G)$: fractional chromatic index of multigraph G

Theorem (Kahn, 1996, 2000)

- G multigraph, with Δ large enough

$$\Longrightarrow \quad ch'(G) \approx \chi'(G) \approx \chi'_F(G)$$

- in fact, Kahn's proofs provide something much more general

Kahn's result

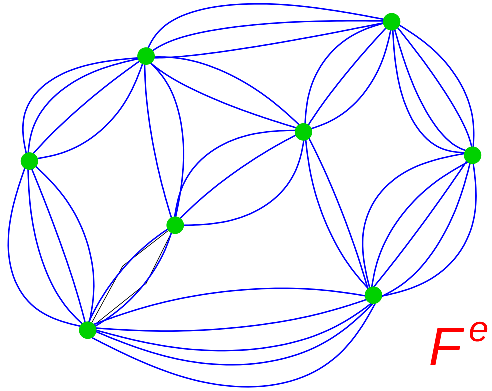
Theorem (Kahn, 2000)

for $0 < \delta < 1$, $\alpha > 0$ there exists $\mathbf{\Delta}_{\delta,\alpha}$ so that if $\mathbf{\Delta} \geq \mathbf{\Delta}_{\delta,\alpha}$:

- G a multigraph with maximum degree $\mathbf{\Delta}$
- each edge e has a list $L(e)$ of colours so that
  - for all edges e: $|L(e)| \geq \alpha\,\mathbf{\Delta}$
  - for all vertices v: $$\sum_{e \ni v} |L(e)|^{-1} \leq 1 \cdot (1-\delta)$$
  - for all $K \subseteq G$ with $|V(K)| \geq 3$ odd : $$\sum_{e \in E(K)} |L(e)|^{-1} \leq \tfrac{1}{2}\,(|V(K)|-1) \cdot (1-\delta)$$
- then there exists a proper colouring of the edges of G so that each edge gets colours from its own list

Kahn's approach for our case

- we have a multigraph F^e :



- so that each edge $e = x_1 x_2$ has a list $L(e)$ of at least

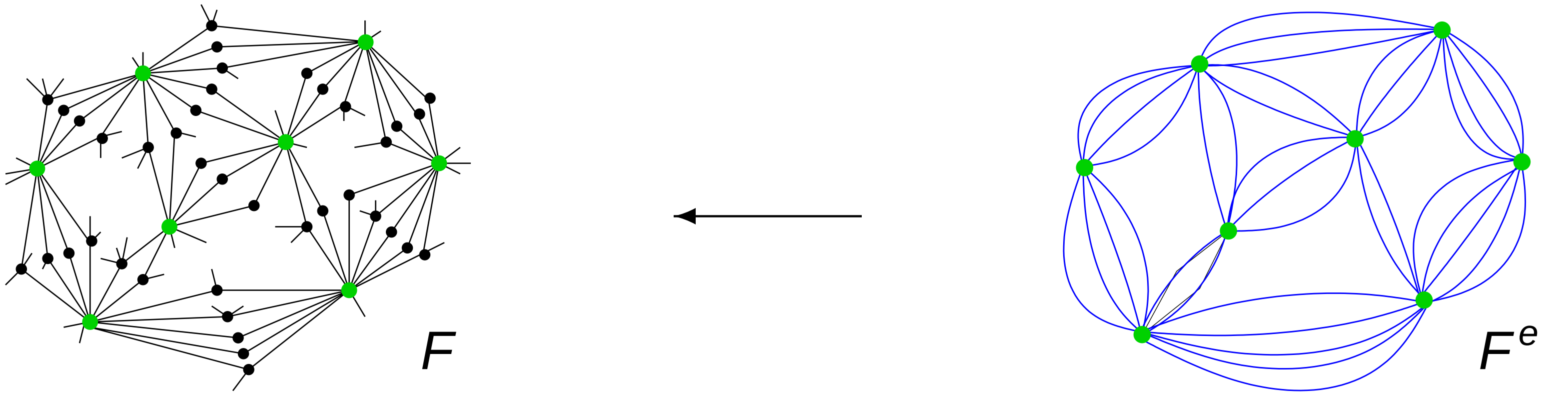
$$(\tfrac{3}{2} + \varepsilon)\,\boldsymbol{\Delta} - (d_G(x_1) - d_F(x_1)) - (d_G(x_2) - d_F(x_2)) - C^2$$

  colours

- and F^e satisfies "some edge density condition"

Extending Kahn's approach

- these conditions guarantee that Kahn's conditions are satisfied for F^e
- $\Longrightarrow$ we can edge-colour F^e



- $\Longrightarrow$ we can colour the Y-vertices in F
  choosing from the left-over colours for each
- also : we found a way to deal with the up to C^2 extra connections from an edge in F^e to other edges in F^e

Some open problems

- prove the next step :

  G planar $\quad\Longrightarrow\quad \chi(G^2) \;\le\; \frac{3}{2}\,\Delta + O(1)$

- what can be said about (A,B)-colourings
  if the sets A and/or B have some structure ?

Theorem

- G planar, $A,B \subseteq V(G)$
  vertices in A have mutual distance at least three

  $$\Longrightarrow\quad ch(G;A,B) \;\le\; \Delta(G;A,B) + 5$$

The problem for small Δ

- for G planar, $\Delta \leq 3$ we know :
  - $\chi(G^2) \leq 7$ (Thomassen, 2007)
  - $ch(G^2) \leq 8$ (Cranston & Kim, 2006)
- what is the right upper bound for $ch(G^2)$ in this case ?
- Wegner's Conjecture for G planar, $4 \leq \Delta \leq 7$:

$$\chi(G^2) \leq \Delta + 5$$

- almost nothing is known in this case

Colouring the square and edge-colouring

- is there some deep relation between edge-colouring multigraphs and colouring the square of graphs ?

- our proof uses edge-colouring to answer questions on colouring the square

- **Theorem** (Shannon, 1949)
  for any multigraph G $\Longrightarrow$ $\chi'(G) \leq \frac{3}{2}\Delta$

- **Conjecture 1** (Vizing, 1975)
  for any multigraph G $\Longrightarrow$ $ch'(G) = \chi'(G)$

- **Conjecture 2** (Kostochka & Woodall, 2001)
  for any graph G $\Longrightarrow$ $ch(G^2) = \chi(G^2)$

Two very general, and very similar, conjectures

Conjecture 1 (Vizing, 1975)

- for any multigraph $G \implies ch'(G) = \chi'(G)$

Conjecture 2 (Kostochka & Woodall, 2001)

- for any graph $G \implies ch(G^2) = \chi(G^2)$

- Conjecture 1 is known for bipartite graphs (Galvin, 1995)
- Conjecture 2 is known for outerplanar graphs with $\Delta \geq 6$
  - but only since we know both $\chi(G^2)$ and $ch(G^2)$

    (Wang & Lih; Hetherington & Woodall, 2008)

- can we prove Conjecture 2 for some other classes of graphs?