# Distance-Two Colouring of Graphs

JAN VAN DEN HEUVEL

joint work with

FRÉDÉRIC HAVET, COLIN MCDIARMID & BRUCE REED

and with OMID AMINI & LOUIS ESPERET

Department of Mathematics
London School of Economics and Political Science

## *The start*

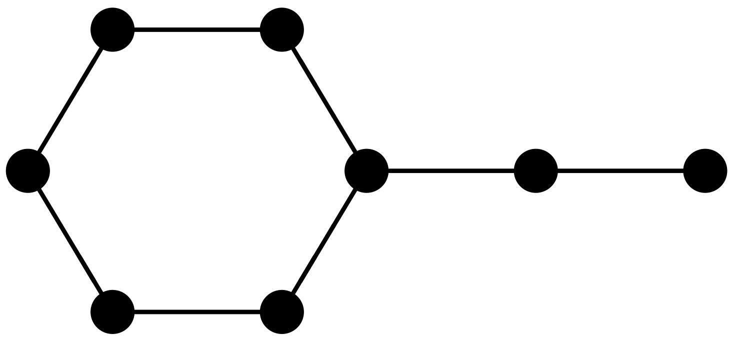
- on way to look at vertex-colouring :
  vertices at distance one must receive different colours

- suppose we require vertices at larger distances
  to receive different colours as well

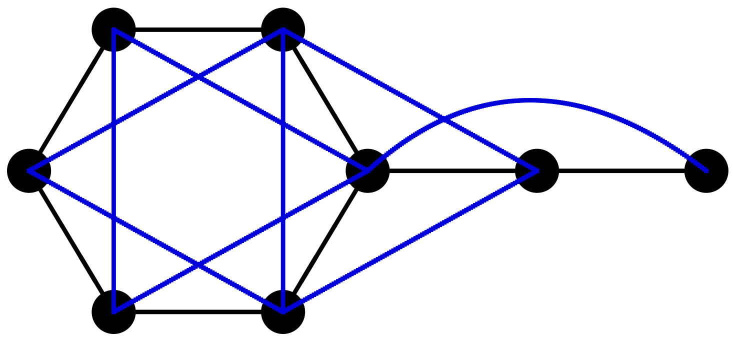
- for today we only look at distance two

## *The square of a graph*

- distance-two colouring can be modelled using the **square $G^2$ of a graph**:
  - same vertex set as $G$
  - edges between vertices with distance at most 2 in $G$
    ( $=$ are adjacent or have a common neighbour )



$G$



$G^2$

## *Colouring the square of a graph*

### Easy facts

- $\chi(G^2) \geq \Delta(G) + 1$

**and**

- $\Delta(G^2) \leq \Delta(G)^2$, **so** $\chi(G^2) \leq \Delta(G)^2 + 1$

### Question

- is the upper bound relevant ?

- there are at most 4 graphs with $\chi(G^2) = \Delta(G)^2 + 1$
- and infinitely many graphs with

$$\chi(G^2) = \Delta(G)^2 - \Delta(G) + 1$$

## *The square of planar graphs*

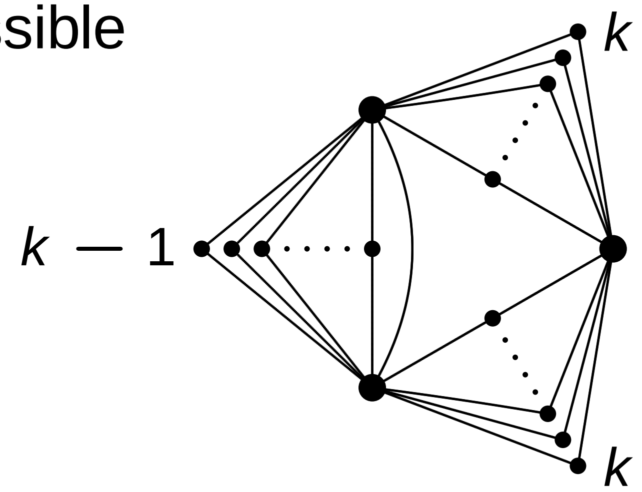
**Conjecture** ( Wegner, 1977 )

- $G$ planar

$$\Longrightarrow \quad \chi(G^2) \le \begin{cases} 7, & \text{if } \Delta = 3 \\ \Delta + 5, & \text{if } 4 \le \Delta \le 7 \\ \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1, & \text{if } \Delta \ge 8 \end{cases}$$

- bounds would be best possible

case $\Delta = 2k \ge 8$ :



## *What is known for large $\Delta$*

$G$ planar $\Longrightarrow$

- $\chi(G^2) \le 8\,\Delta - 22$ (Jonas, PhD, 1993)
- $\chi(G^2) \le 3\,\Delta + 5$ (Wong, MSc, 1996)
- $\chi(G^2) \le 2\,\Delta + 25$ (vdH & McGuinness, 2003)
- $\chi(G^2) \le \left\lceil \frac{9}{5}\,\Delta \right\rceil + 1$ (for $\Delta \ge 47$)
  (Borodin, Broersma, Glebov & vdH, 2001)
- $\chi(G^2) \le \left\lceil \frac{5}{3}\,\Delta \right\rceil + 24$ (for $\Delta \ge 241$)
  (Molloy & Salavatipour, 2005)

## *First new results*

**Theorem**

- $G$ planar $\implies$ $\chi(G^2) \le \left(\frac{3}{2} + o(1)\right) \Delta$ $\qquad (\Delta \to \infty)$

we actually prove the list-colouring version for much larger classes of graphs :

**Theorem**

- graph $G$ $K_{3,k}$-minor free for some fixed $k$

$$\implies ch(G^2) \le \left(\frac{3}{2} + o(1)\right) \Delta$$

## *Even more general results ?*

**Property**

- graph $G$ $H$-minor free for some fixed graph $H$
  $\Longrightarrow \quad ch(G^2) \leq C_H\,\mathbf{\Delta}$ for some constant $C_H$

**Question**

- given $H$, what is the best $C_H$ for large $\mathbf{\Delta}$ ?

**e.g.**

- for $H = K_5$ we have $2 \leq C_{K_5} \leq 9$

## *The clique number*

**Corollary**

- graph $G$ $K_{3,k}$-minor free for some fixed $k$

$$\Longrightarrow \quad \omega(G^2) \;\leq\; \left(\tfrac{3}{2} + o(1)\right) \boldsymbol{\Delta}$$

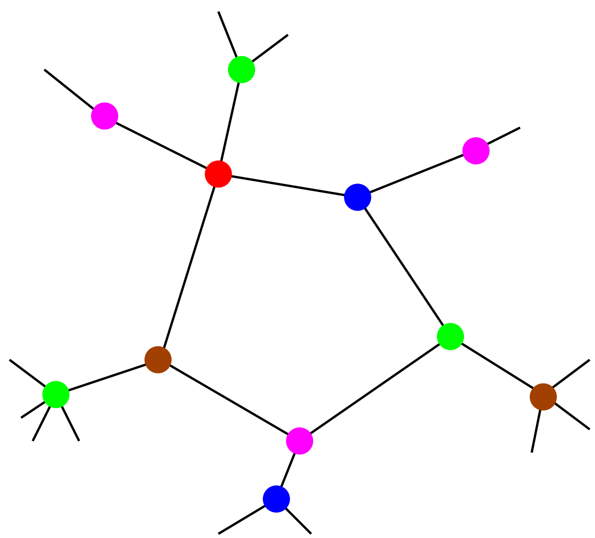
this can be partially improved to

**Theorem**

- $G$ planar $\quad \Longrightarrow \quad \omega(G^2) \;\leq\; \tfrac{3}{2}\,\boldsymbol{\Delta} + O(1)$

# *A related (?) problem*

- **plane graph** : planar graph with a given embedding
- **cyclic colouring** of a plane graph :
  - vertex-colouring so that
  - vertices incident to the same face get a different colour



## *A related (?) problem*

- **plane graph** : planar graph with a given embedding
- **cyclic colouring** of a plane graph :
  - vertex-colouring so that
  - vertices incident to the same face get a different colour
- **cyclic chromatic number** $\chi^*(G)$ :
  minimum number of colours needed for a cyclic colouring
- $\Delta^*(G)$ : size of largest face of $G$

**Easy**

- $G$ plane graph $\Longrightarrow$ $\chi^*(G) \geq \Delta^*(G)$

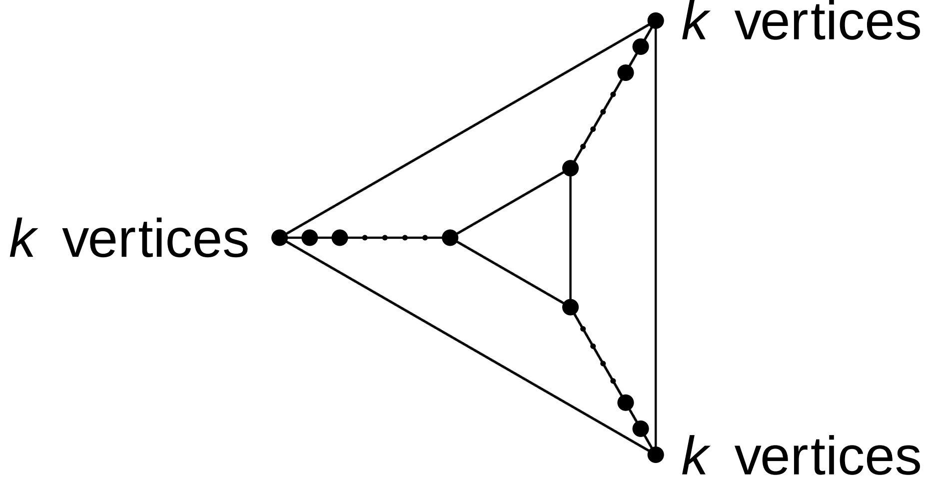
## ***Conjecture for the related(?) problem***

**Conjecture** (Borodin, 1984)

- $G$ plane graph $\Longrightarrow$ $\chi^*(G) \leq \frac{3}{2}\,\Delta^*(G)$

- bound would be best possible

case $\Delta^* = 2\,k$ :



## *Bounds on the cyclic chromatic number*

$G$ plane graph $\Longrightarrow$

- $\chi^*(G) \le 2\,\Delta^*(G)$ (Ore & Plummer, 1969)
- $\chi^*(G) \le \frac{9}{5}\,\Delta^*(G)$ (Borodin, Sanders & Zhao, 1999)
- $\chi^*(G) \le \frac{5}{3}\,\Delta^*(G)$ (Sanders & Zhao, 2001)

**Theorem**

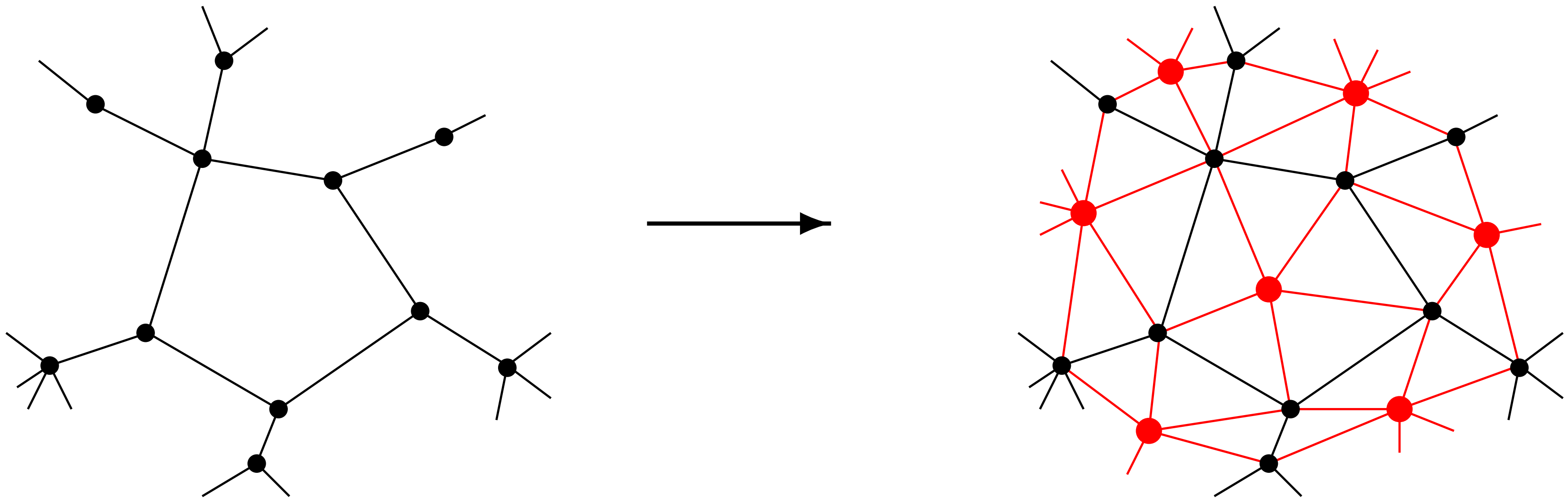
- $G$ plane graph

$$\Longrightarrow \quad \chi^*(G) \le \left(\tfrac{3}{2} + o(1)\right)\Delta^* \quad (\Delta^* \to \infty)$$

(in fact, we again prove the list-colouring version)

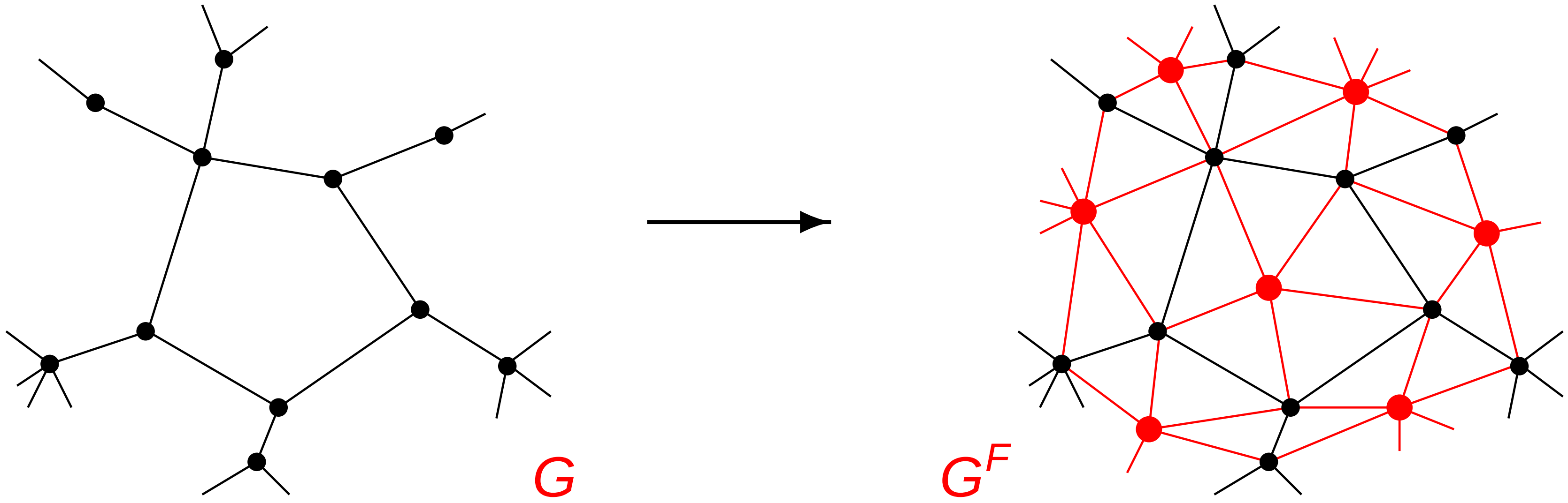
## *From cyclic colouring to “distance-two” colouring*

- given a plane graph $G$, form new graph $G^F$ by
  - adding a vertex in each face
  - adding edges from new vertex to all vertices of the face



$G$ $\qquad\qquad$ $G^F$

# *From cyclic colouring to “square” colouring*




- colouring the square of $G^F$ gives cyclic colouring of $G$
  - but also colours the faces ( not asked for, but o.k. )
  - and degrees of old vertices may be more than $\Delta^*(G)$

    ( serious problem )

## *Both in one go*

**idea** : do not treat all vertices equal :

- some vertices need to be coloured
- some vertices determine distance two

- vertices can be of both types
  - as are all vertices when colouring the square
- or the types can be disjoint
  - as for “faces” and “vertices” for cyclic colouring
- or anything in between

## *Formalising our clever little idea*

- **given** : graph $G$, subsets $A, B \subseteq V(G)$
- $(A, B)$**-colouring of** $G$ : colouring of vertices in $B$ so that
  - adjacent vertices get different colours
  - vertices with a common neighbour in $A$
    get different colours
- **list** $(A, B)$**-colouring of** $G$ :
  - similar, but each vertex in $B$ has its own list
- $\chi(G; A, B)$ / $ch(G; A, B)$ :
  minimum number of colours needed /
  minimum size of each list needed

## *The first baby steps*

- $\chi(G) \;=\; \chi(G; \varnothing, V)$
- $\chi(G^2) \;=\; \chi(G; V, V)$
- $\chi^*(G) \;=\; \chi(G^F; \text{“faces”}, \text{“vertices”})$

- relevant “**degree**” : $d_B(v) =$ number of neighbours in $B$
- “**maximum degree**” :

$$\Delta(G; A, B) = \text{ maximum of } d_B(v) \text{ over all } v \in A$$

**Easy**

- $\Delta(G; A, B) \;\leq\; \chi(G; A, B)$

$$\leq\; \Delta(G) + \Delta(G; A, B) \cdot (\Delta(G; A, B) - 1) + 1$$

## *The obvious(?) conjecture & results*

**Conjecture**

- $G$ planar, $A, B \subseteq V(G)$, $\Delta(G; A, B)$ large enough

$$\Longrightarrow \quad \chi(G; A, B) \ \leq \ \tfrac{3}{2}\,\Delta(G; A, B) + 1$$

**Theorem**

- $G$ planar, $A, B \subseteq V(G)$

$$\Longrightarrow \quad ch(G; A, B) \ \leq \ \left(\tfrac{3}{2} + o(1)\right) \Delta(G; A, B)$$

**Corollary**

(asymptotic list version of Wegner's and Borodin's Conjecture)

- $G$ planar $\quad \Longrightarrow \quad ch(G^2) \ \leq \ \left(\tfrac{3}{2} + o(1)\right) \Delta(G)$
- $G$ plane graph $\quad \Longrightarrow \quad ch^*(G) \ \leq \ \left(\tfrac{3}{2} + o(1)\right) \Delta^*(G)$

# *Sketch of the proof of square of planar graph*

**uses induction on the number of vertices**

- **2-neighbour** : vertex at distance one or two
- $\boldsymbol{d^2(v)}$ : number of 2-neighbours of $v$
  $=$ number of neighbours of $v$ in $G^2$
- we would like to remove a vertex $v$ with $d^2(v) \leq \frac{3}{2}\Delta$
  - but that can change distances in $G - v$
- contraction to a neighbour $u$ will solve the distance problem
  - but may increase maximum degree if $d(u) + d(v) > \Delta$
- easy induction possible if there is an edge $uv$
  with $d(u) + d(v) \leq \Delta$ and $d^2(v) \leq \frac{3}{2}\Delta$

## *When easy induction is not possible*

***S***, **small** vertices: degree at most some constant $C$

***B***, **big** vertices: degree more than $C$

***H***, **huge** vertices: degree at least $\frac{1}{2}\,\Delta$

- small vertices have a least two big neighbours (otherwise for those $v$: $d^2(v) \leq \frac{3}{2}\,\Delta$)
- a planar graph has fewer than $3\,|V|$ edges and fewer than $2\,|V|$ edges if it is bipartite

  **so**:
  - all but $O(|V|/C)$ vertices are small
  - fewer than $2\,|B|$ vertices in $V \setminus B$ have more than two neighbours in $B$

## *When easy induction is not possible*

$S$, **small** vertices : degree at most some constant $C$

$B$, **big** vertices : degree more than $C$

$H$, **huge** vertices : degree at least $\frac{1}{2}\,\boldsymbol{\Delta}$

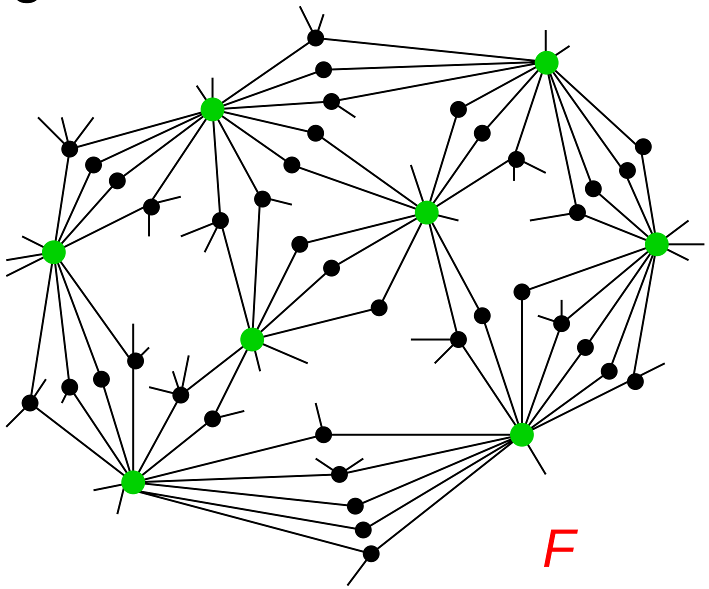
**so** :

- "most" vertices are small
- and these have exactly two big neighbours

  ( in fact two huge neighbours )

## *The structure so far*

- there is a subgraph ***F*** of $G$ looking like :




- green vertices ***X*** have degree at least $\frac{1}{2}\,\mathbf{\Delta}$
- black vertices ***Y*** have degree at most $C$
- all other neighbours of $Y$ -vertices are also small

we can guarantee additionally :

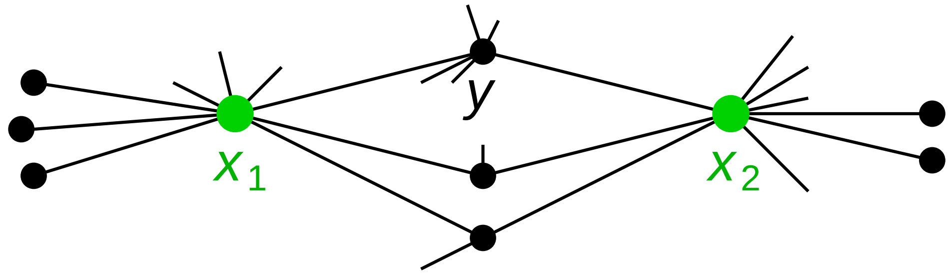
- only "few" edges from $X$ to rest of $G$
- $F$ satisfies "some edge density condition"

# *The other induction step*

- remove the vertices from $Y$ (using contraction)
- colour the smaller graph (which is possible by induction)

# *The other induction step*

- remove the vertices from $Y$ (using contraction)
- colour the smaller graph (which is possible by induction)
- what to do with the uncoloured $Y$-vertices?




- $y$ has
  - a "lot" of 2-neighbours in $Y$ via $x_1$, $x_2$
  - at most $C^2$ other 2-neighbours in $Y$
  - at most $(d_G(x_1) - d_F(x_1)) + (d_G(x_2) - d_F(x_2))$ 2-neighbours outside $Y$ via $x_1$, $x_2$
  - at most $C^2$ other 2-neighbours outside $Y$

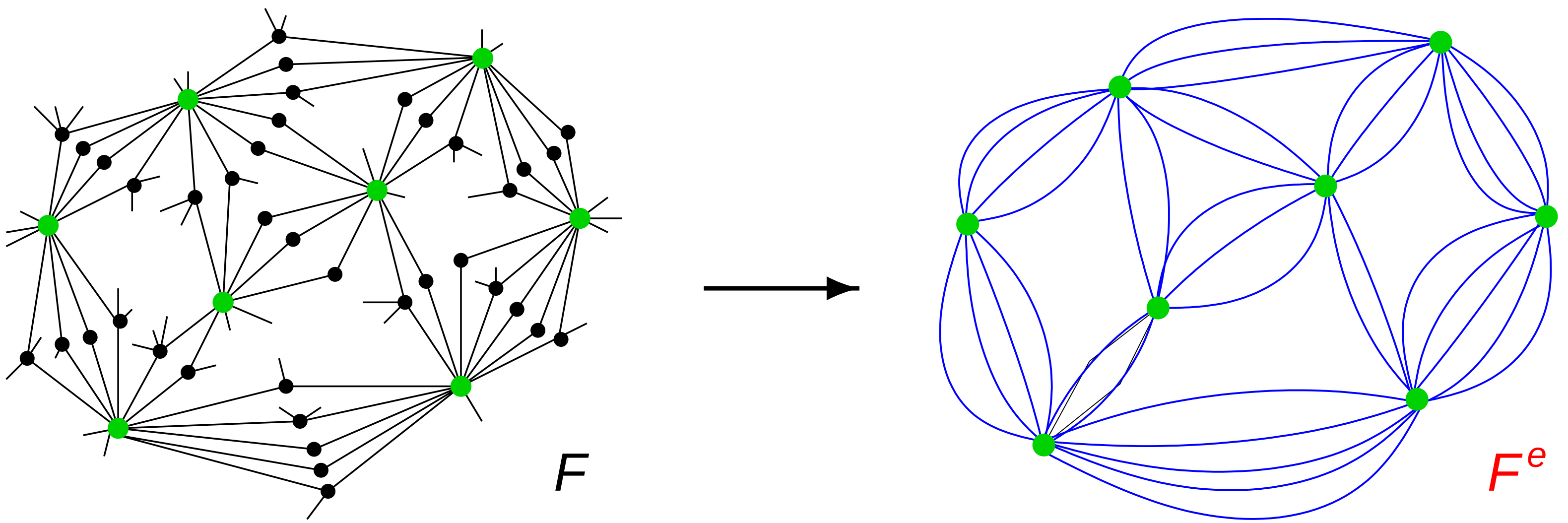
## *Transferring to edge-colouring*

- so a vertex $y$ from $Y$ has at least

  $$\left(\frac{3}{2}+\varepsilon\right)\Delta - (d_G(x_1) - d_F(x_1)) - (d_G(x_2) - d_F(x_2)) - C^2$$

  colours still available

- colouring $Y$ is "almost" like list-colouring edges of the multigraph $\boldsymbol{F^e}$ :



## *Edge-colouring multigraphs*

- $\chi'(G)$ : chromatic index of multigraph $G$
- $ch'(G)$ : list chromatic index of multigraph $G$
- $\chi'_F(G)$ : fractional chromatic index of multigraph $G$

**Theorem** (Kahn, 1996, 2000)

- $G$ multigraph, with $\Delta$ large enough

$$\Longrightarrow \quad ch'(G) \approx \chi'(G) \approx \chi'_F(G)$$

- in fact, Kahn's proofs provide something much more general

## *Kahn's result*

**Theorem** (Kahn, 2000)

for $0 < \delta < 1$, $\alpha > 0$ there exists $\boldsymbol{\Delta}_{\delta,\alpha}$ so that if $\boldsymbol{\Delta} \geq \boldsymbol{\Delta}_{\delta,\alpha}$ :

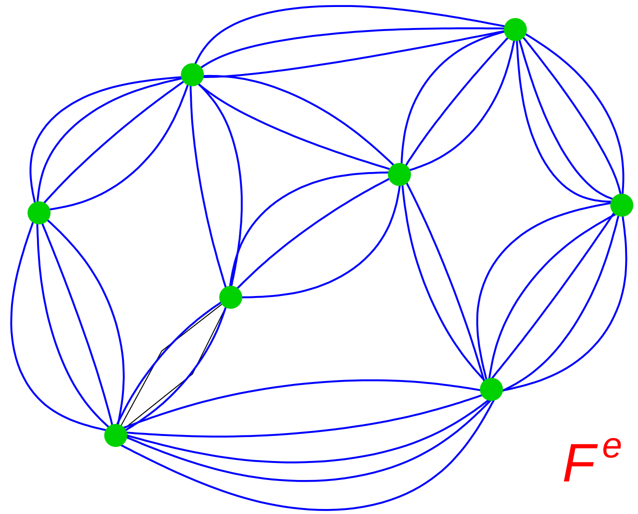
- $G$ a multigraph with maximum degree $\boldsymbol{\Delta}$
- each edge $e$ has a list $L(e)$ of colours so that
  - for all edges $e$: $|L(e)| \geq \alpha\,\boldsymbol{\Delta}$
  - for all vertices $v$: $\displaystyle\sum_{e \ni v} |L(e)|^{-1} \leq 1 \cdot (1-\delta)$
  - for all $K \subseteq G$ with $|V(K)| \geq 3$ odd:

$$\sum_{e \in E(K)} |L(e)|^{-1} \leq \tfrac{1}{2}\,(|V(K)|-1) \cdot (1-\delta)$$

- then there exists a proper colouring of the edges of $G$ so that each edge gets colours from its own list

## *Kahn's approach for our case*

- we have a multigraph $F^e$ :




- so that each edge $e = x_1 x_2$ has a list $L(e)$ of at least

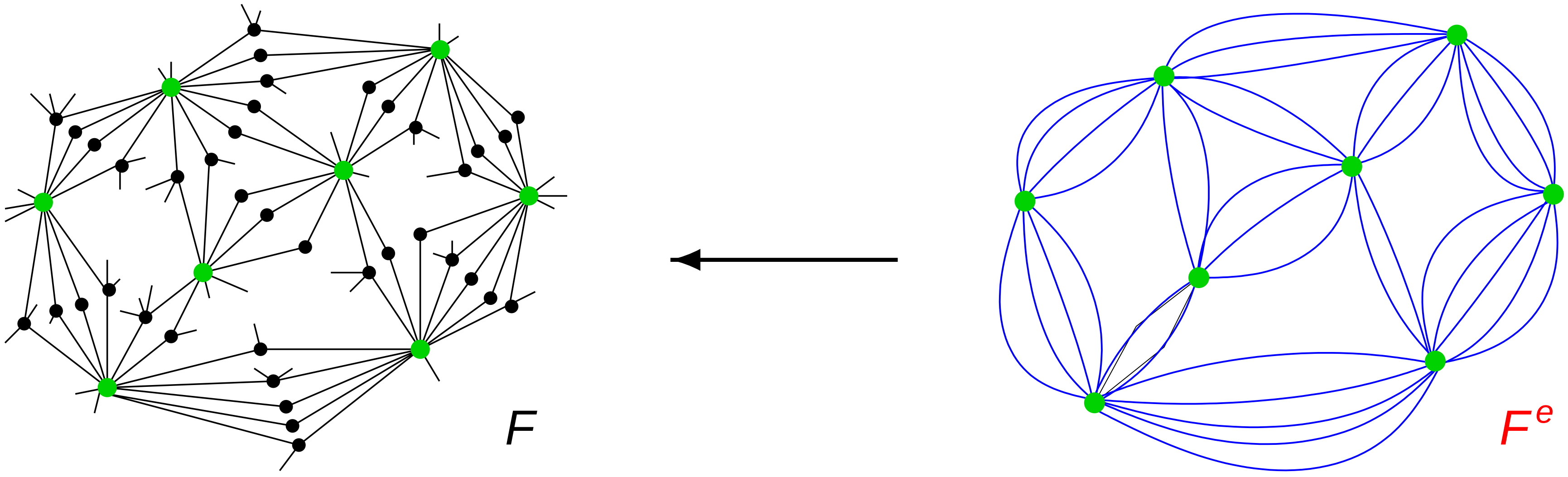
  $(\frac{3}{2} + \varepsilon)\,\Delta - (d_G(x_1) - d_F(x_1)) - (d_G(x_2) - d_F(x_2)) - C^2$

  colours

- and $F^e$ satisfies "some edge density condition"

# *Extending Kahn's approach*

- these conditions guarantee that Kahn's conditions are satisfied for $F^e$
- $\Longrightarrow$ we can edge-colour $F^e$




- $\Longrightarrow$ we can colour the $Y$-vertices in $F$
  choosing from the left-over colours for each
- also : we can deal with the "almost" list-edge colouring

## *Some open problems*

- prove the next step :

  $G$ planar $\quad\Longrightarrow\quad \chi(G^2) \;\le\; \frac{3}{2}\,\Delta + O(1)$

- for $G$ planar, $\Delta \le 3$ we know :
  - $\chi(G^2) \;\le\; 7$ (Thomassen, 2007)
  - $ch(G^2) \;\le\; 8$ (Cranston & Kim, 2006)
  - what is the right upper bound for $ch(G^2)$ in this case ?

- Wegner's Conjecture for $4 \le \Delta \le 7$ :

  $G$ planar $\quad\Longrightarrow\quad \chi(G^2) \;\le\; \Delta + 5$ ?

## *Distance-two colouring and edge-colouring*

- is there some deep relation between edge-colouring multigraphs and distance-two colouring graphs ?

- our proof uses edge-colouring to answer questions on distance-two colouring

- **Theorem** ( Shannon, 1949 )
  for any multigraph $G$ $\Longrightarrow$ $\chi'(G) \leq \frac{3}{2}\,\Delta$

- **Conjecture** ( Wegner, 1977 )
  for a planar graph $G$, $\Delta \geq 8$ $\Longrightarrow$ $\chi(G^2) \leq \frac{3}{2}\,\Delta + 1$

## *Edge-colouring and distance-two colouring*

**Conjecture 1** (Vizing, 1975)

- for any multigraph $G$ $\Longrightarrow$ $ch'(G) = \chi'(G)$

**Conjecture 2** (Kostochka & Woodall, 2001)

- for any graph $G$ $\Longrightarrow$ $ch(G^2) = \chi(G^2)$

- Conjecture 1 is known for bipartite graphs (Galvin, 1995)
  proof doesn't require knowledge of $\chi'(G)$ and $ch'(G)$

- Conjecture 2 is only known for some special graph classes,
  usually since we know both $\chi(G^2)$ and $ch(G^2)$