Distance-Two Colouring of Graphs

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joint work with

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The start

- on way to look at vertex-colouring:
  vertices at distance one must receive different colours

- suppose we require vertices at larger distances to receive different colours as well

- for today we only look at distance two
The square of a graph

- distance-two colouring can be modelled using the square $G^2$ of a graph:
  - same vertex set as $G$
  - edges between vertices with distance at most 2 in $G$
    
    ($=$ are adjacent or have a common neighbour)
**Colouring the square of a graph**

**Easy facts**
- \( \chi(G^2) \geq \Delta(G) + 1 \)

and
- \( \Delta(G^2) \leq \Delta(G)^2 \), so \( \chi(G^2) \leq \Delta(G)^2 + 1 \)

**Question**
- is the upper bound relevant?

- there are at most 4 graphs with \( \chi(G^2) = \Delta(G)^2 + 1 \)

- and infinitely many graphs with

\[
\chi(G^2) = \Delta(G)^2 - \Delta(G) + 1
\]
The square of planar graphs

**Conjecture** (Wegner, 1977)

- $G$ planar

\[ \chi(G^2) \leq \begin{cases} 
7, & \text{if } \Delta = 3 \\
\Delta + 5, & \text{if } 4 \leq \Delta \leq 7 \\
\left\lfloor \frac{3}{2} \Delta \right\rfloor + 1, & \text{if } \Delta \geq 8 
\end{cases} \]

- bounds would be best possible

case $\Delta = 2k \geq 8$:
What is known for large $\Delta$

$G$ planar $\implies$

- $\chi(G^2) \leq 8\Delta - 22$ (Jonas, PhD, 1993)
- $\chi(G^2) \leq 3\Delta + 5$ (Wong, MSc, 1996)
- $\chi(G^2) \leq 2\Delta + 25$ (vdH & McGuinness, 2003)
- $\chi(G^2) \leq \left\lceil \frac{9}{5}\Delta \right\rceil + 1$ (for $\Delta \geq 47$) (Borodin, Broersma, Glebov & vdH, 2001)
- $\chi(G^2) \leq \left\lceil \frac{5}{3}\Delta \right\rceil + 24$ (for $\Delta \geq 241$) (Molloy & Salavatipour, 2005)
First new results

Theorem

- $G$ planar $\implies \chi(G^2) \leq \left(\frac{3}{2} + o(1)\right) \Delta \quad (\Delta \to \infty)$

we actually prove the list-colouring version for much larger classes of graphs:

Theorem

- graph $G$ $K_{3,k}$-minor free for some fixed $k$
  $$\implies ch(G^2) \leq \left(\frac{3}{2} + o(1)\right) \Delta$$
Even more general results?

Property
- graph $G$ $H$-minor free for some fixed graph $H$

$$\implies ch(G^2) \leq C_H \Delta \text{ for some constant } C_H$$

Question
- given $H$, what is the best $C_H$ for large $\Delta$?

e.g.
- for $H = K_5$ we have $2 \leq C_{K_5} \leq 9$
The clique number

Corollary

- graph $G$ $K_{3,k}$-minor free for some fixed $k$

\[ \Rightarrow \quad \omega(G^2) \leq \left( \frac{3}{2} + o(1) \right) \Delta \]

this can be partially improved to

Theorem

- $G$ planar

\[ \Rightarrow \quad \omega(G^2) \leq \frac{3}{2} \Delta + O(1) \]
A related (?) problem

- **plane graph**: planar graph with a given embedding
- **cyclic colouring** of a plane graph:
  - vertex-colouring so that
  - vertices incident to the same face get a different colour
A related (?) problem

- **plane graph**: planar graph with a given embedding

- **cyclic colouring** of a plane graph:
  - vertex-colouring so that
  - vertices incident to the same face get a different colour

- **cyclic chromatic number** $\chi^*(G)$:
  - minimum number of colours needed for a cyclic colouring

- $\Delta^*(G)$: size of largest face of $G$

**Easy**

- $G$ plane graph $\implies \chi^*(G) \geq \Delta^*(G)$
Conjecture for the related(?) problem

Conjecture (Borodin, 1984)

- $G$ plane graph $\implies \chi^*(G) \leq \frac{3}{2} \Delta^*(G)$

- Bound would be best possible

Case $\Delta^* = 2k$:
Bounds on the cyclic chromatic number

\[ \chi^*(G) \leq 2 \Delta^*(G) \]  
(\text{Ore & Plummer, 1969})

\[ \chi^*(G) \leq \frac{9}{5} \Delta^*(G) \]  
(\text{Borodin, Sanders & Zhao, 1999})

\[ \chi^*(G) \leq \frac{5}{3} \Delta^*(G) \]  
(\text{Sanders & Zhao, 2001})

Theorem

\[ \chi^*(G) \leq \left( \frac{3}{2} + o(1) \right) \Delta^* \quad (\Delta^* \to \infty) \]  
(in fact, we again prove the list-colouring version)
From cyclic colouring to “distance-two” colouring

- given a plane graph $G$, form new graph $G^F$ by
  - adding a vertex in each face
  - adding edges from new vertex to all vertices of the face
From cyclic colouring to “square” colouring

- colouring the square of $G^F$ gives cyclic colouring of $G$
- but also colours the faces (not asked for, but o.k.)
- and degrees of old vertices may be more than $\Delta^*(G)$ (serious problem)
Both in one go

idea: do not treat all vertices equal:

- some vertices need to be coloured
- some vertices determine distance two

- vertices can be of both types
  - as are all vertices when colouring the square
- or the types can be disjoint
  - as for “faces” and “vertices” for cyclic colouring
- or anything in between
Formalising our clever little idea

- **given**: graph $G$, subsets $A, B \subseteq V(G)$

- **$(A, B)$-colouring of $G$**: colouring of vertices in $B$ so that
  - adjacent vertices get different colours
  - vertices with a common neighbour in $A$ get different colours

- **list $(A, B)$-colouring of $G$**: similar, but each vertex in $B$ has its own list

- **$\chi(G; A, B) / ch(G; A, B)$**: minimum number of colours needed / minimum size of each list needed
The first baby steps

- \( \chi(G) = \chi(G; \emptyset, V) \)
- \( \chi(G^2) = \chi(G; V, V) \)
- \( \chi^*(G) = \chi(G^F; \text{"faces"}, \text{"vertices"}) \)

- relevant "degree": \( d_B(v) = \) number of neighbours in \( B \)
- "maximum degree": \( \Delta(G; A, B) = \) maximum of \( d_B(v) \) over all \( v \in A \)

Easy

- \( \Delta(G; A, B) \leq \chi(G; A, B) \)
  \[ \leq \Delta(G) + \Delta(G; A, B) \cdot (\Delta(G; A, B) - 1) + 1 \]
The obvious(?) conjecture & results

Conjecture

- $G$ planar, $A, B \subseteq V(G)$, $\Delta(G; A, B)$ large enough

$$\implies \chi(G; A, B) \leq \frac{3}{2} \Delta(G; A, B) + 1$$

Theorem

- $G$ planar, $A, B \subseteq V(G)$

$$\implies ch(G; A, B) \leq \left(\frac{3}{2} + o(1)\right) \Delta(G; A, B)$$

Corollary

( asymptotic list version of Wegner's and Borodin's Conjecture )

- $G$ planar $\implies ch(G^2) \leq \left(\frac{3}{2} + o(1)\right) \Delta(G)$

- $G$ plane graph $\implies ch^*(G) \leq \left(\frac{3}{2} + o(1)\right) \Delta^*(G)$
Sketch of the proof of square of planar graph

uses induction on the number of vertices

- **2-neighbour**: vertex at distance one or two
- \(d^2(v)\): number of 2-neighbours of \(v\)
  \[= \text{number of neighbours of } v \text{ in } G^2\]

we would like to remove a vertex \(v\) with \(d^2(v) \leq \frac{3}{2} \Delta\)

- but that can change distances in \(G - v\)

contraction to a neighbour \(u\) will solve the distance problem

- but may increase maximum degree if \(d(u) + d(v) > \Delta\)

easy induction possible if there is an edge \(uv\)

with \(d(u) + d(v) \leq \Delta\) and \(d^2(v) \leq \frac{3}{2} \Delta\)
When easy induction is not possible

$S$, small vertices: degree at most some constant $C$

$B$, big vertices: degree more than $C$

$H$, huge vertices: degree at least $\frac{1}{2} \Delta$

- small vertices have at least two big neighbours
  (otherwise for those $v$: $d^2(v) \leq \frac{3}{2} \Delta$)

- a planar graph has fewer than $3 |V|$ edges and fewer than $2 |V|$ edges if it is bipartite

so:

- all but $O(|V|/C)$ vertices are small

- fewer than $2 |B|$ vertices in $V \setminus B$ have more than two neighbours in $B$
When easy induction is not possible

$S$, small vertices: degree at most some constant $C$
$B$, big vertices: degree more than $C$
$H$, huge vertices: degree at least $\frac{1}{2} \Delta$

so:

- “most” vertices are small
- and these have exactly two big neighbours (in fact two huge neighbours)
The structure so far

- there is a subgraph $F$ of $G$ looking like:
  - green vertices $X$ have degree at least $\frac{1}{2} \Delta$
  - black vertices $Y$ have degree at most $C$
  - all other neighbours of $Y$-vertices are also small

we can guarantee additionally:
  - only “few” edges from $X$ to rest of $G$
  - $F$ satisfies “some edge density condition”
The other induction step

- remove the vertices from $Y$ (using contraction)
- colour the smaller graph (which is possible by induction)
The other induction step

- remove the vertices from $Y$ (using contraction)
- colour the smaller graph (which is possible by induction)
- what to do with the uncoloured $Y$-vertices?

- $y$ has
  - a “lot” of 2-neighbours in $Y$ via $x_1$, $x_2$
  - at most $C^2$ other 2-neighbours in $Y$
  - at most $(d_G(x_1) - d_F(x_1)) + (d_G(x_2) - d_F(x_2))$ other 2-neighbours outside $Y$ via $x_1$, $x_2$
  - at most $C^2$ other 2-neighbours outside $Y$
Transferring to edge-colouring

- so a vertex $y$ from $Y$ has at least

$\left(\frac{3}{2} + \varepsilon\right) \Delta - (d_G(x_1) - d_F(x_1)) - (d_G(x_2) - d_F(x_2)) - C^2$

colours still available

- colouring $Y$ is “almost” like list-colouring edges of the multigraph $F^e$: 
Edge-colouring multigraphs

- $\chi'(G)$: chromatic index of multigraph $G$
- $ch'(G)$: list chromatic index of multigraph $G$
- $\chi'_F(G)$: fractional chromatic index of multigraph $G$

Theorem (Kahn, 1996, 2000)

- $G$ multigraph, with $\Delta$ large enough

  $\implies ch'(G) \approx \chi'(G) \approx \chi'_F(G)$

- in fact, Kahn’s proofs provide something much more general
Kahn’s result

Theorem (Kahn, 2000)

for $0 < \delta < 1$, $\alpha > 0$ there exists $\Delta_{\delta,\alpha}$ so that if $\Delta \geq \Delta_{\delta,\alpha}$:

- $G$ a multigraph with maximum degree $\Delta$
- each edge $e$ has a list $L(e)$ of colours so that
  - for all edges $e$: $|L(e)| \geq \alpha \Delta$
  - for all vertices $v$:
    $\sum_{e \ni v} |L(e)|^{-1} \leq 1 \cdot (1 - \delta)$
  - for all $K \subseteq G$ with $|V(K)| \geq 3$ odd:
    $\sum_{e \in E(K)} |L(e)|^{-1} \leq \frac{1}{2} (|V(K)| - 1) \cdot (1 - \delta)$
- then there exists a proper colouring of the edges of $G$
  so that each edge gets colours from its own list
Kahn’s approach for our case

- we have a multigraph $F^e$:

- so that each edge $e = x_1x_2$ has a list $L(e)$ of at least
  \[
  \left(\frac{3}{2} + \varepsilon\right) \Delta - (d_G(x_1) - d_F(x_1)) - (d_G(x_2) - d_F(x_2)) - C^2
  \]
  colours

- and $F^e$ satisfies “some edge density condition”
Extending Kahn’s approach

- these conditions guarantee that Kahn’s conditions are satisfied for $F^e$
  
- $\implies$ we can edge-colour $F^e$
  
- $\implies$ we can colour the $Y$-vertices in $F$
  choosing from the left-over colours for each

- also: we can deal with the “almost” list-edge colouring
Some open problems

- prove the next step:
  \[ G \text{ planar} \implies \chi(G^2) \leq \frac{3}{2} \Delta + O(1) \]

- for \( G \) planar, \( \Delta \leq 3 \) we know:
  - \( \chi(G^2) \leq 7 \) (Thomassen, 2007)
  - \( ch(G^2) \leq 8 \) (Cranston & Kim, 2006)
  - what is the right upper bound for \( ch(G^2) \) in this case?

- Wegner's Conjecture for \( 4 \leq \Delta \leq 7 \):
  \[ G \text{ planar} \implies \chi(G^2) \leq \Delta + 5 \]
Distance-two colouring and edge-colouring

- is there some deep relation between edge-colouring multigraphs and distance-two colouring graphs?

- our proof uses edge-colouring to answer questions on distance-two colouring

- **Theorem** (Shannon, 1949)
  
  for any multigraph $G \implies \chi'(G) \leq \frac{3}{2} \Delta$

- **Conjecture** (Wegner, 1977)
  
  for a planar graph $G$, $\Delta \geq 8 \implies \chi(G^2) \leq \frac{3}{2} \Delta + 1$
**Edge-colouring and distance-two colouring**

**Conjecture 1** (Vizing, 1975)
- for any multigraph $G$ $\Rightarrow ch'(G) = \chi'(G)$

**Conjecture 2** (Kostochka & Woodall, 2001)
- for any graph $G$ $\Rightarrow ch(G^2) = \chi(G^2)$

- Conjecture 1 is known for bipartite graphs (Galvin, 1995)
  - proof doesn’t require knowledge of $\chi'(G)$ and $ch'(G)$

- Conjecture 2 is only known for some special graph classes,
  - usually since we know both $\chi(G^2)$ and $ch(G^2)$