Degree Sequences and Graph Properties

JAN VAN DEN HEUVEL

joint work (ongoing) with DOUG BAUER, HAJO BROERSMA,
NATHAN KAHL, ED SCHMEICHEL

Department of Mathematics

London School of Economics and Political Science

The basics

- $G = (V, E)$: finite, simple, undirected, graph
- n : number of vertices of G

- **degree sequence** $\boldsymbol{d}(G) = (d_1, \ldots, d_n)$:
  sequence of vertex degrees, with $d_1 \leq \cdots \leq d_n$

- **sequence** $\boldsymbol{d} = (d_1, \ldots, d_n)$:
  any sequence of integers with $0 \leq d_1 \leq \cdots \leq d_n \leq n - 1$

- **graphical sequence** $\boldsymbol{d}$:
  sequence for which there is a graph G with $\boldsymbol{d}(G) = \boldsymbol{d}$

Order of sequences

- $\boldsymbol{d} \leq \boldsymbol{d}'$: for all i : $d_i \leq d'_i$
- so $\boldsymbol{d} \not\leq \boldsymbol{d}'$: $d_1 \geq d'_1 + 1$ or $d_2 \geq d'_2 + 1$ or $\ldots$ $\ldots$ or $d_n \geq d'_n + 1$

<u>Easy</u>

- G spanning subgraph of G' $\Longrightarrow$ $\boldsymbol{d}(G) \leq \boldsymbol{d}(G')$

Some old results I

<u>**Theorem**</u> (Chvátal, 1972)

- G graph with $\boldsymbol{d}(G) = (d_1, \ldots, d_n)$ so that

  for $i = 1, \ldots, \left\lfloor \frac{1}{2}(n-1) \right\rfloor$: $\quad d_i \leq i \;\Rightarrow\; d_{n-i} \geq n - i$

  $\Longrightarrow$ G contains a Hamilton cycle

- the condition is **best possible** :
  - for every $\boldsymbol{d}'$ failing the condition there is a graph G with $\boldsymbol{d}(G) \geq \boldsymbol{d}'$ and G has no Hamilton cycle

- **not**
  - for every graphical sequence $\boldsymbol{d}'$ failing the condition there is a graph G with $\boldsymbol{d}(G) = \boldsymbol{d}'$ without Ham. cycle

Some old results II

Theorem (Bondy, 1969; Boesch, 1974)

- k positive integer, G graph with $\boldsymbol{d}(G) = (d_1, \ldots, d_n)$

  for $i = 1, \ldots, \left\lfloor \frac{1}{2}(n-k+1) \right\rfloor$:

  $$d_i \leq i + k - 2 \;\Rightarrow\; d_{n-k+1} \geq n - i$$

  $\Longrightarrow$ G is k-connected

- the condition is best possible

Some not so old results I

<u>**Theorem**</u> (Kriesell, 2007)

- G graph with $\boldsymbol{d}(G) = (d_1, \ldots, d_n)$
  - for $i = 2, \ldots, \left\lfloor \frac{1}{2}(n-2) \right\rfloor$:

    $$(\; d_{i-2} \le i-1 \;\wedge\; d_i \le i \;) \\ \Rightarrow \; (\; d_{n-2} \ge n-i \;\vee\; d_n \ge n-i+1 \;)$$
  - for $i = 1, \ldots, \left\lfloor \frac{1}{2}(n-2) \right\rfloor$:

    $$(\; d_{i-1} \le i-1 \;\wedge\; d_i \le i+1 \;) \\ \Rightarrow \; (\; d_{n-2} \ge n-i \;\vee\; d_n \ge n-i+1 \;)$$
  - for $i = 2, \ldots, \left\lfloor \frac{1}{2}(n-1) \right\rfloor$:

    $$(\; d_{i-2} \le i-1 \;\wedge\; d_i \le i \;) \\ \Rightarrow \; (\; d_{n-1} \ge n-i \;\vee\; d_n \ge n-i+2 \;)$$

cont. ...

Some not so old results I

Theorem (Kriesell, 2007)

- . . . cont.
  - $n \geq 3$ odd: $d_{(n-3)/2} \leq \frac{1}{2}(n-3)$
    $\Rightarrow \; (\; d_{n-3} \geq \frac{1}{2}(n+1) \;\vee\; d_n \geq \frac{1}{2}(n+3) \;)$
  - $n \geq 4$ even: $d_{n-4} \geq \frac{1}{2}n \;\vee\; d_n \geq \frac{1}{2}(n+2)$
  - $n \geq 4$ even:
    $d_{n-3} \geq \frac{1}{2}n \;\vee\; d_{n-1} \geq \frac{1}{2}(n+2) \;\vee\; d_n \geq \frac{1}{2}(n+4)$
  - $n \geq 5$ odd: $d_{(n-5)/2} \leq \frac{1}{2}(n-3)$
    $\Rightarrow \; (\; d_{n-2} \geq \frac{1}{2}(n+1) \;\vee\; d_n \geq \frac{1}{2}(n+3) \;)$

  $\Longrightarrow$ G is 3-edge-connected

- the condition is best possible

Some not so old results II

Theorem (Kriesell, 2007)

- k positive integer
  then there exist a best possible condition
  implying k-edge-connectivity
- where the number of conditions is $C_k\,n + O(1)$
- with C_k superpolynomial in k

A first observation

- $(\; d_{i-2} \leq i-1 \;\; \wedge \;\; d_i \leq i \;)$
  $\Rightarrow \; (\; d_{n-2} \geq n-i \;\; \vee \;\; d_n \geq n-i+1 \;)$
  - is equivalent to :

    $d_{i-2} \geq i \;\; \vee \;\; d_i \geq i+1 \;\; \vee \;\; d_{n-2} \geq n-i$
    $\vee \;\; d_n \geq n-i+1$

same with the others :

- Chvátal : $\;\; d_i \leq i \;\; \Rightarrow \;\; d_{n-i} \geq n-i$
  - equivalent to : $\;\; d_i \geq i+1 \;\; \vee \;\; d_{n-i} \geq n-i$
- Bondy : $\;\; d_i \leq i+k-2 \;\; \Rightarrow \;\; d_{n-k+1} \geq n-i$
  - equivalent to : $\;\; d_i \geq i+k-1 \;\; \vee \;\; d_{n-k+1} \geq n-i$

Conditions for graph properties

- given graph property P

  we want a degree sequence condition C so that

  - C is monotone

    ($\boldsymbol{d}$ satisfies C and $\boldsymbol{d}' \geq \boldsymbol{d} \quad \Longrightarrow \quad \boldsymbol{d}'$ satisfies C)
  - $\boldsymbol{d}(G)$ satisfies $C \quad \Longrightarrow \quad$ graph G has property P
  - $\boldsymbol{d}$ fails C

    $\Longrightarrow \quad$ there is a G without P and with $\boldsymbol{d}(G) \geq \boldsymbol{d}$

Conditions for graph properties

- given graph property P

  we want a degree sequence condition C so that

  - C is monotone

    ($\boldsymbol{d}$ satisfies C and $\boldsymbol{d}' \geq \boldsymbol{d} \implies \boldsymbol{d}'$ satisfies C)

  - $\boldsymbol{d}(G)$ satisfies $C \implies$ graph G has property P

  - $\boldsymbol{d}$ fails C

    $\implies$ there is a G without P and with $\boldsymbol{d}(G) \geq \boldsymbol{d}$

Theorem

- for all graph properties P
  - there exists such a condition
  - and it is unique

Proof

- - $\boldsymbol{d}$ satisfies C and $\boldsymbol{d}' \geq \boldsymbol{d} \quad \Longrightarrow \quad \boldsymbol{d}'$ satisfies C
  - $\boldsymbol{d}(G)$ satisfies $C \quad \Longrightarrow \quad$ graph G has property P
  - $\boldsymbol{d}$ fails C

    $\Longrightarrow \quad$ there is a G with $\boldsymbol{d}(G) \geq \boldsymbol{d}$ but without P

- uniqueness follows just from these properties

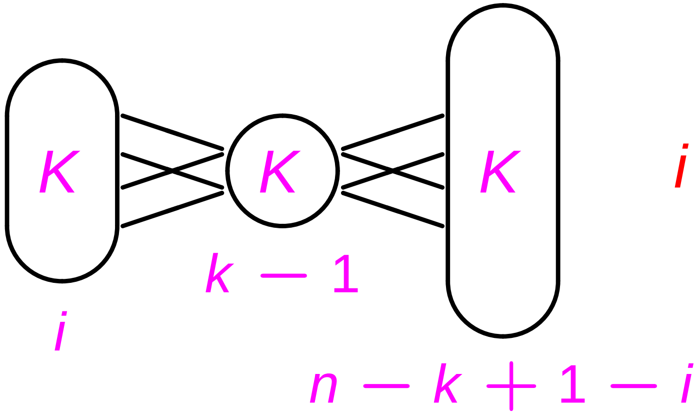
- existence :
  - let $\overline{\mathcal{P}}$ be the set of all graphs failing P
  - condition C : for all $H \in \overline{\mathcal{P}}$: $\boldsymbol{d} \not\leq \boldsymbol{d}(H)$
  - in fact : enough to consider set $\overline{\mathcal{P}}^{\mathrm{m}}$ of all

    edge-maximal graphs failing P

Simple but it works

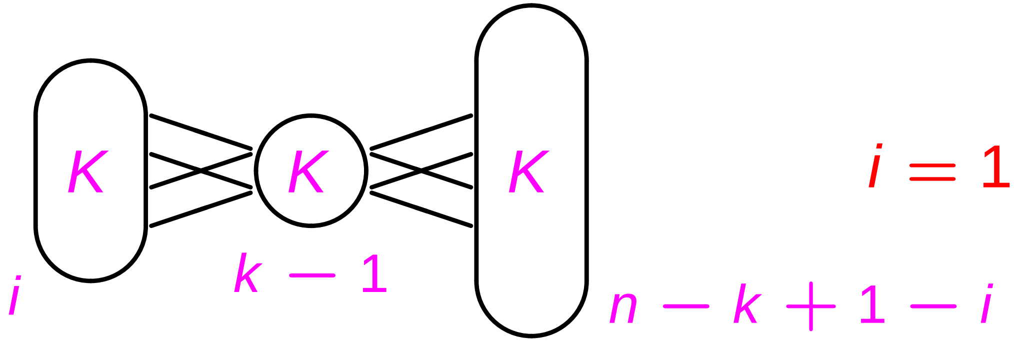
- property : G is k-connected
- an edge-maximal graph that is not k-connected
  - has cutset S of size $k - 1$
  - exactly two components in $G - S$
  - both components are complete
  - vertices in S are adjacent to all other vertices

hence looks like :



$$i = 1, \ldots, \left\lfloor \tfrac{1}{2}(n - k + 1) \right\rfloor$$

Condition for k-connected

- 

  $i = 1, \ldots, \left\lfloor \frac{1}{2}(n-k+1) \right\rfloor$

- these have degree sequence

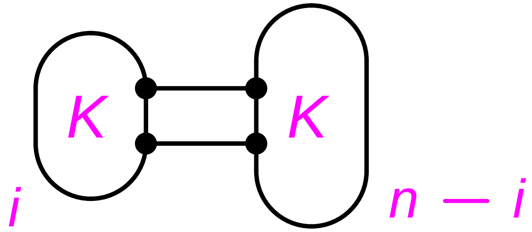
$$((i+k-2)^{i},\ (n-1-i)^{n-k+1-i},\ (n-1)^{k-1})$$

- condition that guarantees k-connected:

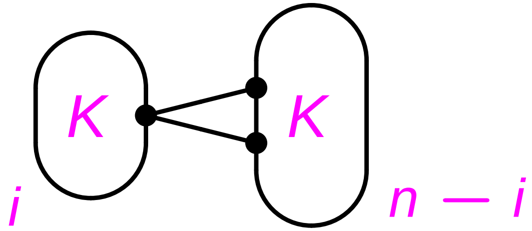
  $d_1 \geq i+k-1$ or $\ldots$ or $d_i \geq i+k-1$
  or $d_{i+1} \geq n-i$ or $\ldots$ or $d_{n-k+1} \geq n-i$
  or $d_{n-k+2} \geq n$ or $\ldots$ or $d_n \geq n$

- equivalent to: $d_i \geq i+k-1$ or $d_{n-k+1} \geq n-i$

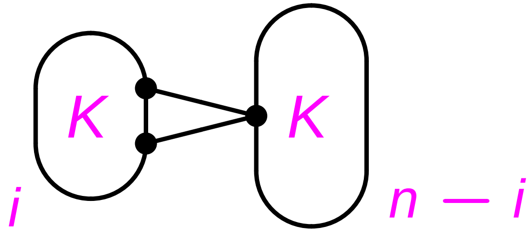
Edge-maximal graphs that are not 3-edge-connected

- 

$$\boldsymbol{d}(G) = ((i-1)^{i-2}, i^2, (n-i-1)^{n-i-2}, (n-i)^2)$$



$$\boldsymbol{d}(G) = ((i-1)^{i-1}, i+1, (n-i-1)^{n-i-2}, (n-i)^2)$$



$$\boldsymbol{d}(G) = ((i-1)^{i-2}, i^2, (n-i-1)^{n-i-1}, n-i+1)$$

$$i = 1, \ldots, \approx \lfloor \tfrac{1}{2} n \rfloor$$

- degree sequences immediately give Kriesell's result

Colouring

- property : $\chi(G) \geq k$
- edge-maximal graphs that fail :

  can be coloured with $k-1$ colours

  - are the complete $(k-1)$-partite graphs $K_{m_1,\ldots,m_{k-1}}$

    $$(\, 1 \leq m_1 \leq \ldots \leq m_{k-1}, \; m_1 + \cdots + m_{k-1} = n \,)$$

  - with degree sequence

    $$(\, (n-m_{k-1})^{m_{k-1}}, \; (n-m_{k-2})^{m_{k-2}}, \; \ldots, \; (n-m_1)^{m_1} \,)$$

Colouring

Property

- k positive integer, G graph with $\boldsymbol{d}(G) = (d_1, \ldots, d_n)$
- for all $m_1, \ldots, m_{k-1}$

  with $1 \le m_1 \le \ldots \le m_{k-1}$ and $m_1 + \cdots + m_{k-1} = n$:

$$\begin{array}{rll} & d_{m_{k-1}} & \ge n - m_{k-1} + 1 \\ \vee & d_{m_{k-1}+m_{k-2}} & \ge n - m_{k-2} + 1 \\ \vee & d_{m_{k-1}+m_{k-2}+m_{k-3}} & \ge n - m_{k-3} + 1 \\ \vee & \ldots & \\ \vee & d_{m_{k-1}+\cdots+m_1} & \ge n - m_1 + 1 \end{array}$$

$$\Longrightarrow \quad \chi(G) \ge k$$

- the condition is best possible

Cliques

Theorem

- for all $m_1, \ldots, m_{k-1}$
  with $1 \le m_1 \le \ \ldots \ \le m_{k-1}$ and $m_1 + \cdots + m_{k-1} = n$:

$$
\begin{aligned}
& d_{m_{k-1}} && \ge n - m_{k-1} + 1 \\
\vee\ & d_{m_{k-1}+m_{k-2}} && \ge n - m_{k-2} + 1 \\
\vee\ & \qquad \ldots && \\
\vee\ & d_{m_{k-1}+\cdots+m_1} && \ge n - m_1 + 1
\end{aligned}
$$

$\Longrightarrow$ G has a clique of size k

- the condition is best possible

- generalises Turán's Theorem

The clique result

- main idea appears more or less in
  - Erdős' proof (1970) of Turán's Theorem
  - degree sequence condition for stable sets

(Murphy, 1991)

- **algorithmic aspect**:

  there exists a linear algorithm

  that, given graph G and positive integer k
  - finds a clique of size k **or**
  - finds integers $m_1, \ldots, m_{k-1}$ for which $\boldsymbol{d}(G)$ violates the condition

When edge-maximal is too much

- for certain properties P, the set of all edge-maximal failing graphs may be hard to find and/or more than required

- enough to have **sequence-maximal set for failing P**:

  a set $\{H_1, \ldots, H_m\}$ so that

  - all H_i fail P
  - G fails $P \implies \boldsymbol{d}(G) \leq \boldsymbol{d}(H_i)$ for some H_i
  - pairs in $\{\boldsymbol{d}(H_1), \ldots, \boldsymbol{d}(H_m)\}$ are incomparable

- gives best possible condition for P: for all H_i: $\boldsymbol{d} \not\leq \boldsymbol{d}(H)$

- such a set always exists, but not necessarily unique

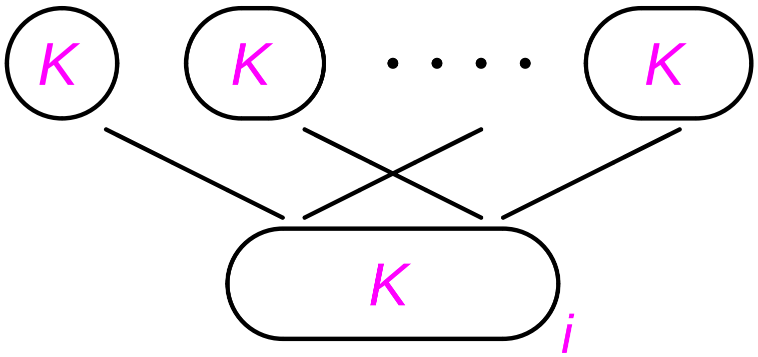
  but always gives same sequence $\{\boldsymbol{d}(H_1), \ldots, \boldsymbol{d}(H_m)\}$

Toughness

- real number $t > 0$, then graph G is **t-tough** if
  - for every cut set S:

    number of components of $G - S$: $c(G - S) \leq \frac{|S|}{t}$

obvious edge-maximal graphs that are not t-tough:

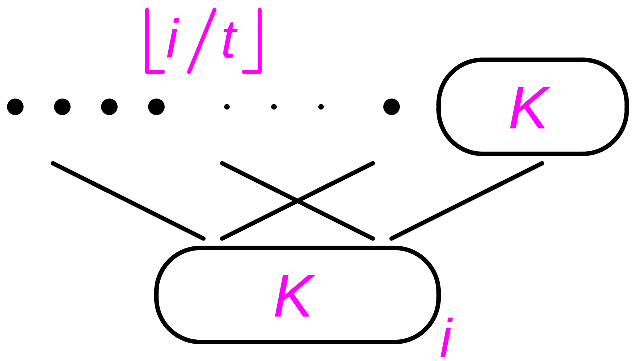


$\lfloor i/t \rfloor + 1$

complete components

Special edge-maximal graphs for toughness

H_i :



$i \geq 1, \; i + \lfloor i/t \rfloor + 1 \leq n$

Theorem

- for $t \geq 1$ the set $\{\, H_i \mid i \geq 1, \; i + \lfloor i/t \rfloor + 1 \leq n \,\}$ is sequence-maximal for failing t-tough

Corollary $t \geq 1$ real, G graph with $\boldsymbol{d}(G) = (d_1, \ldots, d_n)$

- for $i \geq 1, \; i + \lfloor i/t \rfloor + 1 \leq n$:

$$d_{\lfloor i/t \rfloor} \geq i + 1 \;\; \vee \;\; d_{n-i} \geq n - \lfloor i/t \rfloor$$

$\Longrightarrow$ G is t-tough

Special edge-maximal graphs for toughness

Theorem

- for $t \geq 1$ the set $\{ H_i \mid i \geq 1,\ i + \lfloor i/t \rfloor + 1 \leq n \}$ is sequence-maximal for failing t-tough

Theorem

- for $t < 1$ every sequence-maximal set for failing t-tough contains a superpolynomial (in n) number of graphs

Question

- Does this mean that there is no simple best possible degree sequence condition for t-tough, $t < 1$?

Matchings

Theorem

- n even, G graph with $\boldsymbol{d}(G) = (d_1, \ldots, d_n)$
  for $i = 1, \ldots, \frac{1}{2}n - 1$:
  $$d_{i+1} \geq i + 1 \;\vee\; d_{n-i} \geq n - i - 1$$
  $\Longrightarrow$ G contains a perfect matching

more general :

- $\beta \equiv n \pmod 2$, G graph with $\boldsymbol{d}(G) = (d_1, \ldots, d_n)$
  for $i = \beta, \ldots, \frac{1}{2}(n + \beta) - 1$:
  $$d_{i+1} \geq i - \beta + 1 \;\vee\; d_{n+\beta-i} \geq n - i - 1$$
  $\Longrightarrow$ G contains a matching missing at most β vertices

k-Factors

- ***k*-factor**: k-regular spanning subgraph
- we know the structure of graphs without a k-factor

  (Belck, 1950; Tutte, 1952)
- can we derive a sequence-maximal set for failing to have a k-factor from this?
- yes for $k = 2$
- but for $k \geq 3$??

A final, sobering, thought

- property: G has a cycle
- every sequence-maximal set for failing this property contains

$$\sim \frac{1}{4\, n \sqrt{3}}\, e^{\pi \sqrt{2n/3}} \text{ graphs}$$

(all trees with different degree sequences)

- but a simple best possible condition is

$$d_1 + \cdots + d_n \geq 2\, n - 1 \quad \Longrightarrow \quad G \text{ has a cycle}$$