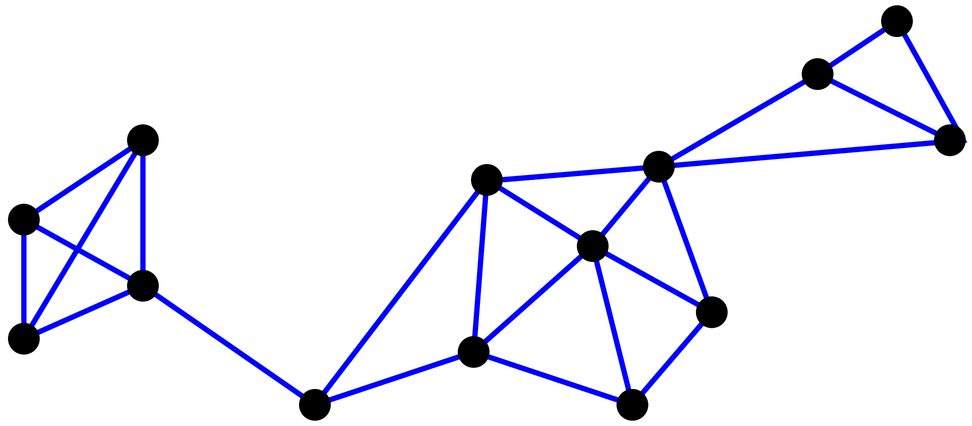
The External Network Problem and the Source Location Problem

JAN VAN DEN HEUVEL

joint work with MATTHEW JOHNSON

Department of Mathematics
London School of Economics and Political Science

Simple model of communication networks



- communication goes over paths along links
- communication should be possible between all / some pairs of nodes
- links or nodes can fail
  - network should be able to survive after a few failures

General problem

Given :

- existing network
- with certain communication requirements
- and a minimum required measure of reliability

Task :

- if necessary, change the network
- to guarantee required reliability

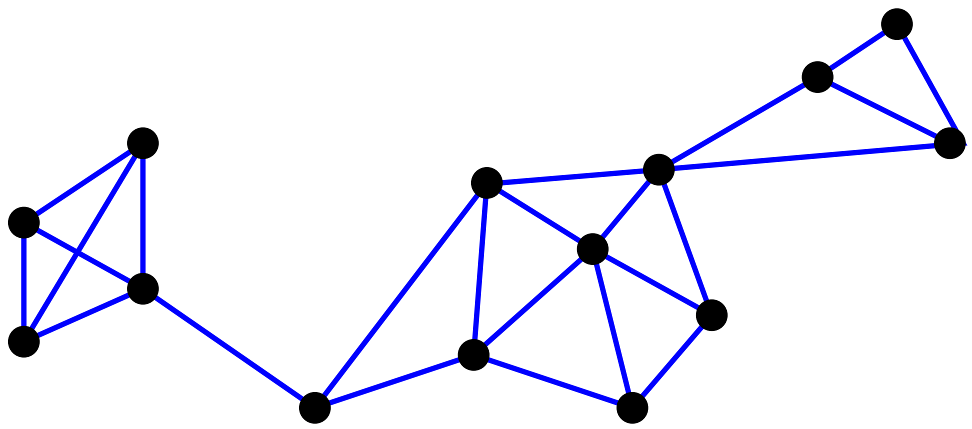
Constraints :

- only certain type of changes allowed
- must be done minimising “cost”

Elements to consider

- type of communication required
- undirected / directed links (**edges / arcs**)
- node and / or link failure and measure of reliability :
  - **vertex connectivity** and / or **edge / arc connectivity**
- how is cost determined
  - **today** : all changes **equal** cost : **minimise** number
- what type of changes can be made to the network
- cost of computing optimal / approximate solutions

Well-studied problem : edge / arc augmentation



- how many edges need to be added to make this graph, say, 3-edge-connected ?
- and what if we would like to make it 4-vertex-connected ?

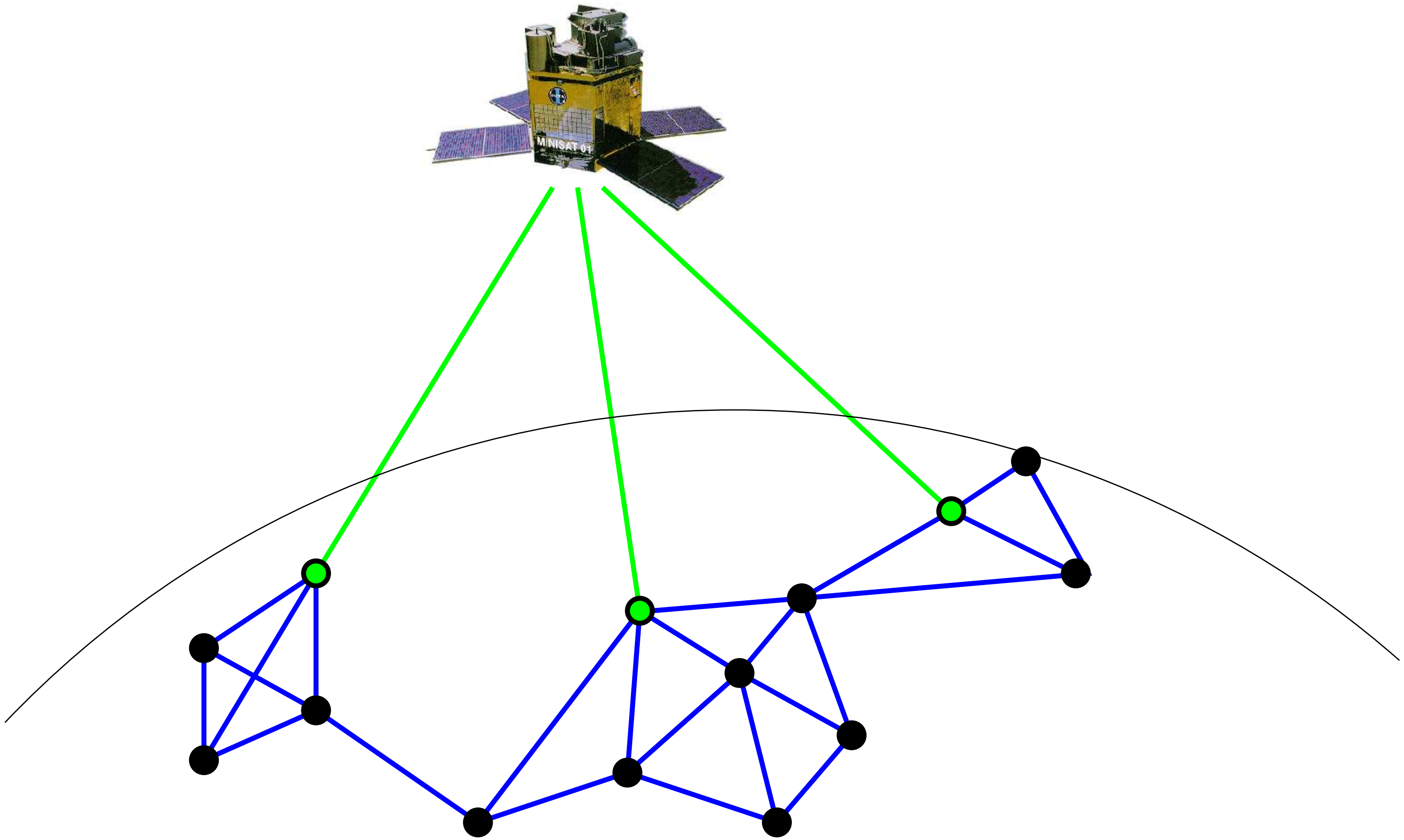
New problem : External Connectivity

- suppose it is not possible to add edges or arcs
  - adding links may be too expensive
  - many wireless networks can only form links if nodes are within a certain distance

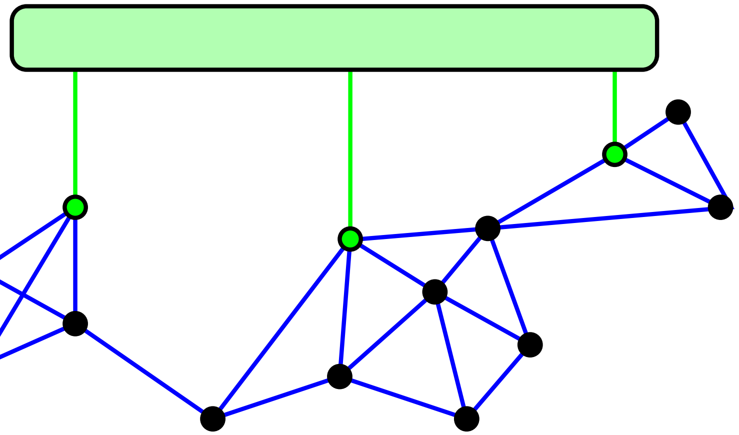
instead :

- an **external network** can be used
- cost is involved for
  each node that connects to the external network

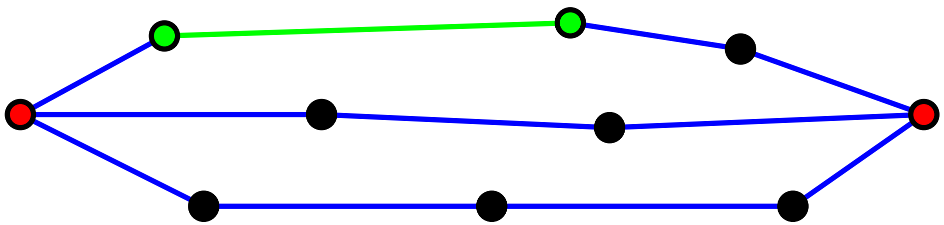
Example : wireless network with satellite



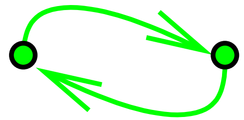
Using the external network – assumptions



- paths can use **external links**



- external links can work as arc in both directions



- external network never fails and has sufficient capacity
- fixed cost per node that connects to the external network

Related problem : Source Location (Ito et al.)

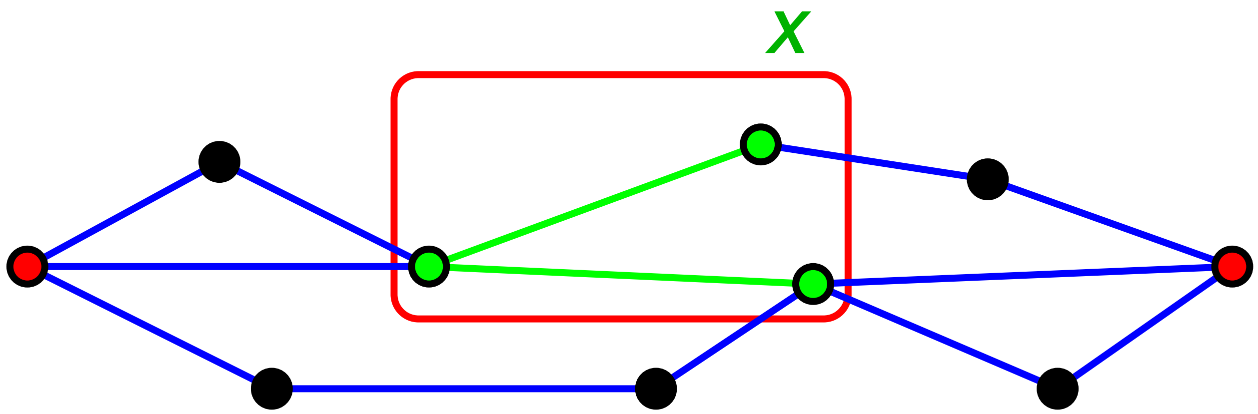
- certain set of nodes must be chosen : the **Sources**
- all other nodes must be able to communicate with at least one source with a certain guaranteed reliability
  - i.e., minimum required number of vertex / edge / arc disjoint paths to at least one source
- fixed cost per node to make it a source

Undirected network & edge-connectivity – UE

External Network problem

Given: undirected graph $\boldsymbol{G} = (V, E)$ and positive integer $\boldsymbol{k}$

Task: find set $\boldsymbol{X} \subseteq V$ of minimum order
so that between any pair of vertices
there are k edge-disjoint paths
where vertices in X are considered pairwise connected

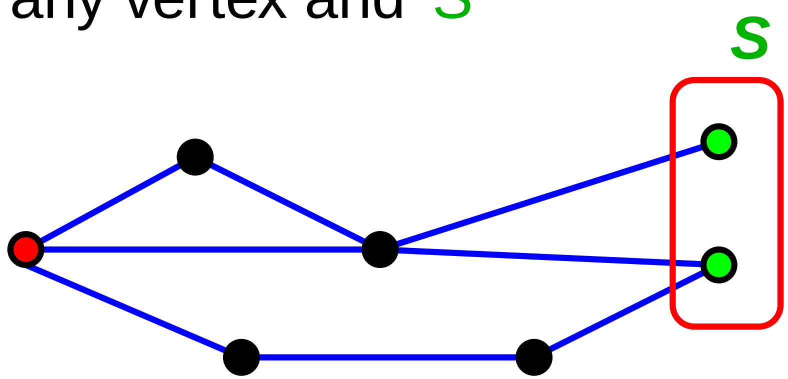


UE – *Undirected network & edge-connectivity*

Source Location problem

Given : undirected graph $\boldsymbol{G} = (V, E)$ and positive integer $\boldsymbol{k}$

Task : find set $\boldsymbol{S} \subseteq V$ of minimum order
so that there are k edge-disjoint paths
between any vertex and S



- for this case the External Network problem and the Source Location problem are equivalent

UE – *Solving the Source Location problem*

- for $T \subseteq V$:
  - $\boldsymbol{d(T)}$: number of edges between T and $V \setminus T$
  - T is ***k*-deficient** : $d(T) < k$
- *every k-deficient set should contain at least one source*

Theorem (Ito *et al.*)

- *min* $\{$ *order of set of sources for* G $\}$

  $=$ *max* $\{$ *number of disjoint k-deficient sets in* G $\}$

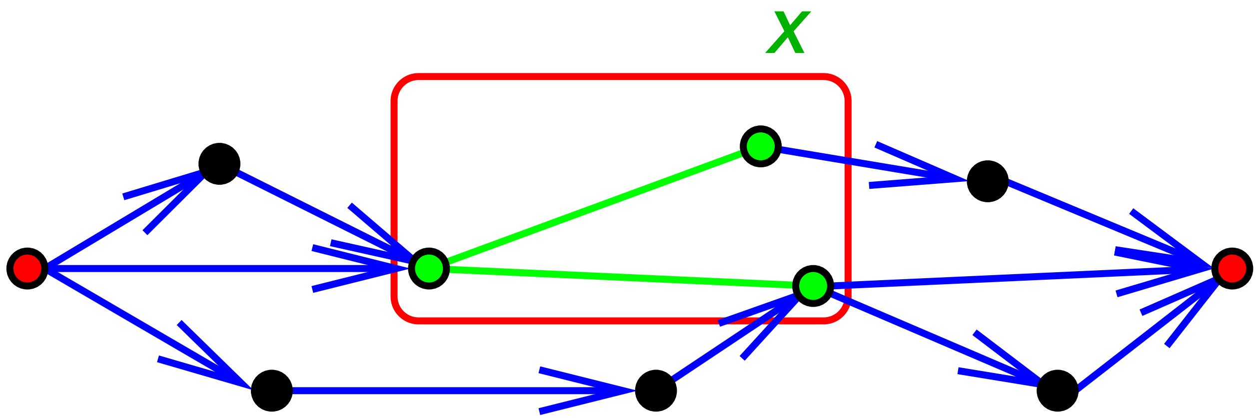
- plus : polynomial algorithm to find minimum set of sources
- also solves External Network problem for this instance

Directed network & arc-connectivity – DA

External Network

Given: directed graph $G = (V, E)$ and positive integer k

Task: find set $X \subseteq V$ of minimum order
so that between any pair of vertices
there are k arc-disjoint directed paths
where vertices in X are considered pairwise connected

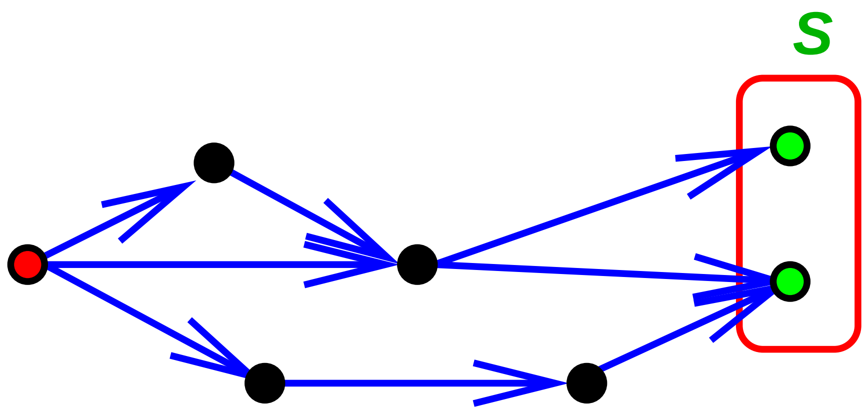


DA – Directed network & arc-connectivity

Source Location

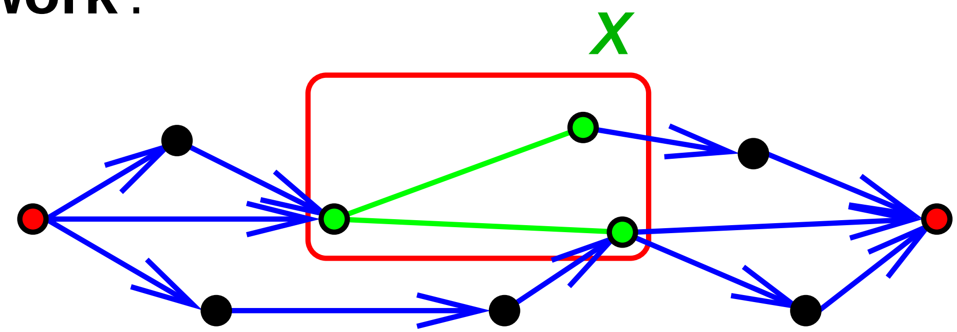
Given : directed graph $\boldsymbol{G} = (V, E)$ and positive integer $\boldsymbol{k}$

Task : find set $\boldsymbol{S} \subseteq V$ of minimum order
so that there are k arc-disjoint directed paths
from any vertex to S

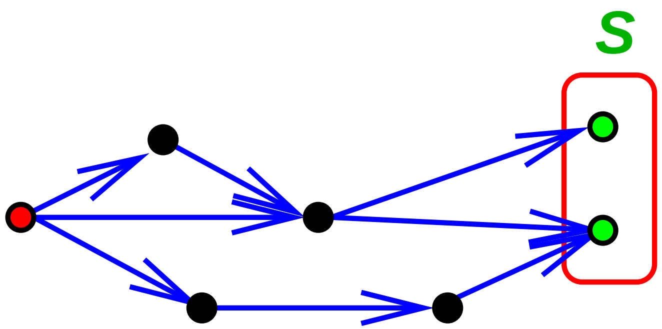


DA – *Comparing the two problems*

External Network :



Source Location :



- this version of External Network and Source Location are **not** equivalent

DA – *A stronger Source Location problem*

- for equivalence we would need Source Location to be :
  - k arc-disjoint paths from any vertex to S **and**
  - k arc-disjoint paths from S to any vertex

Generalised Source Location

Given : directed graph $\boldsymbol{G} = (V, E)$, positive integers $\boldsymbol{k}$ and $\boldsymbol{m}$

Task : find set $\boldsymbol{S} \subseteq V$ of minimum order so that there are

k arc-disjoint directed paths from any vertex to S

and m arc-disjoint directed paths from S to any vertex

- case $k = m$ is equivalent to External Network problem

DA – *Solving the one-sided Source Location problem*

- for $T \subset V$:
  - $\boldsymbol{d^+(T)}$: number of arcs from T to $V \backslash T$
  - T is ***k*-out-deficient** : $d^+(T) < k$
- *every k-out-deficient set should contain at least one source*

Theorem (Ito *et al.*)

- *min* $\{$ *order of set of sources for* D $\}$

  $=$ *max* $\{$ *number of disjoint k-out-deficient sets in* D $\}$

- but : proof gives no efficient (polynomial in $|V|$) algorithm !

UE – *Sketch of proof for undirected case*

Task : find minimum set $\boldsymbol{S} \subseteq V$ so that there are $\boldsymbol{k}$ edge-disjoint paths between any vertex and S

- equivalent to : S needs to cover every k-deficient set

critical set : minimal (for set inclusion) k-deficient set

- hence equivalent to : S needs to cover every critical set
- fairly easy to prove : *all critical sets are disjoint*
- immediately gives : *min* { *order of set of sources for* G } $=$ *max* { *number of disjoint* k*-deficient sets in* G }

DA – *Different for the directed case*

Task : find minimum set $\boldsymbol{S} \subseteq V$ so that there are $\boldsymbol{k}$ directed arc-disjoint paths from any vertex to S

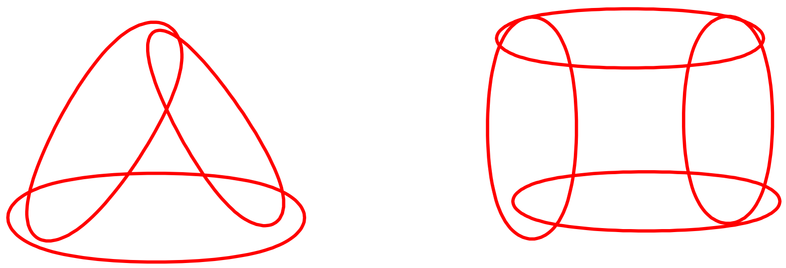
- equivalent to : S needs to cover every k-out-deficient set

out-critical set : minimal (for set inclusion) k-out-deficient set

- so equivalent to : S needs to cover every out-critical set
- **but . . .**
  - out-critical sets need not be disjoint !
  - in fact, there can be exponentially many out-critical sets

DA – *Structure of out-critical sets*

- the collection of out-critical sets satisfies :
  - overlapping out-critical sets can't form cycles in which only consecutive pairs overlap

so no  etc., . . .

  - $C_1, \ldots, C_t$ a collection of out-critical sets, then :

$$\left[\, \forall\, i, j \,:\, C_i \cap C_j \neq \varnothing \,\right] \implies \bigcap_i C_i \neq \varnothing$$

(Helly property)

- such a set system is called a **Subtree Hypergraph**

DA – *Structure of out-critical sets*

- the Subtree Hypergraph structure allows to prove

$$\textit{min}\ \{\ \textit{number of vertices to cover all out-critical sets}\ \} = \textit{max}\ \{\ \textit{number of disjoint out-critical sets}\ \}$$

- but, since the number of out-critical sets can be exponentially large, no efficient algorithm can explore the full structure of the out-critical sets

DA – *Result on Source Location*

Theorem (vdH & Johnson)

- *there exists a polynomial algorithm (in $|V|$)*
  to find a minimum order source set in a directed graph

- proof exploits :
  - the Subtree Hypergraph structure of the out-critical sets
  - it is easy to check if a set $S \subseteq V$ is a set of sources

- at about the same time :
  similar result found by Bárász, Becker & Frank
  using a completely different algorithm

A more general setting

Given : finite set $\boldsymbol{V}$

collection $\boldsymbol{\mathcal{C}}$ of subsets of V

Task : find set $\boldsymbol{X} \subseteq V$ of minimum order

so that X intersects every subset in $\mathcal{C}$

(i.e., X is a **transversal** of $\mathcal{C}$)

- NP-hard in general

(dominating set in graphs is an example)

Our result in the more general setting

Theorem (vdH & Johnson)

given:

- *finite set* $\boldsymbol{V}$ *and a collection* $\boldsymbol{\mathcal{C}}$ *of subsets of* V
  - *so that* $(V, \mathcal{C})$ *is a Subtree Hypergraph*
- *an* ***oracle*** *that for any subset* $T \subseteq V$ *decides if* T *is a transversal of* $\mathcal{C}$ *or not*

then:

- *a minimum size transversal of* $\mathcal{C}$ *can be found using at most* $O(|V|^3)$ *calls to the oracle*

DA – *More results*

Theorems (vdH & Johnson; Bárász, Becker & Frank)

- *similar results (min-max relation and polynomial algorithm) for*
  - *External Network problem*
  - *Generalised Source Location problem*

    (k paths to S , m paths from S)

UE & DA – Another equivalent problem

equivalent are :

- External Network problem for some k
- Generalised Source Location problem with $k = m$

- **Given** : graph (or directed graph) and positive integer $\boldsymbol{k}$

  Task : add new edges /arcs

  so that resulting graph is k-edge / arc-connected

  and with the minimum number of vertices

  incident with the new edges / arcs

UE & DA – *Augmentation and External Network*

both : add new edges / arcs
to achieve required edge / arc-connectivity

- **Edge / Arc Augmentation problem** :
  task : minimise number of edges / arcs
- **External Network problem** :
  task : minimise number of vertices incident
  with new edges / arcs

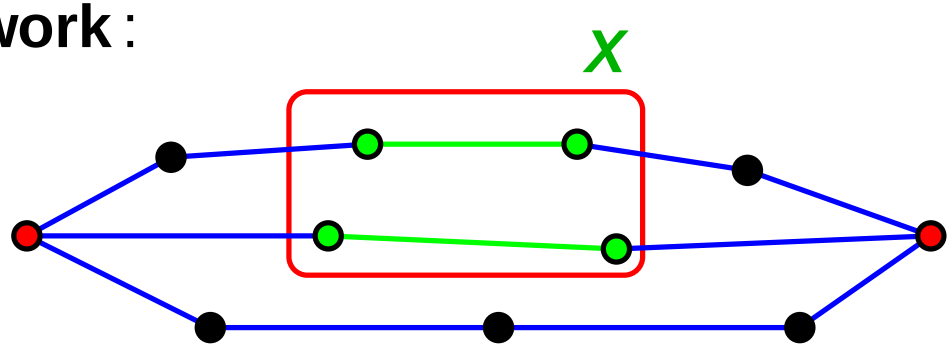
Theorem

- *there exist a set of required edges / arcs minimising **both***
- *such a set can be found in polynomial time*

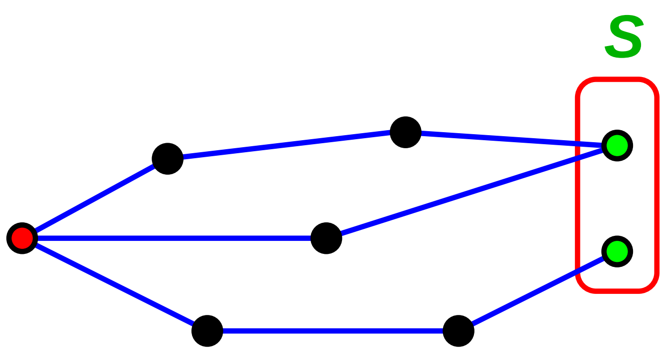
Undirected network & vertex-connectivity – UV

- required : ***k*** internally vertex-disjoint paths

External Network :



Source Location :



- equivalent ?

UV – *Undirected network & vertex-connectivity*

Theorem (Ito *et al.*)

- ***minimum Source Location*** *is NP-complete for all* $k \geq 3$

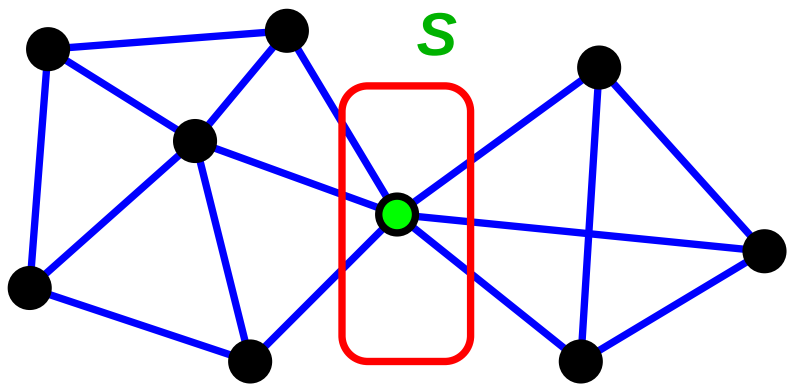
Theorem (vdH & Johnson)

- ***minimum External Network problem*** *can be done in polynomial time for* $k \leq 3$

why the difference ?

UV – *The double role of sources*

- an allowed Source Location solution for $k = 3$:



- this set is useless for the External Network problem for $k = 3$

- in this setting, finding a small source set is more like finding a “clever” transversal of small cut sets

UV – *Another result*

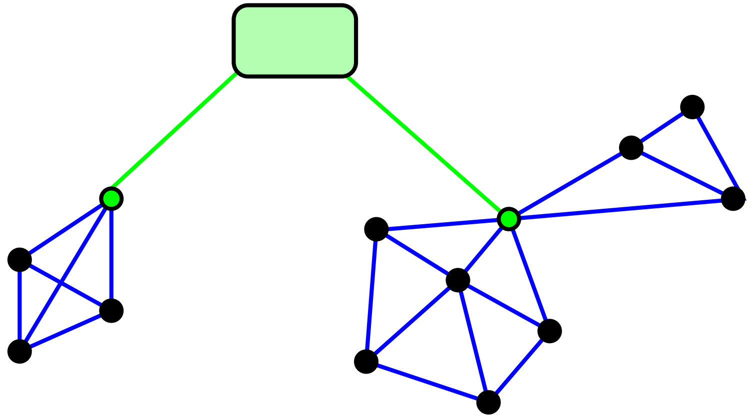
- for vertex-connectivity, what about larger k ?

Theorem (vdH & Johnson)

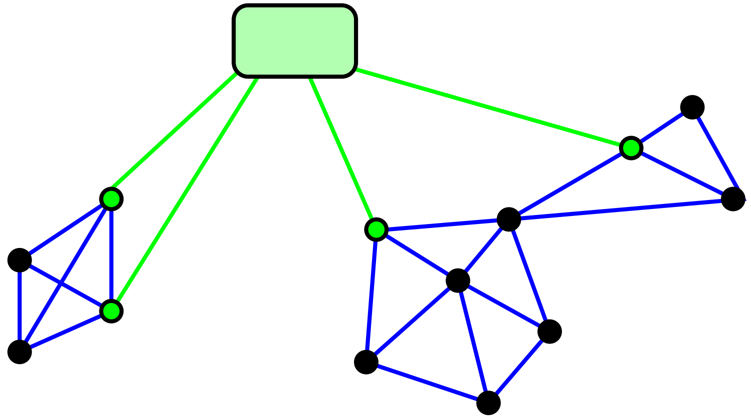
- ***External Network problem*** *can be done in polynomial time for fixed* k *if* G *is already* $(k-1)$*-connected*

UV – Problems with this result

- you can't just go from $k-2$ to $k-1$ to k
- optimal for $k=1$:



- optimal for $k=2$:



Open problems

- is the **External Network problem**,

  with vertex-connectivity, **polynomial for all** k ?

- what can be done for **directed graphs** and **vertex-connectivity** ?

Note : this are hard cases for edge / arc augmentation as well

- what if we have **non-uniform connectivity requirements** ?
- or **non-uniform costs** ?