Distance-Two Colouring of Graphs

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joint work with

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The basics of graph colouring

- **vertex-colouring** with k colours

  adjacent vertices must receive different colours

- **chromatic number** $\chi(G)$

  minimum k so that a vertex-colouring exists

- **list-colouring**: as vertex-colouring,

  but each vertex v has its own list $L(v)$ of colours

- **choice number** $ch(G)$:

  minimum k so that if $|L(v)| \geq k$,

  then a proper list vertex-colouring exists

Another way to look at vertex-colouring

- vertex-colouring :

  vertices at distance one must receive different colours

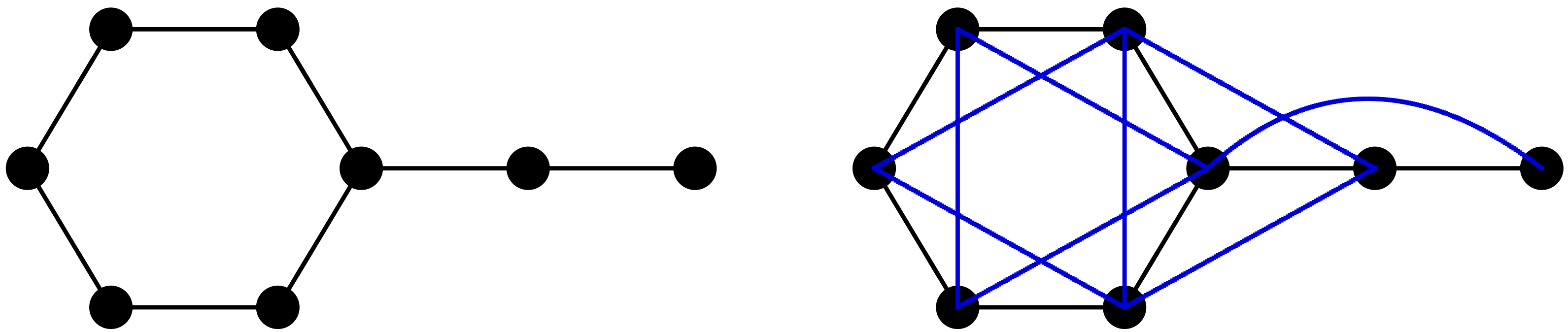
- suppose we require vertices at larger distances

  to receive different colours as well

- for today we only look at distance two

The square of a graph

- distance-two colouring can be modelled using the **square G^2 of a graph**:
  - same vertex set as G
  - edges between vertices with distance at most 2 in G
    ($=$ are adjacent or have a common neighbour)



G $\qquad\qquad$ G^2

Colouring the square of a graph

Easy facts

- $\chi(G^2) \;\geq\; \Delta(G) + 1$

and

- $\Delta(G^2) \;\leq\; \Delta(G)^2$, **so** $\chi(G^2) \;\leq\; \Delta(G)^2 + 1$

(Δ : maximum degree of the graph)

about the upper bound :

- there are at most 4 graphs with $\chi(G^2) = \Delta(G)^2 + 1$
- and infinitely many graphs with
$$\chi(G^2) = \Delta(G)^2 - \Delta(G) + 1$$

The square of planar graphs

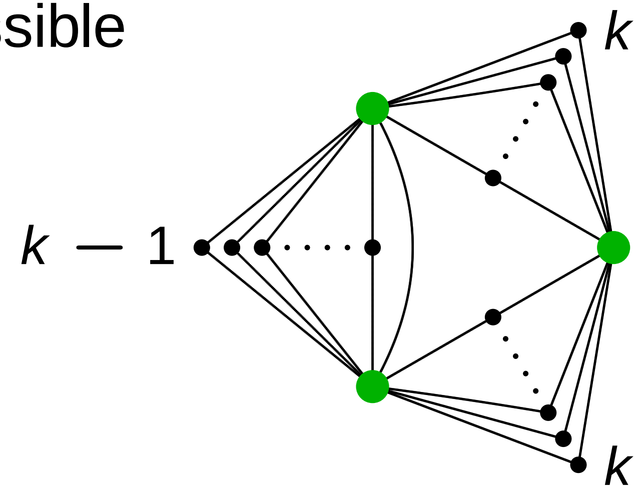
Conjecture (Wegner, 1977)

- G planar

$$\Longrightarrow \quad \chi(G^2) \le \begin{cases} 7, & \text{if } \Delta = 3 \\ \Delta + 5, & \text{if } 4 \le \Delta \le 7 \\ \lfloor 3\Delta/2 \rfloor + 1, & \text{if } \Delta \ge 8 \end{cases}$$

- bounds would be best possible

case $\Delta = 2k \ge 8$:



What was known for large Δ

G planar $\Longrightarrow$

- $\chi(G^2) \leq 8\,\Delta - 22$ (Jonas, PhD, 1993)
- $\chi(G^2) \leq 3\,\Delta + 5$ (Wong, MSc, 1996)
- $\chi(G^2) \leq 2\,\Delta + 25$ (vdH & McGuinness, 2003)
- $\chi(G^2) \leq {}^{9}/_{5}\,\Delta + 1$ (for $\Delta \geq 47$)
  (Borodin, Broersma, Glebov & vdH, 2001)
- $\chi(G^2) \leq {}^{5}/_{3}\,\Delta + 24$ (for $\Delta \geq 241$)
  (Molloy & Salavatipour, 2005)

First new results

Theorem

- G planar $\implies \chi(G^2) \le \left({}^3\!/_2 + o(1)\right)\Delta \quad (\Delta \to \infty)$

we actually prove the list-colouring version

and for much larger classes of graphs :

Theorem

- graph G $K_{3,k}$-minor free for some fixed k

$$\implies ch(G^2) \le \left({}^3\!/_2 + o(1)\right)\Delta$$

Even more general results ?

Property

- graph G H-minor free for some fixed graph H

$$\Longrightarrow \quad ch(G^2) \leq C_H \, \mathbf{\Delta}, \text{ for some constant } C_H$$

Question

- given H, what is the best C_H for large $\mathbf{\Delta}$?

e.g.

- for $H = K_{3,k}$ we know $C_{K_{3,k}} = 3/2$
- for $H = K_5$ we have $2 \leq C_{K_5} \leq 9$
- for $H = K_{4,4}$ we have $C_{K_{4,4}} \geq 7/3$

The clique number

Corollary

- graph G $K_{3,k}$-minor free for some fixed k

$$\Longrightarrow \quad \omega(G^2) \ \le \ \left({}^3\!/_2 + o(1)\right)\Delta$$

can be partially improved to

Theorem

- G embeddable on a fixed surface S

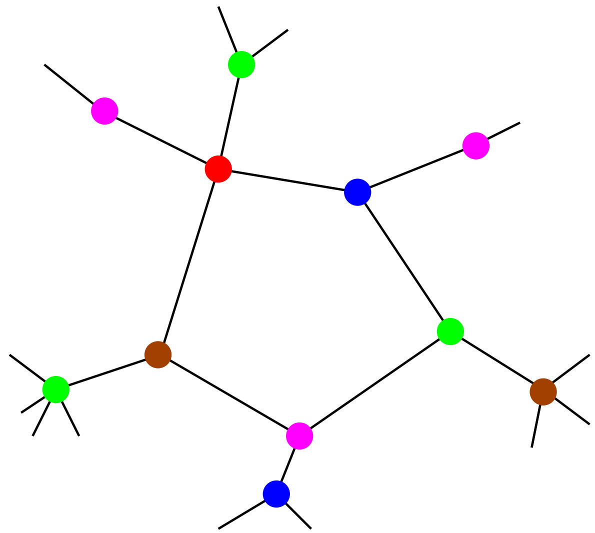
$$\Longrightarrow \quad \omega(G^2) \ \le \ {}^3\!/_2\,\Delta + O(1)$$

Theorem (Cohen & vdH, 2009+)

- G planar and $\Delta(G) \ge 41 \quad \Longrightarrow \quad \omega(G^2) \ \le \ \left\lfloor {}^3\!/_2\,\Delta \right\rfloor + 1$

A related (?) problem

- **plane graph** : planar graph with a given embedding
- **cyclic colouring** of a plane graph :
  - vertex-colouring so that
  - vertices incident to the same face get a different colour



A related (?) problem

- **plane graph**: planar graph with a given embedding
- **cyclic colouring** of a plane graph:
  - vertex-colouring so that
  - vertices incident to the same face get a different colour
- **cyclic chromatic number** $\chi^*(G)$:
  minimum number of colours needed for a cyclic colouring
- $\Delta^*(G)$: size of largest face of G

Easy

- G plane graph $\implies$ $\chi^*(G) \geq \Delta^*(G)$

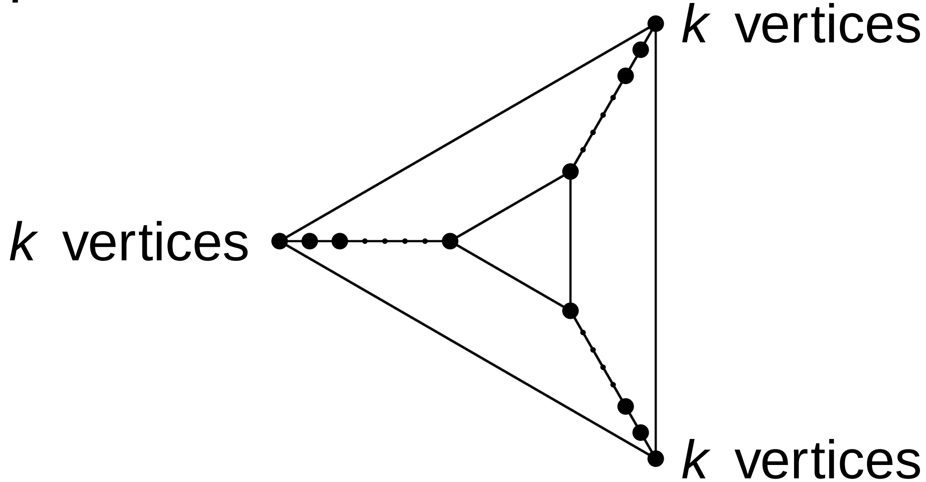
Conjecture for the related(?) problem

Conjecture (Borodin, 1984)

- G plane graph $\Longrightarrow$ $\chi^*(G) \le {}^3\!/_2\,\Delta^*(G)$

- bound would be best possible

case $\Delta^* = 2\,k$:



Bounds on the cyclic chromatic number

G plane graph $\Longrightarrow$

- $\chi^*(G) \le 2\,\Delta^*(G)$ (Ore & Plummer, 1969)
- $\chi^*(G) \le {}^9\!/_5\,\Delta^*(G)$ (Borodin, Sanders & Zhao, 1999)
- $\chi^*(G) \le {}^5\!/_3\,\Delta^*(G)$ (Sanders & Zhao, 2001)

Theorem

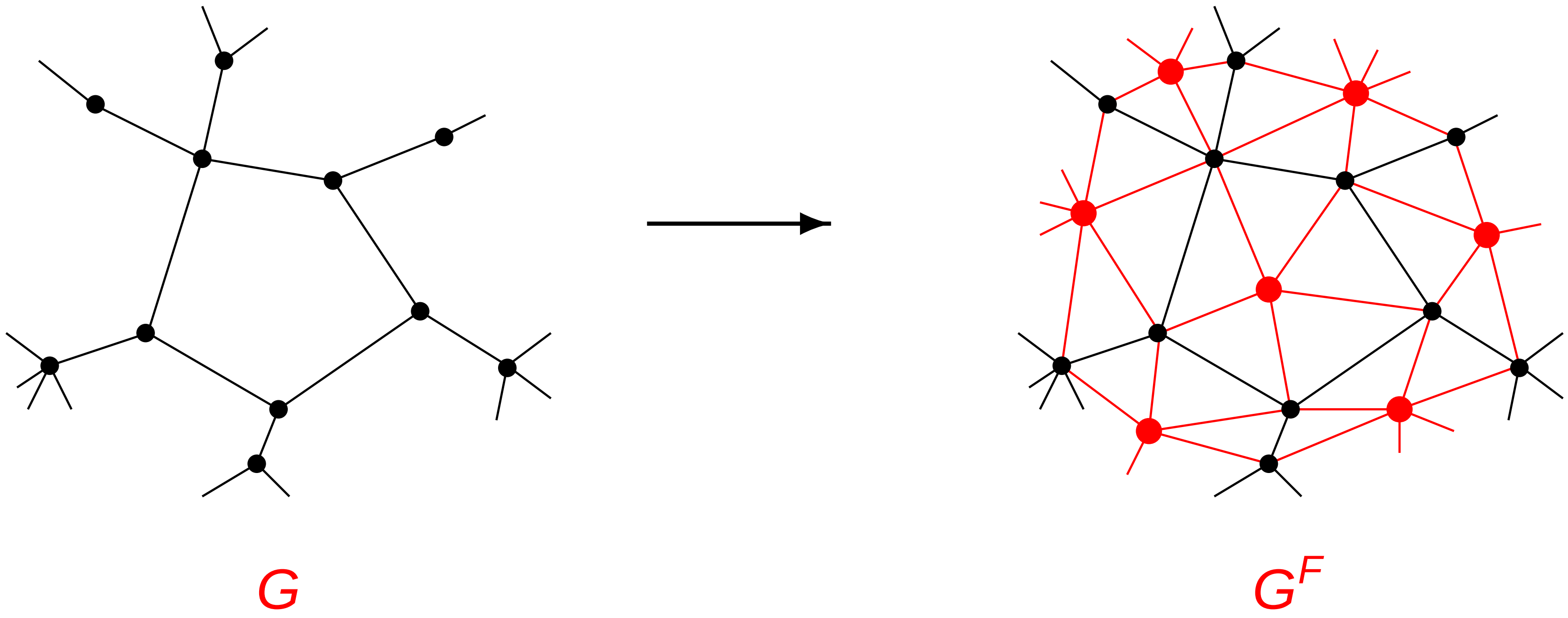
- G plane graph

$$\Longrightarrow \quad \chi^*(G) \le \big({}^3\!/_2 + o(1)\big)\,\Delta^* \qquad (\Delta^* \to \infty)$$

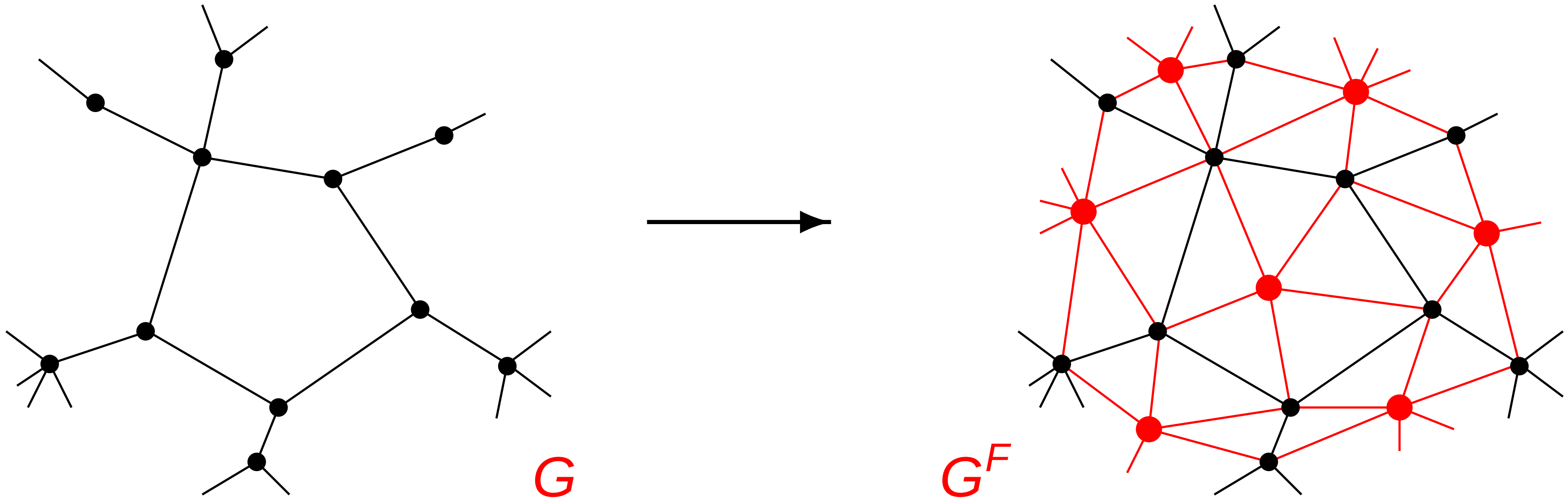
(in fact, we again prove the list-colouring version)

From cyclic colouring to "distance-two" colouring

- given a plane graph G, form new graph G^F by
  - adding a vertex in each face
  - adding edges from new vertex to all vertices of the face



From cyclic colouring to "square" colouring



- colouring the square of G^F gives cyclic colouring of G
  - but also colours the faces (not asked for, but o.k.)
  - and degrees of old vertices may be more than $\Delta^*(G)$

    (serious problem)

Both in one go

idea : do not treat all vertices equal :

- some vertices need to be coloured
- some vertices determine distance two

- vertices can be of both types
  - as are all vertices when colouring the square
- or the types can be disjoint
  - as for “faces” and “vertices” for cyclic colouring
- or anything in between

Formalising our clever little idea

- **given** : graph G, subsets $A, B \subseteq V(G)$
- $(\boldsymbol{A}, \boldsymbol{B})$**-colouring of** $\boldsymbol{G}$: colouring of vertices in B so that
  - adjacent vertices get different colours
  - vertices with a common neighbour in A get different colours
- **list** $(\boldsymbol{A}, \boldsymbol{B})$**-colouring of** $\boldsymbol{G}$:
  - similar, but each vertex in B has its own list
- $\chi(G; A, B)$ / $ch(G; A, B)$:
  minimum number of colours needed / minimum size of each list needed

The first baby steps

- $\chi(G) = \chi(G; \varnothing, V)$
- $\chi(G^2) = \chi(G; V, V)$
- $\chi^*(G) = \chi(G^F; \text{“faces”}, \text{“vertices”})$

- relevant “**degree**”: $d_B(v) =$ number of neighbours in B
- “**maximum degree**”:

$$\mathbf{\Delta}(G; A, B) = \text{ maximum of } d_B(v) \text{ over all } v \in A$$

Easy

- $\mathbf{\Delta}(G; A, B) \le \chi(G; A, B)$

$$\le \mathbf{\Delta}(G) + \mathbf{\Delta}(G; A, B) \cdot (\mathbf{\Delta}(G; A, B) - 1) + 1$$

The obvious(?) conjecture & results

Conjecture

- G planar, $A, B \subseteq V(G)$, $\mathbf{\Delta}(G;A,B)$ large enough

$$\Longrightarrow \quad \chi(G;A,B) \;\leq\; {}^{3}\!/_{2}\,\mathbf{\Delta}(G;A,B)+1$$

Theorem

- G embeddable on a fixed surface S, $A, B \subseteq V(G)$

$$\Longrightarrow \quad ch(G;A,B) \;\leq\; \left({}^{3}\!/_{2}+o(1)\right)\mathbf{\Delta}(G;A,B)$$

Corollary

(asymptotic list version of Wegner's and Borodin's Conjecture)

- G planar $\quad \Longrightarrow \quad ch(G^2) \;\leq\; \left({}^{3}\!/_{2}+o(1)\right)\mathbf{\Delta}(G)$
- G plane graph $\quad \Longrightarrow \quad ch^*(G) \;\leq\; \left({}^{3}\!/_{2}+o(1)\right)\mathbf{\Delta}^*(G)$

Sketch of the proof of square of planar graph

uses induction on the number of vertices

- **2-neighbour** : vertex at distance one or two
- $d^2(v)$: number of 2-neighbours of v

  $=$ number of neighbours of v in G^2
- we would like to remove a vertex v with $d^2(v) \leq {}^3\!/_2\,\Delta$
  - but that can change distances in $G - v$
- contraction to a neighbour u will solve the distance problem
  - but may increase maximum degree if $d(u) + d(v) > \Delta$
- easy induction possible if there is an edge uv

  with $d(u) + d(v) \leq \Delta$ and $d^2(v) \leq {}^3\!/_2\,\Delta$

When easy induction is not possible

S, **small** vertices: degree at most some constant C

B, **big** vertices: degree more than C

H, **huge** vertices: degree at least $\frac{1}{2}\,\Delta$

- small vertices have a least two big neighbours
  (otherwise for those v: $d^2(v) \leq {}^3\!/_2\,\Delta$)
- a planar graph has fewer than $3\,|V|$ edges
  and fewer than $2\,|V|$ edges if it is bipartite
  so:
  - all but $O(|V|/C)$ vertices are small
  - fewer than $2\,|B|$ vertices in $V \setminus B$
    have more than two neighbours in B

When easy induction is not possible

$\boldsymbol{S}$, **small** vertices: degree at most some constant C

$\boldsymbol{B}$, **big** vertices: degree more than C

$\boldsymbol{H}$, **huge** vertices: degree at least $\frac{1}{2}\,\boldsymbol{\Delta}$

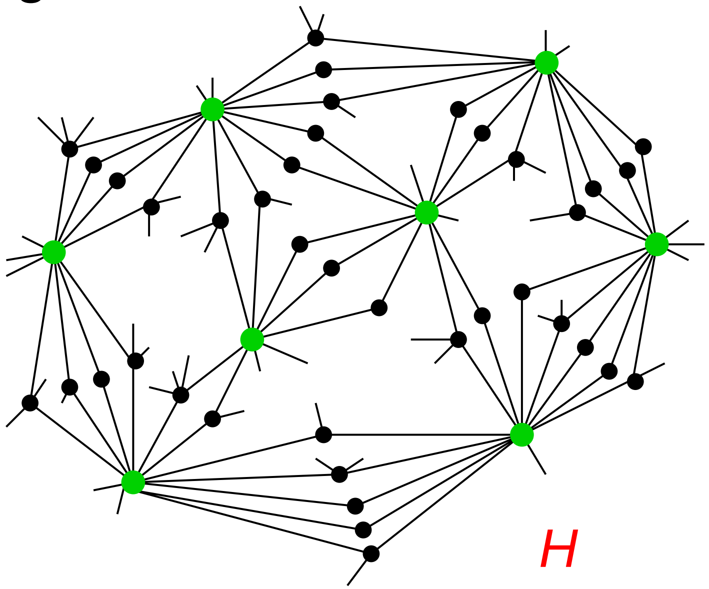
so:

- "most" vertices are small
- and these have exactly two big neighbours

(in fact two huge neighbours)

The structure so far

- there is a subgraph ***H*** of G looking like :



- green vertices ***X***
  have degree at least $\frac{1}{2}\,\Delta$
- black vertices ***Y***
  have degree at most C
- all other neighbours of Y -vertices are also small

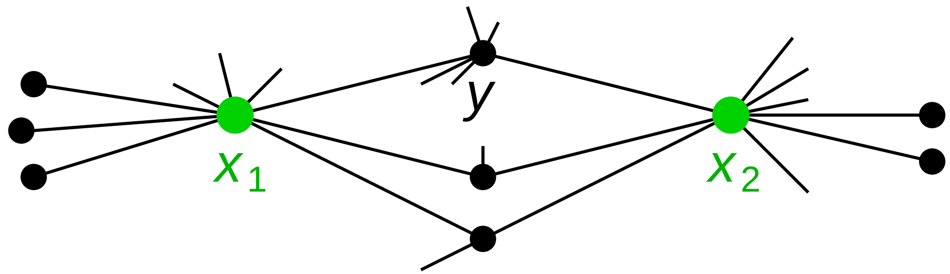
inside H there is a subgraph ***H*** which satisfies additionally :

- only “few” edges from X to rest of G
- H satisfies “some edge density condition”

The other induction step

- remove the vertices from Y in H (using contraction)
- colour the smaller graph (can be done by induction)
- in the original graph G :

  what to do with the uncoloured Y-vertices?



- 2-neighbours of y already coloured (i.e., outside Y):
  - at most $(d_G(x_1) - d_H(x_1)) + (d_G(x_2) - d_H(x_2))$

    2-neighbours outside Y via x_1, x_2
  - at most C^2 other 2-neighbours outside Y

Transferring to edge-colouring

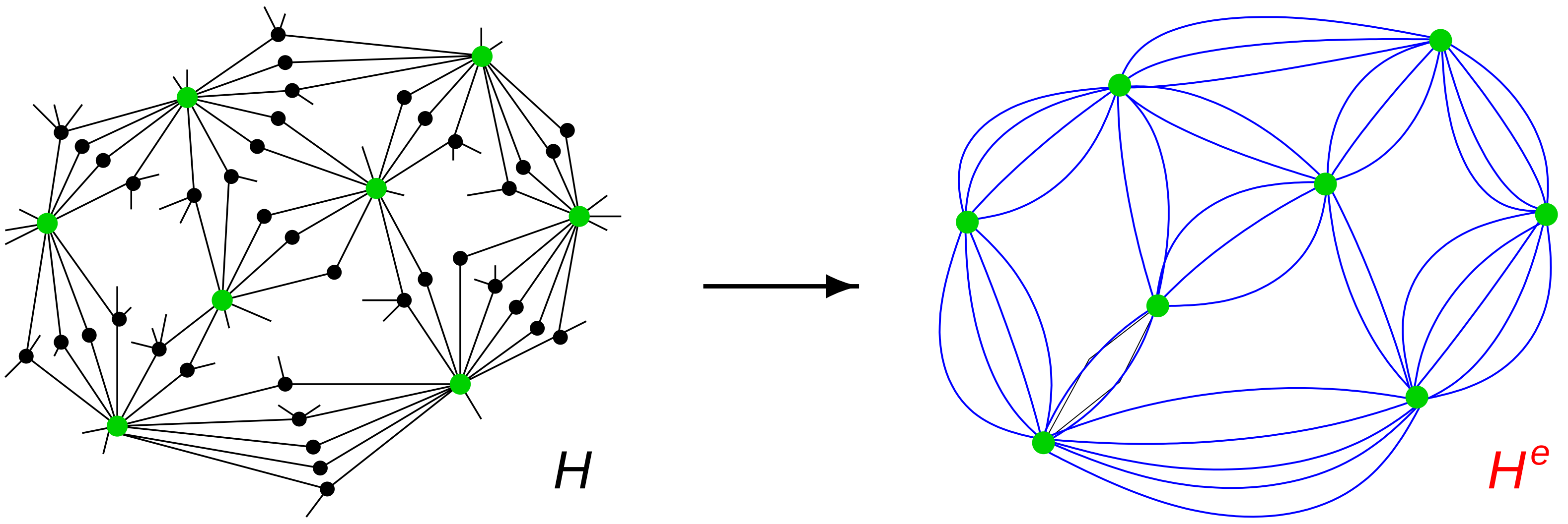
- so a vertex y from Y has at least

$$(3/2+\varepsilon)\,\boldsymbol{\Delta} - (d_G(x_1) - d_H(x_1)) - (d_G(x_2) - d_H(x_2)) - C^2$$

  colours still available

- colouring Y is "almost" like

  list-colouring edges of the multigraph $\boldsymbol{H^e}$:



Edge-colouring multigraphs

- $\chi'(G)$: chromatic index of multigraph G
- $ch'(G)$: list chromatic index of multigraph G
- $\chi'_f(G)$: fractional chromatic index of multigraph G

Theorem (Kahn, 1996, 2000)

- G multigraph, with Δ large enough

$$\Longrightarrow \quad ch'(G) \approx \chi'(G) \approx \chi'_f(G)$$

- in fact, Kahn's proofs provide something much more general

Kahn's result

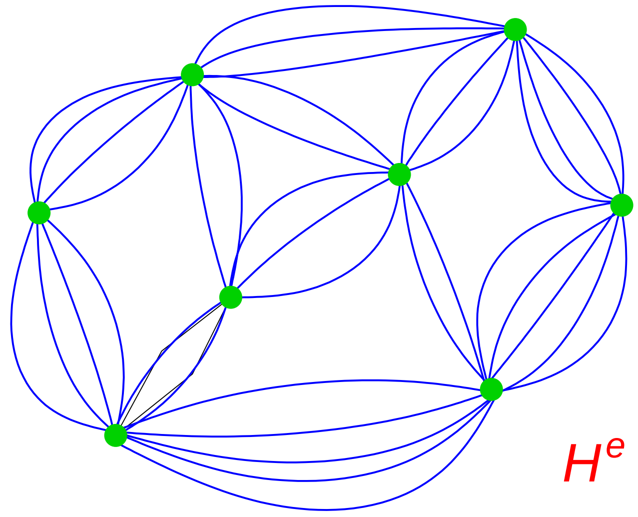
Theorem (Kahn, 2000)

for $0 < \delta < 1$, $\alpha > 0$ there exists $\mathbf{\Delta}_{\delta,\alpha}$ so that if $\mathbf{\Delta} \geq \mathbf{\Delta}_{\delta,\alpha}$:

- G a multigraph with maximum degree $\mathbf{\Delta}$
- each edge e has a list $L(e)$ of colours so that
  - for all edges e: $|L(e)| \geq \alpha\,\mathbf{\Delta}$
  - for all vertices v: $$\sum_{e \ni v} |L(e)|^{-1} \leq 1 \cdot (1 - \delta)$$
  - for all $K \subseteq G$ with $|V(K)| \geq 3$ odd: $$\sum_{e \in E(K)} |L(e)|^{-1} \leq \tfrac{1}{2}\,(|V(K)| - 1) \cdot (1 - \delta)$$
- then there exists a proper colouring of the edges of G so that each edge gets colours from its own list

Kahn's approach for our case

- we have a multigraph H^e :

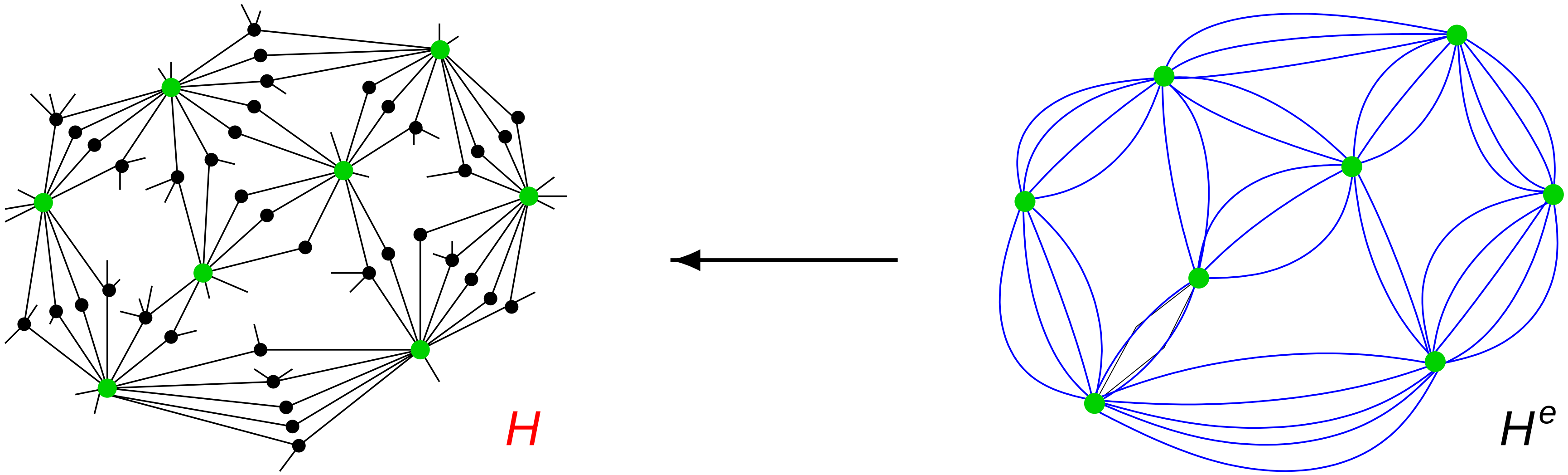


- so that each edge $e = x_1 x_2$ has a list $L(e)$ of at least
$$({}^3\!/\!_2 + \varepsilon)\,\boldsymbol{\Delta} - (d_G(x_1) - d_H(x_1)) - (d_G(x_2) - d_H(x_2)) - C^2$$
colours

- and H^e satisfies "some edge density condition"

Extending Kahn's approach

- these conditions guarantee that Kahn's conditions are satisfied for H^e
- $\Longrightarrow$ we can edge-colour H^e



- $\Longrightarrow$ we can colour the Y-vertices in H
  choosing from the left-over colours for each
- also : we can deal with the "almost" list-edge colouring

Some open problems

- prove the next step :

  G planar $\Longrightarrow$ $\chi(G^2) \le {}^3\!/_2\,\Delta + O(1)$

- for G planar, $\Delta \le 3$ we know :
  - $\chi(G^2) \le 7$ (Thomassen, 2007)
  - $ch(G^2) \le 8$ (Cranston & Kim, 2006)
  - what is the right upper bound for $ch(G^2)$ in this case ?

- Wegner's Conjecture for $4 \le \Delta \le 7$:

  G planar $\Longrightarrow$ $\chi(G^2) \le \Delta + 5$?

Conjectures, conjectures, conjectures, . . .

List-Square-Colouring Conjecture (Kostochka & Woodall, 2001)

- for any graph G $\Longrightarrow$ $ch(G^2) = \chi(G^2)$

this would imply :

List-Total-Colouring Conjecture

(Borodin, Kostochka & Woodall, 1997)

- for any graph G $\Longrightarrow$ $ch''(G) = \chi''(G)$

related (?) to :

List-Edge-Colouring Conjecture (several authors, $\approx$ 1975)

- for any graph G $\Longrightarrow$ $ch'(G) = \chi'(G)$