Degrees of Perfectness

for **Jack** at his 75th birthday

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We use nice, but not always standard, terminology

- **set system** $(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$: a finite set S with a collection $\mathcal{F}$ of subsets of S

- a set system is **nice** if :
  - $\mathcal{F}$ is closed under taking subsets, and
  - $\mathcal{F}$ covers all of S

- $G = (V_G, E_G)$ a graph, $\boldsymbol{\mathcal{S}_G}$ the collection of all stable sets (sets containing no adjacent pairs of vertices)
  - then $(V_G, \mathcal{S}_G)$ is a nice set system

Coverings

- a **covering** of $(S, \mathcal{F})$:

  a collection of sets from $\mathcal{F}$ whose union is S

- **covering number** $\mathbf{Cov}(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$:

  the minimum number of elements in a covering

- for a graph G : $\mathrm{Cov}(V_G, \mathcal{S}_G)$ is just the chromatic number

That's easy, so let's make it more complicated

- the covering number is also the solution of the IP problem :

$$\begin{array}{lll} \text{minimise} & \displaystyle\sum_{F\in\mathcal{F}} x_F & \\ \text{subject to} & \displaystyle\sum_{F\ni s} x_F \ \geq\ 1, & \text{for all } s\in S \\ & x_F\ \in\ \{0,1,2,\ldots\}, & \text{for all } F\in\mathcal{F} \end{array}$$

The fractional version

- removing the integrality condition :

$$\text{minimise} \quad \sum_{F \in \mathcal{F}} x_F$$

$$\text{subject to} \quad \sum_{F \ni s} x_F \geq 1, \qquad \text{for all } s \in S$$

$$x_F \geq 0, \qquad \text{for all } F \in \mathcal{F}$$

- gives the **fractional covering number** $\mathbf{Cov}_f(S, \mathcal{F})$
  - and we obviously have : $\mathrm{Cov}_f(S, \mathcal{F}) \leq \mathrm{Cov}(S, \mathcal{F})$

Rule 1 of Linear Programming : dualise

- the dual LP problem of the fractional covering number is :

$$\text{maximise} \quad \sum_{s \in S} y_s$$

$$\text{subject to} \quad \sum_{s \in F} y_s \leq 1, \qquad \text{for all } F \in \mathcal{F}$$

$$y_s \geq 0, \qquad \text{for all } s \in S$$

- this gives the **fractional packing number** $\mathbf{Pack}_f(S, \mathcal{F})$
  - and by LP-duality : $\mathrm{Pack}_f(S, \mathcal{F}) = \mathrm{Cov}_f(S, \mathcal{F})$

The packing number

- the integral version is the **packing number** $\mathbf{Pack}(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$:
  - the maximum size $|T|$ of a subset T of S so that no two elements of T appear together in a set from $\mathcal{F}$
  - i.e.: the maximum size $|T|$ of some $T \subseteq S$ so that $|T \cap F| \leq 1$, for all $F \in \mathcal{F}$
- for a graph G : $\text{Pack}(V_G, \mathcal{S}_G)$ is just the clique number
  - the maximum size of a set of vertices $U \subseteq V_G$ so that all pairs in U are adjacent

The status so far

- for any nice set system $(S, \mathcal{F})$ we have

$$\mathrm{Pack}(S,\mathcal{F}) \;\le\; \mathrm{Pack}_{f}(S,\mathcal{F}) \;=\; \mathrm{Cov}_{f}(S,\mathcal{F}) \;\le\; \mathrm{Cov}(S,\mathcal{F})$$

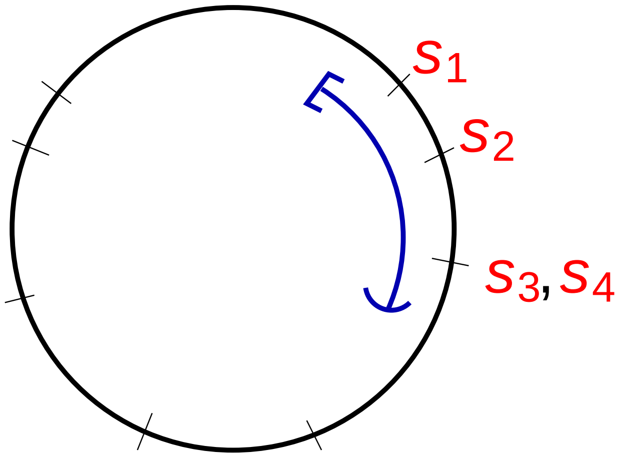
- we will add one more parameter :

  the **circular covering number** $\mathbf{Cov}_c(S, \mathcal{F})$

The circular covering number

- map the elements of S to a circle so that :
  - for every unit interval $[x, x+1)$ along the circle

    elements mapped into that interval form a set from $\mathcal{F}$

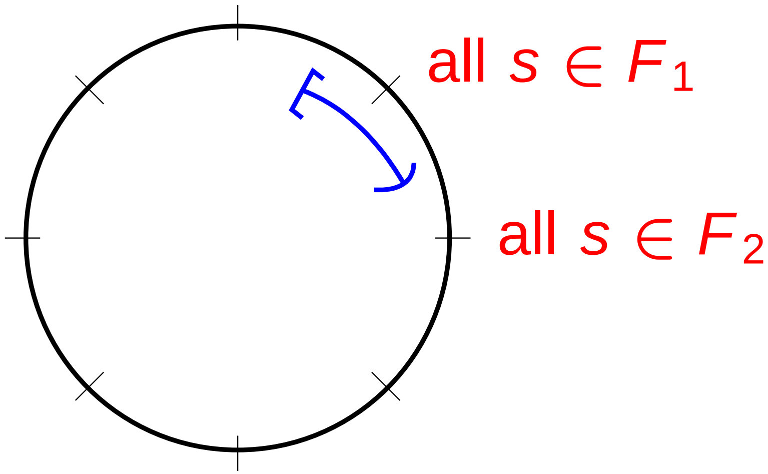


- **circular covering number** $\mathbf{Cov}_c(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$:

  minimum circumference of a circle for which this is possible

Let's put in in the right place - I

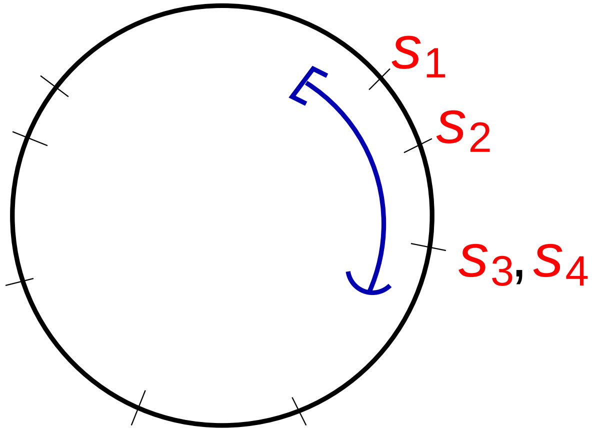
- for a nice set system : $\mathrm{Cov}_c(S, \mathcal{F}) \leq \mathrm{Cov}(S, \mathcal{F})$
  - take a disjoint cover $F_1, \ldots, F_k$ of $(S, \mathcal{F})$
  - put the elements of each F_i together at unit distance around a circle with circumference k :



  - gives a circular cover with circumference k

Let's put in in the right place - II

- for a nice set system : $\mathrm{Cov}_{\boldsymbol{f}}(S, \mathcal{F}) \leq \mathrm{Cov}_{\boldsymbol{c}}(S, \mathcal{F})$
  - take a circular cover along a circle



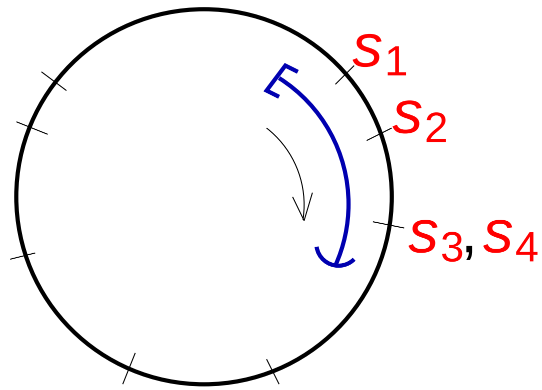
  - "move" the unit interval with "unit speed" round the circle
  - for a set F that appears in the interval at some point :

    denote by x_F the "length of time" it appears

Let's put in in the right place - II

- for a nice set system : $\mathrm{Cov}_{\boldsymbol{f}}(S,\mathcal{F}) \leq \mathrm{Cov}_{\boldsymbol{c}}(S,\mathcal{F})$
  - take a circular cover along some circle
  - for a set F that appears in the interval at some point :

    denote by x_F the "length of time" it appears
  - then for all $s \in S$: $\sum_{F \ni s} x_F = 1$
  - and $\sum_{F \in \mathcal{F}} x_F =$ circumference
  - this gives a fractional cover with value the circumference



Inequalities, inequalities, and more inequalities

- so now we know :

$$\text{Pack} \ \le \ \text{Pack}_{\boldsymbol{f}} \ = \ \text{Cov}_{\boldsymbol{f}} \ \le \ \text{Cov}_{\boldsymbol{c}} \ \le \ \text{Cov}$$

- can we say for which nice set systems we have equality for one of the inequalities ?
  - probably too hard
- what about those that satisfy an equality "through and through" ?

$$\text{Pack} \ \le \ \begin{matrix}\text{Cov}_f \\ \text{Pack}_f\end{matrix} \ \le \ \text{Cov}_c \ \le \ \text{Cov}$$

Through and through $=$ *induced*

- $(S, \mathcal{F})$ a nice set system and $T \subseteq S$, then define :

$$\mathcal{F}_T = \{ F \cap T \mid F \in \mathcal{F} \} = \{ F \in \mathcal{F} \mid F \subseteq T \}$$

- $(T, \mathcal{F}_T)$ is again a nice set system
  - called an **induced** set system

- for a graph G with $U \subseteq V_G$:

  $(\mathcal{S}_G)_U$ are the stable sets of the subgraph induced by U

$$\text{Pack} \le \begin{matrix}\text{Cov}_f \\ \text{Pack}_f\end{matrix} \le \text{Cov}_c \le \text{Cov}$$

Degrees of perfectness

- a nice set system is $(\boldsymbol{A} = \boldsymbol{B})$ **-perfect** :
  - the system and all its induced systems satisfy $A = B$
- note that we have **six** degrees of perfectness
- by definition, perfect graphs are exactly those graphs G for which $(V_G, \mathcal{S}_G)$ is $(\text{Pack} = \text{Cov})$ -perfect
  - that makes them perfect for all inequalities !

$$\text{Pack} \le \begin{matrix}\text{Cov}_f \\ \text{Pack}_f\end{matrix} \le \text{Cov}_c \le \text{Cov}$$

What about the other set systems ?

- we know non-perfect graphs very well :

Strong Perfect Graph Theorem

- G **not** a perfect graph $\qquad \Longleftrightarrow$

  G contains an induced copy :

  - of an odd cycle C_{2k+1} , $k \geq 2$, or
  - of the complement $\overline{C_{2k+1}}$ of an odd cycle, $k \geq 2$

$$\text{Pack} \ \leq \ \begin{matrix} \text{Cov}_f \\ \text{Pack}_f \end{matrix} \ \leq \ \text{Cov}_c \ \leq \ \text{Cov}$$

What about other "graphical" set systems ?

- for an odd cycle C_{2k+1}, $k \geq 2$, it's easy to check :
  - $\mathrm{Pack}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 2$
  - $\mathrm{Cov}_{\boldsymbol{f}}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = \mathrm{Cov}_{\boldsymbol{c}}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 2 + \dfrac{1}{k}$
  - $\mathrm{Cov}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 3$

- similar things happen for

  the complement $\overline{C_{2k+1}}$ of an odd cycle, $k \geq 2$

$$\mathrm{Pack} \ \leq \ \begin{matrix} \mathrm{Cov}_f \\ \mathrm{Pack}_f \end{matrix} \ \leq \ \mathrm{Cov}_c \ \leq \ \mathrm{Cov}$$

Perfect graphs are very perfect

so:

- a nice set system of the form $(V_G, \mathcal{S}_G)$ is

  $(\text{Pack} = \text{Cov}_{\boldsymbol{f}})$-perfect, or $(\text{Pack} = \text{Cov}_{\boldsymbol{c}})$-perfect, or
  $(\text{Pack} = \text{Cov})$-perfect, or $(\text{Cov}_{\boldsymbol{f}} = \text{Cov})$-perfect, or
  $(\text{Cov}_{\boldsymbol{c}} = \text{Cov})$-perfect

  $\Longleftrightarrow$ G is perfect

problem:

- prove this for $(\text{Cov}_{\boldsymbol{c}} = \text{Cov})$-perfectness,

  without using the Strong Perfect Graph Theorem

$$\text{Pack} \leq \begin{matrix} \text{Cov}_f \\ \text{Pack}_f \end{matrix} \leq \text{Cov}_c \leq \text{Cov}$$

And what about non-graphical set systems ?

- suppose $(S, \mathcal{F})$ is a nice set system such that
  - all minimal sets outside $\mathcal{F}$ have size 2

    (smaller than 2 is not possible, as $\mathcal{F}$ covers S)

- then form the graph G with $V_G = S$ by setting

$$s_1 s_2 \in E_G \quad \Longleftrightarrow \quad \{s_1, s_2\} \notin \mathcal{F}$$

- easy to check : $(S, \mathcal{F}) = (V_G, \mathcal{S}_G)$

$$\text{Pack} \leq \frac{\text{Cov}_f}{\text{Pack}_f} \leq \text{Cov}_c \leq \text{Cov}$$

That's that about non-graphical set systems!

- $(S, \mathcal{F})$ is a non-graphical nice set system $\iff$

  there is a subset $T \subseteq S$ with $|T| = k \geq 3$ so that:

  - $T \notin \mathcal{F}$
  - but every **proper** subset of T is in $\mathcal{F}$

- for such a T, the induced set system $(T, \mathcal{F}_T)$ satisfies:

  - $\mathrm{Pack}(T, \mathcal{F}_T) = 1$
  - $\mathrm{Cov}_{\boldsymbol{f}}(T, \mathcal{F}_T) = \mathrm{Cov}_{\boldsymbol{c}}(T, \mathcal{F}_T) = 1 + \dfrac{1}{k-1}$
  - $\mathrm{Cov}(T, \mathcal{F}_T) = 2$

$$\mathrm{Pack} \leq \begin{matrix} \mathrm{Cov}_f \\ \mathrm{Pack}_f \end{matrix} \leq \mathrm{Cov}_c \leq \mathrm{Cov}$$

Perfect graphs are really, really perfect!

<u>**so**</u>:

- a nice nice set system $(S, \mathcal{F})$ is

  $(\text{Pack} = \text{Cov}_{\boldsymbol{f}})$-perfect, or $(\text{Pack} = \text{Cov}_{\boldsymbol{c}})$-perfect, or $(\text{Pack} = \text{Cov})$-perfect, or $(\text{Cov}_{\boldsymbol{f}} = \text{Cov})$-perfect, or $(\text{Cov}_{\boldsymbol{c}} = \text{Cov})$-perfect

$$\Longleftrightarrow$$

$(S, \mathcal{F}) = (V_G, \boldsymbol{\mathcal{S}}_G)$ for some perfect graph G

$$\text{Pack} \leq \begin{matrix} \text{Cov}_f \\ \text{Pack}_f \end{matrix} \leq \text{Cov}_c \leq \text{Cov}$$

The bit that's left to do

- what nice set systems $(S, \mathcal{F})$ are $(\mathrm{Cov}_{\boldsymbol{f}} = \mathrm{Cov}_{\boldsymbol{c}})$-perfect ?
- well ...
  - stable sets of perfect graphs
  - stable sets of odd cycles or complements of odd cycles
  - loopless matroids (vdH & Thomassé)
  - and a lot more

$\mathrm{Cov}_f \ \leq \ \mathrm{Cov}_c$

What the $\ast\ast\ast\ast$ is a loopless matroid?

- a set system $(S, \mathcal{F})$ is a **loopless matroid** if
  - $(S, \mathcal{F})$ is nice
  - for each $F_1, F_2 \in \mathcal{F}$ with $|F_1| > |F_2|$:

    there is an $s \in F_1 \setminus F_2$ so that $F_2 \cup \{s\} \in \mathcal{F}$

by the way:

- a stable set system $(V_G, \mathcal{S}_G)$ is a loopless matroid

  $\Longleftrightarrow$ G is the disjoint union of cliques

$\mathrm{Cov}_f \leq \mathrm{Cov}_c$

This looks likes it's going to be complicated

- so nice set systems that are $(\mathrm{Cov}_{f} = \mathrm{Cov}_{c})$-perfect include
  - stable sets of perfect graphs
  - stable sets of odd cycles or complements of odd cycles
  - loopless matroids
  - disjoint unions of the above
  - and probably a lot more . . .

question :

- can we characterise $(\mathrm{Cov}_{f} = \mathrm{Cov}_{c})$-perfect set systems ?

$$\mathrm{Cov}_{f} \leq \mathrm{Cov}_{c}$$