Degrees of Perfectness

for Jack at his 75th birthday

JAN VAN DEN HEUVEL

Department of Mathematics
London School of Economics and Political Science
We use nice, but not always standard, terminology

- **set system** \((S, \mathcal{F})\): a finite set \(S\) with a collection \(\mathcal{F}\) of subsets of \(S\)

- a set system is **nice** if:
  - \(\mathcal{F}\) is closed under taking subsets, and
  - \(\mathcal{F}\) covers all of \(S\)

- \(G = (V_G, E_G)\) a graph, \(\mathcal{S}_G\) the collection of all stable sets (sets containing no adjacent pairs of vertices)

- then \((V_G, \mathcal{S}_G)\) is a nice set system
Coverings

a covering of \((S, \mathcal{F})\):

a collection of sets from \(\mathcal{F}\) whose union is \(S\)

covering number \(\text{Cov}(S, \mathcal{F})\):

the minimum number of elements in a covering

for a graph \(G\): \(\text{Cov}(V_G, S_G)\) is just the chromatic number
That’s easy, so let’s make it more complicated

- the covering number is also the solution of the IP problem:

$$\text{minimise } \sum_{F \in \mathcal{F}} x_F$$

subject to

$$\sum_{F \supset s} x_F \geq 1, \quad \text{for all } s \in S$$

$$x_F \in \{0, 1, 2, \ldots \}, \quad \text{for all } F \in \mathcal{F}$$
The fractional version

- removing the integrality condition:

\[
\begin{align*}
\text{minimise} & \quad \sum_{F \in \mathcal{F}} x_F \\
\text{subject to} & \quad \sum_{F \ni s} x_F \geq 1, \quad \text{for all } s \in S \\
& \quad x_F \geq 0, \quad \text{for all } F \in \mathcal{F}
\end{align*}
\]

- gives the fractional covering number \( \text{Cov}_f(S, \mathcal{F}) \)

- and we obviously have: \( \text{Cov}_f(S, \mathcal{F}) \leq \text{Cov}(S, \mathcal{F}) \)
Rule 1 of Linear Programming: dualise

- the dual LP problem of the fractional covering number is:

  maximise \[ \sum_{s \in S} y_s \]

  subject to \[ \sum_{s \in F} y_s \leq 1, \text{ for all } F \in \mathcal{F} \]

  \[ y_s \geq 0, \text{ for all } s \in S \]

- this gives the fractional packing number \( \text{Pack}_f(S, \mathcal{F}) \)

- and by LP-duality: \( \text{Pack}_f(S, \mathcal{F}) = \text{Cov}_f(S, \mathcal{F}) \)
The packing number

- the integral version is the **packing number** \( \text{Pack}(S, \mathcal{F}) \):  
  - the maximum size \( |T| \) of a subset \( T \) of \( S \) so that no two elements of \( T \) appear together in a set from \( \mathcal{F} \)
  - i.e.: the maximum size \( |T| \) of some \( T \subseteq S \) so that \( |T \cap F| \leq 1 \), for all \( F \in \mathcal{F} \)

- for a graph \( G \): \( \text{Pack}(V_G, \mathcal{S}_G) \) is just the **clique number**
  - the maximum size of a set of vertices \( U \subseteq V_G \) so that all pairs in \( U \) are adjacent
The status so far

- for any nice set system \((S, \mathcal{F})\) we have

\[
\text{Pack}(S, \mathcal{F}) \leq \text{Pack}_f(S, \mathcal{F}) = \text{Cov}_f(S, \mathcal{F}) \leq \text{Cov}(S, \mathcal{F})
\]

- we will add one more parameter:

the **circular covering number** \(\text{Cov}_c(S, \mathcal{F})\)
The circular covering number

- map the elements of $S$ to a circle so that:
  - for every unit interval $[x, x + 1)$ along the circle
    elements mapped into that interval form a set from $F$

- circular covering number $\text{Cov}_c(S, F)$:
  minimum circumference of a circle for which this is possible
Let's put it in the right place - I

- for a nice set system: \( \text{Cov}_c(S, \mathcal{F}) \leq \text{Cov}(S, \mathcal{F}) \)

- take a disjoint cover \( F_1, \ldots, F_k \) of \( (S, \mathcal{F}) \)

- put the elements of each \( F_i \) together at unit distance around a circle with circumference \( k \):

  - gives a circular cover with circumference \( k \)
Let’s put in in the right place - II

- for a nice set system: \( \text{Cov}_f(S, \mathcal{F}) \leq \text{Cov}_c(S, \mathcal{F}) \)

- take a **circular cover** along a circle

- “move” the unit interval with “unit speed” round the circle

- for a set \( F \) that appears in the interval at some point:
  
  denote by \( x_F \) the “length of time” it appears
Let's put in in the right place - II

- for a nice set system: $\text{Cov}_f(S, \mathcal{F}) \leq \text{Cov}_c(S, \mathcal{F})$

- take a circular cover along some circle

- for a set $F$ that appears in the interval at some point:
  - denote by $x_F$ the “length of time” it appears

- then for all $s \in S$: $\sum_{F \ni s} x_F = 1$

- and $\sum_{F \in \mathcal{F}} x_F =$ circumference

- this gives a fractional cover with value the circumference
Inequalities, inequalities, and more inequalities

so now we know:

\[ \text{Pack} \leq \text{Pack}_f = \text{Cov}_f \leq \text{Cov}_c \leq \text{Cov} \]

can we say for which nice set systems we have equality for one of the inequalities?

probably too hard

what about those that satisfy an equality “through and through”?

\[ \text{Pack} \leq \frac{\text{Cov}_f}{\text{Pack}_f} \leq \text{Cov}_c \leq \text{Cov} \]
Through and through = induced

- \((S, \mathcal{F})\) a nice set system and \(T \subseteq S\), then define:

\[
\mathcal{F}_T = \{ F \cap T \mid F \in \mathcal{F} \} = \{ F \in \mathcal{F} \mid F \subseteq T \}
\]

- \((T, \mathcal{F}_T)\) is again a nice set system
  - called an induced set system

- for a graph \(G\) with \(U \subseteq V_G\):
  - \((S_G)_U\) are the stable sets of the subgraph induced by \(U\)

\[
\text{Pack} \leq \text{Cov}_f \leq \text{Cov}_c \leq \text{Cov}
\]
a nice set system is \((A = B)\)-perfect:

- the system and all its induced systems satisfy \(A = B\)

- note that we have six degrees of perfectness

- by definition, perfect graphs are exactly those graphs \(G\) for which \((V_G, \mathcal{S}_G)\) is \((\text{Pack} = \text{Cov})\)-perfect

- that makes them perfect for all inequalities!

\[
\text{Pack} \leq \frac{\text{Cov}}{\text{Pack}} \leq \frac{\text{Cov}}{\text{Pack}} \leq \text{Cov}
\]
What about the other set systems?

- we know non-perfect graphs very well:

**Strong Perfect Graph Theorem**

- $G$ not a perfect graph $\iff$ $G$ contains an induced copy:
  - of an odd cycle $C_{2k+1}$, $k \geq 2$, or
  - of the complement $\overline{C_{2k+1}}$ of an odd cycle, $k \geq 2$

$\text{Pack} \leq \frac{\text{Cov}_f}{\text{Pack}_f} \leq \frac{\text{Cov}_c}{\text{Cov}}$
What about other “graphical” set systems?

■ for an odd cycle $C_{2k+1}$, $k \geq 2$, it’s easy to check:

- $\text{Pack}(V_{C_{2k+1}}, S_{C_{2k+1}}) = 2$
- $\text{Cov}_f(V_{C_{2k+1}}, S_{C_{2k+1}}) = \text{Cov}_c(V_{C_{2k+1}}, S_{C_{2k+1}}) = 2 + \frac{1}{k}$
- $\text{Cov}(V_{C_{2k+1}}, S_{C_{2k+1}}) = 3$

■ similar things happen for

the complement $\overline{C_{2k+1}}$ of an odd cycle, $k \geq 2$

$\text{Pack} \leq \frac{\text{Cov}_f}{\text{Pack}_f} \leq \text{Cov}_c \leq \text{Cov}$
Perfect graphs are very perfect

so:

- a nice set system of the form \((V_G, S_G)\) is
  - (\(\text{Pack} = \text{Cov}_f\))-perfect, or (\(\text{Pack} = \text{Cov}_c\))-perfect, or
  - (\(\text{Pack} = \text{Cov}\))-perfect, or (\(\text{Cov}_f = \text{Cov}\))-perfect, or
  - (\(\text{Cov}_c = \text{Cov}\))-perfect

\[ \iff \quad G \text{ is perfect} \]

problem:

- prove this for (\(\text{Cov}_c = \text{Cov}\))-perfectness,

\[ \text{without using the Strong Perfect Graph Theorem} \]

\[ \text{Pack} \leq \text{Cov}_f \leq \text{Cov}_c \leq \text{Cov} \]
And what about non-graphical set systems?

- Suppose \((S, \mathcal{F})\) is a nice set system such that
  - all minimal sets outside \(\mathcal{F}\) have size 2
    (smaller than 2 is not possible, as \(\mathcal{F}\) covers \(S\))

- Then form the graph \(G\) with \(V_G = S\) by setting
  \[s_1 s_2 \in E_G \iff \{s_1, s_2\} \notin \mathcal{F}\]

- Easy to check: \((S, \mathcal{F}) = (V_G, S_G)\)

\[
\text{Pack} \leq \text{Pack}_f \leq \text{Cov}_c \leq \text{Cov}
\]
That’s that about non-graphical set systems!

- \((S, \mathcal{F})\) is a non-graphical nice set system \(\iff\)
  - there is a subset \(T \subseteq S\) with \(|T| = k \geq 3\) so that:
    - \(T \notin \mathcal{F}\)
    - but every proper subset of \(T\) is in \(\mathcal{F}\)

- for such a \(T\), the induced set system \((T, \mathcal{F}_T)\) satisfies:
  - \(\text{Pack}(T, \mathcal{F}_T) = 1\)
  - \(\text{Cov}_f(T, \mathcal{F}_T) = \text{Cov}_c(T, \mathcal{F}_T) = 1 + \frac{1}{k - 1}\)
  - \(\text{Cov}(T, \mathcal{F}_T) = 2\)

\[
\text{Pack} \leq \frac{\text{Cov}_f}{\text{Pack}_f} \leq \text{Cov}_c \leq \text{Cov}
\]
Perfect graphs are really, really perfect!

so:

- a nice nice set system \((S, \mathcal{F})\) is
  - \((\text{Pack} = \text{Cov}_f)\)-perfect, or \((\text{Pack} = \text{Cov}_c)\)-perfect, or
  - \((\text{Pack} = \text{Cov})\)-perfect, or \((\text{Cov}_f = \text{Cov})\)-perfect, or
  - \((\text{Cov}_c = \text{Cov})\)-perfect

\[
\iff
\]

\[(S, \mathcal{F}) = (V_G, S_G)\] for some perfect graph \(G\)

\[
\text{Pack} \leq \frac{\text{Cov}_f}{\text{Pack}_f} \leq \text{Cov}_c \leq \text{Cov}
\]
The bit that’s left to do

- what nice set systems \((S, \mathcal{F})\) are \((\text{Cov}_f = \text{Cov}_c)\)-perfect?

- well ... 

  - stable sets of perfect graphs 
  - stable sets of odd cycles or complements of odd cycles 
  - loopless matroids \((\text{vdH & Thomassé})\) 
  - and a lot more 

\[
\text{Cov}_f \leq \text{Cov}_c
\]
What the **** is a loopless matroid?

- A set system \((S, \mathcal{F})\) is a **loopless matroid** if
  - \((S, \mathcal{F})\) is nice
  - For each \(F_1, F_2 \in \mathcal{F}\) with \(|F_1| > |F_2|\):
    there is an \(s \in F_1 \setminus F_2\) so that \(F_2 \cup \{s\} \in \mathcal{F}\)

By the way:

- A stable set system \((V_G, \mathcal{S}_G)\) is a loopless matroid
  \(\iff\) \(G\) is the disjoint union of cliques

\[\text{Cov}_f \leq \text{Cov}_c\]
This looks like it’s going to be complicated

- so nice set systems that are \((\text{Cov}_f = \text{Cov}_c)\)-perfect include
  - stable sets of perfect graphs
  - stable sets of odd cycles or complements of odd cycles
  - loopless matroids
  - disjoint unions of the above
  - and probably a lot more . . .

question:
- can we characterise \((\text{Cov}_f = \text{Cov}_c)\)-perfect set systems?

\[
\text{Cov}_f \leq \text{Cov}_c
\]