Graph Colouring with Distances

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Queen Mary, London, 19 May 2010

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The basics of graph colouring

- **vertex-colouring** with k colours :
  adjacent vertices must receive different colours

- **chromatic number** $\chi(G)$:
  minimum k so that a vertex-colouring exists

- **list-colouring** : as vertex-colouring,
  but each vertex v has its own list $L(v)$ of colours

- **choice number** $\mathrm{ch}(G)$:
  minimum k so that if all $|L(v)| \geq k$,
  then a proper list vertex-colouring exists

Another way to look at vertex-colouring

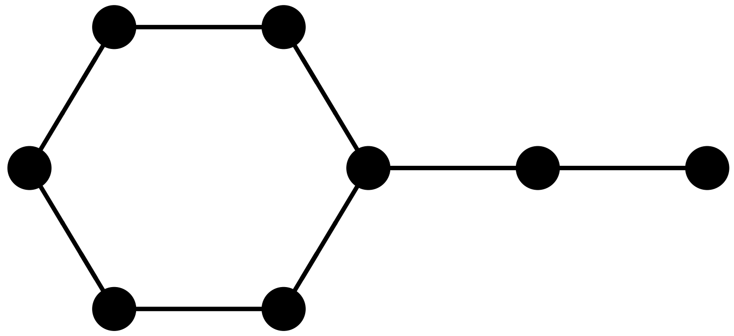
- vertex-colouring :

  vertices at distance one must receive different colours

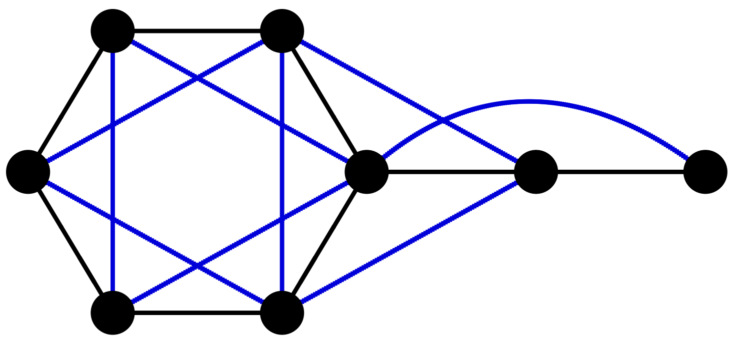
- now suppose we want vertices at larger distances
  (say, up to distance D) to receive different colours as well

- can be modelled using the **D -th power G^D of a graph** :
  - same vertex set as G
  - edges between vertices with distance at most D in G

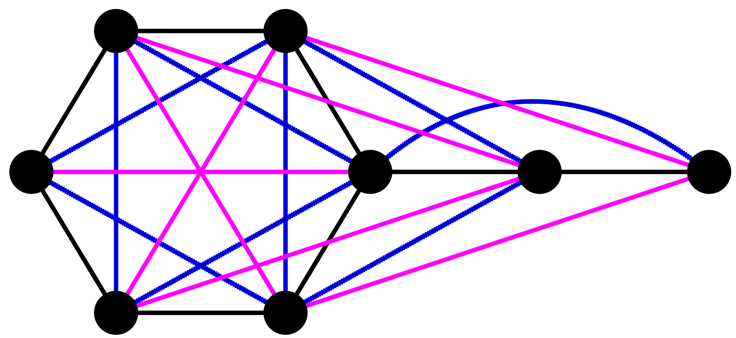
Powers of a graph



G



G^2



G^3

A first conjecture / problem

- powers of graphs seem to have more structure than graphs in general

<u>**List-Square-Colouring Conjecture**</u> (Kostochka & Woodall, 2001)

- for any graph G : $\quad \mathrm{ch}(G^2) = \chi(G^2)$

- if true, then $\mathrm{ch}(G^D) = \chi(G^D)$ for all even D

  (since $G^{2d} = (G^d)^2$)

- what about $\mathrm{ch}(G^D) = \chi(G^D)$ for odd D ?

Colouring powers of a graph

<u>**easy fact**</u>

- $$\Delta(G^D) \le \sum_{i=0}^{D-1} \Delta(G)\,(\Delta(G)-1)^i = O(\Delta(G)^D)$$

  ($\Delta = \Delta(G)$: maximum degree of G)

  - **so** : $\chi(G^D) \le O(\Delta(G)^D)$

- but for very few graphs you would expect to need that many colours

The square of planar graphs

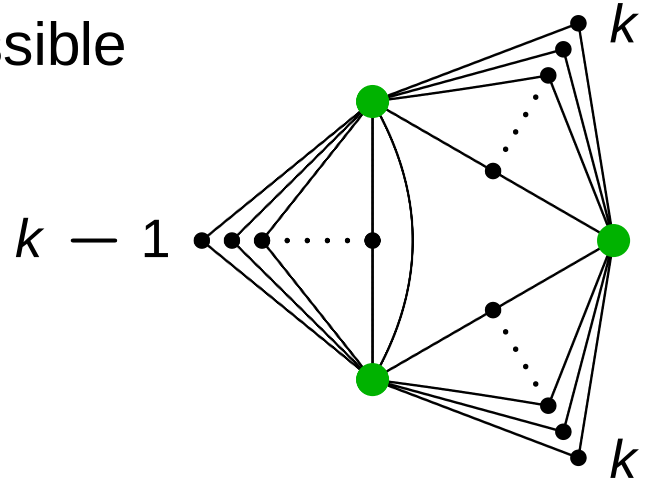
<u>**Conjecture**</u> (Wegner, 1977)

- G planar

$$\Longrightarrow \quad \chi(G^2) \le \begin{cases} 7, & \text{if } \Delta = 3 \\ \Delta + 5, & \text{if } 4 \le \Delta \le 7 \\ \lfloor {}^3\!/_2\, \Delta \rfloor + 1, & \text{if } \Delta \ge 8 \end{cases}$$

- bounds would be best possible

case $\Delta = 2k \ge 8$:



Towards Wegner's Conjecture

G planar $\Longrightarrow$

- $\chi(G^2) \leq 8\,\Delta - 22$ (Jonas, PhD, 1993)
- $\chi(G^2) \leq 3\,\Delta + 5$ (Wong, MSc, 1996)
- $\chi(G^2) \leq 2\,\Delta + 25$ (vdH & McGuinness, 2003)
- $\chi(G^2) \leq {}^{9}\!/_{5}\,\Delta + 1$ (for $\Delta \geq 47$)
  (Borodin, Broersma, Glebov & vdH, 2001)
- $\chi(G^2) \leq {}^{5}\!/_{3}\,\Delta + 24$ (for $\Delta \geq 241$)
  (Molloy & Salavatipour, 2005)

Towards Wegner's Conjecture

Theorem (Havet, vdH, McDiarmid & Reed, 2008+)

- G planar $\Longrightarrow$ $\chi(G^2) \le \left({}^3/_2 + o(1)\right)\Delta$ $(\Delta \to \infty)$

- - we actually prove the list-colouring version
  - and for much larger classes of graphs :

Theorem

- G graph, $K_{3,k}$ -minor free for some fixed k

$$\Longrightarrow \quad \mathrm{ch}(G^2) \le \left({}^3/_2 + o(1)\right)\Delta$$

What about larger D ?

Theorem (Agnarsson & Halldórsson, 2003)

- G planar $\implies$ $\chi(G^D) \leq c_D \, \Delta^{\lfloor D/2 \rfloor}$
- best possible : take Δ -regular tree with radius $\left\lfloor \frac{1}{2} D \right\rfloor$

What about larger D ?

Theorem (Agnarsson & Halldórsson, 2003)

- G planar $\Longrightarrow$ $\chi(G^D) \leq c_D\, \Delta^{\lfloor D/2 \rfloor}$

in fact, their proof gives something much more general:

Theorem

- G k-degenerate $\Longrightarrow$ $\chi(G^D) \leq c_{k,D}\, \Delta^{\lfloor D/2 \rfloor}$

- G is ***k*-degenerate**: every subgraph of G has a vertex of degree at most k

Main ideas of a simple proof

- G is ***m*-orientable**: G has an orientation in which every vertex has outdegree at most m

  - G is k-degenerate $\implies$ G is k-orientable
  - G is m-orientable $\implies$ G is $2m$-degenerate $\implies$ $\chi(G) \leq 2m+1$

Theorem

- G is m-orientable $\implies$ G^D is $c_{m,D}\,\Delta^{\lfloor D/2 \rfloor}$-orientable

Main ideas of a simple proof

- fix an orientation $\vec{G}$ of G with maximum outdegree m, and fix $D \geq 1$
- let uv be an edge in G^D
  - so there is uv-path $u = x_0, x_1, \ldots, x_\ell = v$ of length $\ell \leq D$
- orient uv in G^D according to the majority of the orientation of the edges in that uv-path (when going from u to v) (arbitrarily if a tie)

Main ideas of a simple proof

- so outdegree in oriented G^D of a vertex u is at most :
  - the number of uv -paths of length $\ell \leq D$ in G with at least $\left\lceil \frac{1}{2}\,\ell \right\rceil$ edges oriented $x_i \longrightarrow x_{i+1}$ in $\vec{G}$
- and the number of such paths is at most :

$$\sum_{\ell=1}^{D} \sum_{i=\lceil \ell/2 \rceil}^{\ell} \binom{\ell}{i} \cdot m^{i} \cdot \Delta^{\ell-i}$$

$$= \sum_{\ell=1}^{D} \sum_{j=0}^{\lfloor \ell/2 \rfloor} \binom{\ell}{j} \cdot m^{\ell-j} \cdot \Delta^{j} \leq c_{m,D}\,\Delta^{\lfloor D/2 \rfloor}$$

Colouring the cube of planar graphs

- so now we know there is some constant c_3 so that:

  G planar $\implies \chi(G^3) \leq c_3\,\Delta + O(1)$

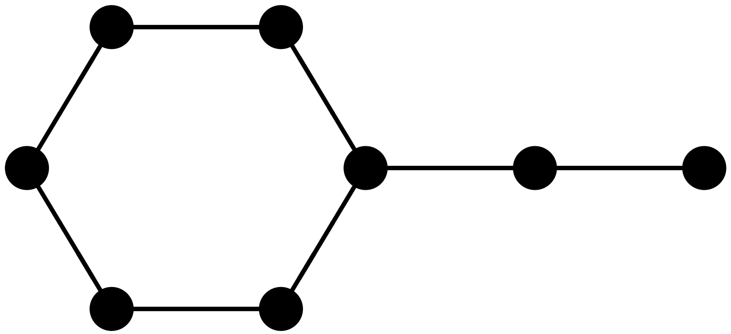
  - but what is the best c_3 ?
  - we only know: $4 \leq c_3 \leq 68$

- and what about distance $D > 3$?

A variant with exact distances

- suppose we only want vertices at distance exactly D to have different colours
- can be modelled using the **exact distance graph** $G^{=D}$ of G :
  - same vertex set as G
  - edges between vertices with distance exactly D in G

Exact distance graphs



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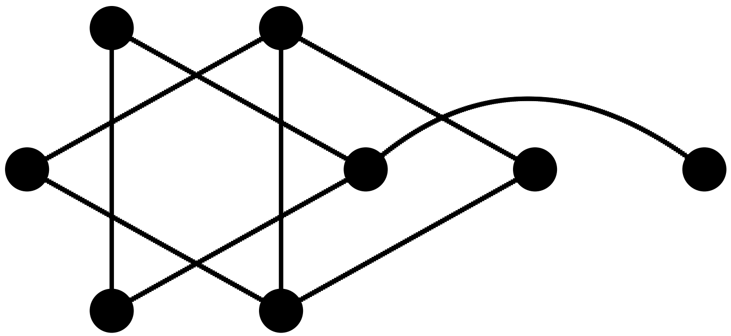
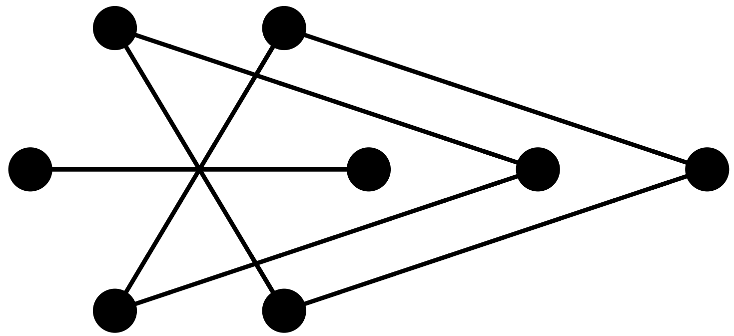


$G^{=2}$



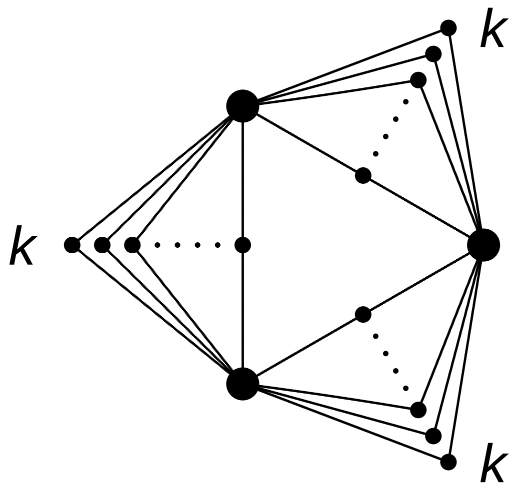
$G^{=3}$

Colouring exact distance graphs of planar graphs

- obviously :

  G planar $\implies$ $\chi(G^{=D}) \leq \chi(G^{D}) \leq O(\Delta^{\lfloor D/2 \rfloor})$

- and for $D = 2$ we can have $\chi(G^{=2}) = {}^{3}\!/_{2}\,\Delta$:



- in fact, for all <u>even</u> D , the bound seems to be ${}^{3}\!/_{2}\,\Delta$

Colouring exact distance graphs

- for odd D , the situation is very different :

Theorem (Nešetřil & Ossona de Mendez, 2008)

- $\mathcal{K}$ a graph class with "bounded expansion", D odd
  - then there exists a constant $c_{\mathcal{K},D}$ so that :

$$G \in \mathcal{K} \quad \Longrightarrow \quad \chi(G^{=D}) \ \leq \ c_{\mathcal{K},D}$$

- a proper minor-closed class is of bounded expansion
  - hence planar graphs are of bounded expansion

Colouring exact distance graphs

- the result is best possible in many senses :
  - not true for even D
  - not true for k-degenerate graphs :
    - consider $S_{n,D}$: complete graph K_n
      with edges replaced by paths of length D
    - $S_{n,D}$ is 2-degenerate, but $\chi((S_{n,D})^{=D}) = n$
  - not true if "u, v have distance exactly D"
    replaced by "there is a uv-path of length D"
    - consider wheel W_n with n spokes

The exact cube of planar graphs

- so now we know : G planar $\Longrightarrow$ $\chi(G^{=3}) \le c'_3$
  - short proof ?
  - what can we say about c'_3 ?
- more general : what can we say about the structure of $G^{=3}$ for planar G ?
  - does not contain K_5 as a subgraph
  - can contain any complete bipartite $K_{n,n}$ as a subgraph
    - hence arbitrarily large K_n as a minor

List-colouring conjectures

- Kostochka & Woodall conjectured : $\mathrm{ch}(G^2) = \chi(G^2)$

Conjecture

- for any graph G : $\mathrm{ch}(G^{=2}) = \chi(G^{=2})$

Shortly after the talk, I realised that simple counterexamples to this conjecture exist.

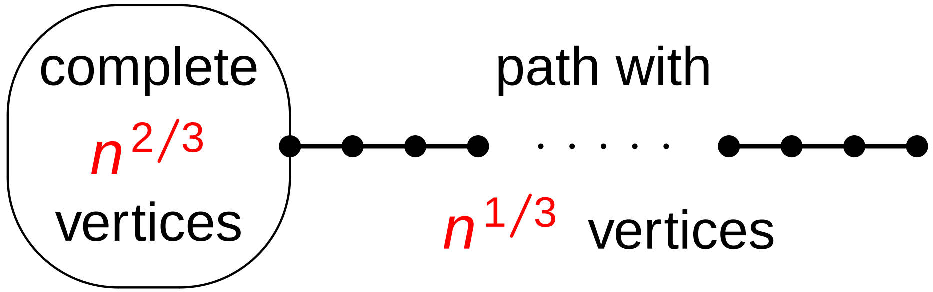
More fun with powers of graphs

- going from G to G^D, how many edges do we gain?
  - in particular: do we have $e(G^D) \geq (1 + \varepsilon_D)\, e(G)$ for some $\varepsilon_D > 0$?

- natural to assume:
  - G is connected
  - $\mathrm{diam}(G) \geq D$

More fun with powers of graphs

- do we have $e(G^D) \geq (1+\varepsilon_D)\,e(G)$ for some $\varepsilon_D > 0$?
- connected and large diameter is still not enough :
  - take G :



  - then : $e(G) \approx \binom{n^{2/3}}{2} = \Theta(n^{4/3})$
  - while G^D gains only about $D\,n = O(n)$ new edges
- so let's require regular as well

More fun with powers of graphs

Theorem (Hegarty, 2009)

- for $D \geq 3$, there exist $\varepsilon_D > 0$ so that:
  G connected, regular, and $\mathrm{diam}(G) \geq D$
  $$\Longrightarrow \quad e(G^D) \geq (1 + \varepsilon_D)\, e(G)$$
- for $D = 2$, no such ε_2 exists (Hegarty)
- $\varepsilon_3 \geq 0.087$ (Hegarty), $\varepsilon_3 \geq 1/6$ (Pokrovskiy, 2010)
- for $D \geq 4$: $\varepsilon_D \geq \left\lceil \frac{1}{3} D \right\rceil - 2$ (Pokrovskiy, 2010)
  - can't be increased to αD for $\alpha > \frac{1}{3}$

One final problem

- **given** : graph G
  question : is there an H so that $G = L(H)$?
  - can be done in polynomial time

- **given** : graph G
  question : is there an H so that $G = H^2$?
  - is NP-complete (Motwani & Rajeev, 1994)

- what about checking if there is an H so that $G = H^{=2}$?

I have no found an argument that shows
this can be done in polynomial time.