Graph Colouring with Distances

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The basics of graph colouring

- **vertex-colouring** with \( k \) colours:
  adjacent vertices must receive different colours

- **chromatic number** \( \chi(G) \):
  minimum \( k \) so that a vertex-colouring exists

- **list-colouring**:
  as vertex-colouring,
  but each vertex \( v \) has its own list \( L(v) \) of colours

- **choice number** \( \text{ch}(G) \):
  minimum \( k \) so that if all \( |L(v)| \geq k \),
  then a proper list vertex-colouring exists
Another way to look at vertex-colouring

- vertex-colouring:
  vertices at distance one must receive different colours

- now suppose we want vertices at larger distances (say, up to distance $D$) to receive different colours as well

- can be modelled using the $D$-th power $G^D$ of a graph:
  - same vertex set as $G$
  - edges between vertices with distance at most $D$ in $G$
Powers of a graph

$G$

$G^2$

$G^3$
A first conjecture/problem

- powers of graphs seem to have more structure than graphs in general

List-Square-Colouring Conjecture (Kostochka & Woodall, 2001)

- for any graph $G$: $\text{ch}(G^2) = \chi(G^2)$

- if true, then $\text{ch}(G^D) = \chi(G^D)$ for all even $D$
  (since $G^{2d} = (G^d)^2$)

- what about $\text{ch}(G^D) = \chi(G^D)$ for odd $D$?
Colouring powers of a graph

easy fact

\[ \Delta(G^D) \leq \sum_{i=0}^{D-1} \Delta(G) (\Delta(G) - 1)^i = O(\Delta(G)^D) \]

( \( \Delta = \Delta(G) \) : maximum degree of \( G \) )

so:

\[ \chi(G^D) \leq O(\Delta(G)^D) \]

but for very few graphs you would expect to need that many colours
The square of planar graphs

Conjecture (Wegner, 1977)

- $G$ planar

\[ \chi(G^2) \leq \begin{cases} 
7, & \text{if } \Delta = 3 \\
\Delta + 5, & \text{if } 4 \leq \Delta \leq 7 \\
\left\lfloor \frac{3}{2} \Delta \right\rfloor + 1, & \text{if } \Delta \geq 8
\end{cases} \]

- bounds would be best possible

case $\Delta = 2k \geq 8$:
Towards Wegner’s Conjecture

\[ G \text{ planar} \implies \]

- \[ \chi(G^2) \leq 8 \Delta - 22 \]  
  (Jonas, PhD, 1993)

- \[ \chi(G^2) \leq 3 \Delta + 5 \]  
  (Wong, MSc, 1996)

- \[ \chi(G^2) \leq 2 \Delta + 25 \]  
  (vdH & McGuinness, 2003)

- \[ \chi(G^2) \leq \frac{9}{5} \Delta + 1 \]  
  (for \( \Delta \geq 47 \))  
  (Borodin, Broersma, Glebov & vdH, 2001)

- \[ \chi(G^2) \leq \frac{5}{3} \Delta + 24 \]  
  (for \( \Delta \geq 241 \))  
  (Molloy & Salavatipour, 2005)
Towards Wegner’s Conjecture

**Theorem**  \((Havet, vdH, McDiarmid & Reed, 2008+)\)

- \(G\) planar \(\implies\) \(\chi(G^2) \leq \left(\frac{3}{2} + o(1)\right) \Delta\) (\(\Delta \to \infty\))

- we actually prove the list-colouring version
- and for much larger classes of graphs:

**Theorem**

- \(G\) graph, \(K_{3,k}\)-minor free for some fixed \(k\)

\[\implies\] \(\text{ch}(G^2) \leq \left(\frac{3}{2} + o(1)\right) \Delta\)
What about larger $D$?

**Theorem** (Agnarsson & Halldórsson, 2003)

- $G$ planar $\implies \chi(G^D) \leq c_D \Delta^{\lfloor D/2 \rfloor}$

- best possible: take $\Delta$-regular tree with radius $\lfloor \frac{1}{2} D \rfloor$
What about larger $D$?

**Theorem** (Agnarsson & Halldórsson, 2003)

- $G$ planar $\implies \chi(G^D) \leq c_D \Delta^{[D/2]}$

in fact, their proof gives something much more general:

**Theorem**

- $G$ $k$-degenerate $\implies \chi(G^D) \leq c_{k,D} \Delta^{[D/2]}$

- $G$ is $k$-degenerate: every subgraph of $G$

  has a vertex of degree at most $k$
Main ideas of a simple proof

- \( G \) is \( m \)-orientable: \( G \) has an orientation in which every vertex has outdegree at most \( m \)

- \( G \) is \( k \)-degenerate \( \implies \) \( G \) is \( k \)-orientable

- \( G \) is \( m \)-orientable \( \implies \) \( G \) is \( 2m \)-degenerate
  \( \implies \) \( \chi(G) \leq 2m + 1 \)

Theorem

- \( G \) is \( m \)-orientable \( \implies \) \( G^D \) is \( c_{m,D} \Delta^{[D/2]} \)-orientable
Main ideas of a simple proof

- fix an orientation $\vec{G}$ of $G$ with maximum outdegree $m$, and fix $D \geq 1$

- let $uv$ be an edge in $G^D$
  - so there is $uv$-path $u = x_0, x_1, \ldots, x_\ell = v$ of length $\ell \leq D$

- orient $uv$ in $G^D$ according to the majority of the orientation of the edges in that $uv$-path (when going from $u$ to $v$) (arbitrarily if a tie)
Main ideas of a simple proof

- so outdegree in oriented $G^D$ of a vertex $u$ is at most:
  - the number of $uv$-paths of length $\ell \leq D$ in $G$ with at least $\lceil \frac{1}{2} \ell \rceil$ edges oriented $x_i \rightarrow x_{i+1}$ in $\vec{G}$

- and the number of such paths is at most:

$$
\sum_{\ell=1}^{D} \sum_{i=\lfloor \ell/2 \rfloor}^{\ell} \binom{\ell}{i} \cdot m^i \cdot \Delta^{\ell-i} \\
= \sum_{\ell=1}^{D} \sum_{j=0}^{\lfloor \ell/2 \rfloor} \binom{\ell}{j} \cdot m^{\ell-j} \cdot \Delta^j \leq c_{m,D} \Delta^{\lfloor D/2 \rfloor}
$$
**Colouring the cube of planar graphs**

- so now we know there is some constant $c_3$ so that:
  
  \[ G \text{ planar} \implies \chi(G^3) \leq c_3 \Delta + O(1) \]

- but what is the best $c_3$?

- we only know: $4 \leq c_3 \leq 68$

- and what about distance $D > 3$?
A variant with exact distances

- suppose we only want vertices at distance exactly $D$ to have different colours

- can be modelled using the exact distance graph $G^{=D}$ of $G$:
  - same vertex set as $G$
  - edges between vertices with distance exactly $D$ in $G$
Exact distance graphs

$G$

$G=2$

$G=3$
Colouring exact distance graphs of planar graphs

- obviously:
  $$G \text{ planar} \implies \chi(G^{=D}) \leq \chi(G^D) \leq O(\Delta^{\left\lfloor \frac{D}{2} \right\rfloor})$$

- and for $$D = 2$$ we can have $$\chi(G^{=2}) = \frac{3}{2} \Delta$$:

- in fact, for all even $$D$$, the bound seems to be $$\frac{3}{2} \Delta$$
Colouring exact distance graphs

- for odd $D$, the situation is very different:

**Theorem**  (Nešetřil & Ossona de Mendez, 2008)

- $\mathcal{K}$ a graph class with “bounded expansion”, $D$ odd
  - then there exists a constant $c_{\mathcal{K},D}$ so that:
    \[ G \in \mathcal{K} \implies \chi(G^D) \leq c_{\mathcal{K},D} \]

- a proper minor-closed class is of bounded expansion
- hence planar graphs are of bounded expansion
Colouring exact distance graphs

- the result is best possible in many senses:
  - not true for even \( D \)
  - not true for \( k \)-degenerate graphs:
    - consider \( S_{n,D} \): complete graph \( K_n \) with edges replaced by paths of length \( D \)
    - \( S_{n,D} \) is 2-degenerate, but \( \chi((S_{n,D})=D) = n \)
  - not true if “\( u, v \) have distance exactly \( D \)” replaced by “there is a \( uv \)-path of length \( D \)”
    - consider wheel \( W_n \) with \( n \) spokes
The exact cube of planar graphs

- so now we know: \( G \) planar \( \implies \chi(G^{=3}) \leq c'_3 \)
  - short proof?
  - what can we say about \( c'_3 \) ?

- more general: what can we say about the structure of \( G^{=3} \) for planar \( G \) ?
  - does not contain \( K_5 \) as a subgraph
  - can contain any complete bipartite \( K_{n,n} \) as a subgraph
    - hence arbitrarily large \( K_n \) as a minor
Kostochka & Woodall conjectured: \( \text{ch}(G^2) = \chi(G^2) \)

**Conjecture**

for any graph \( G \): \( \text{ch}(G^{-2}) = \chi(G^{-2}) \)

Shortly after the talk, I realised that simple counterexamples to this conjecture exist.
More fun with powers of graphs

- going from $G$ to $G^D$, how many edges do we gain?
  - in particular: do we have $e(G^D) \geq (1 + \varepsilon_D) e(G)$ for some $\varepsilon_D > 0$?

- natural to assume:
  - $G$ is connected
  - $\text{diam}(G) \geq D$
More fun with powers of graphs

- do we have \( e(G^D) \geq (1 + \varepsilon_D) e(G) \) for some \( \varepsilon_D > 0 \)?

- connected and large diameter is still not enough:
  - take \( G \):
    - complete \( n^{2/3} \) vertices
    - path with \( n^{1/3} \) vertices
  - then: \( e(G) \approx \binom{n^{2/3}}{2} = \Theta(n^{4/3}) \)
  - while \( G^D \) gains only about \( Dn = O(n) \) new edges

- so let’s require regular as well
More fun with powers of graphs

Theorem (Hegarty, 2009)

- For \( D \geq 3 \), there exist \( \varepsilon_D > 0 \) so that:
  
  \[
  G \text{ connected, regular, and } \text{diam}(G) \geq D
  \implies e(G^D) \geq (1 + \varepsilon_D) e(G)
  \]

- For \( D = 2 \), no such \( \varepsilon_2 \) exists (Hegarty)

- \( \varepsilon_3 \geq 0.087 \) (Hegarty), \( \varepsilon_3 \geq 1/6 \) (Pokrovskiy, 2010)

- For \( D \geq 4 \):
  
  \[
  \varepsilon_D \geq \left\lceil \frac{1}{3} D \right\rceil - 2
  \]

  Can’t be increased to \( \alpha D \) for \( \alpha > \frac{1}{3} \) (Pokrovskiy, 2010)
One final problem

- **given**: graph $G$
  - **question**: is there an $H$ so that $G = L(H)$?
  - can be done in polynomial time

- **given**: graph $G$
  - **question**: is there an $H$ so that $G = H^2$?
  - is **NP-complete** (Motwani & Rajeev, 1994)

- what about checking if there is an $H$ so that $G = H^{-2}$?

  I have no found an argument that shows this can be done in polynomial time.