Degrees of Perfection

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PCC, 9 July 2010

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Some good terminology

- set system \((S, \mathcal{F})\) : a finite set \(S\) with a collection \(\mathcal{F}\) of subsets of \(S\)

- a set system is “good” if :
  - \(\mathcal{F}\) is closed under taking subsets, and
  - \(\mathcal{F}\) covers all of \(S\)
    
    \((\text{for all } s \in S \text{ there is an } F \in \mathcal{F} \text{ with } s \in F)\)
Two important examples

- $G = (V_G, E_G)$ a graph
  - take $\mathcal{S}_G$ the collection of all stable sets (sets containing no adjacent pairs of vertices)
  - then $(V_G, \mathcal{S}_G)$ is a good set system

- $V$ a vector space, and $U$ a subset of $V \setminus \{0\}$
  - take $\mathcal{I}_U$ the collection of all linearly independent subsets of $U$
  - then $(U, \mathcal{I}_U)$ is a good set system
Coverings

- **a covering** of \((S, \mathcal{F})\) :
  a collection of sets from \(\mathcal{F}\) whose union is \(S\)

- **covering number** \(\text{Cov}(S, \mathcal{F})\) :
  the *minimum* number of elements in a covering

- for a **graph** \(G\) : \(\text{Cov}(V_G, \mathcal{S}_G)\) is the *minimum* number of stable sets needed to cover all vertices

  - so \(\text{Cov}(V_G, \mathcal{S}_G)\) is just the **chromatic number**
Let's make life more complicated

the covering number is also the solution of the IP problem:

\[
\begin{align*}
\text{minimise} & \quad \sum_{F \in \mathcal{F}} x_F \\
\text{subject to} & \quad \sum_{F \ni s} x_F \geq 1, \quad \text{for all } s \in S \\
& \quad x_F \in \{0, 1\}, \quad \text{for all } F \in \mathcal{F}
\end{align*}
\]
The fractional version

removing the integrality condition:

\[
\text{minimise } \sum_{F \in \mathcal{F}} x_F \\
\text{subject to } \sum_{F \ni s} x_F \geq 1, \text{ for all } s \in S \\
x_F \geq 0, \text{ for all } F \in \mathcal{F}
\]

gives the fractional covering number \( \text{F-Cov}(S, \mathcal{F}) \)

and we obviously have: \( \text{F-Cov}(S, \mathcal{F}) \leq \text{Cov}(S, \mathcal{F}) \)
Rule 1 of Linear Programming: dualise

The dual LP problem of the fractional covering number is:

\[
\text{maximise } \sum_{s \in S} y_s \\
\text{subject to } \sum_{s \in F} y_s \leq 1, \text{ for all } F \in \mathcal{F} \\
y_s \geq 0, \text{ for all } s \in S
\]

This gives the fractional packing number \( \text{F-Pack}(S, \mathcal{F}) \)

And by LP-duality: \( \text{F-Pack}(S, \mathcal{F}) = \text{F-Cov}(S, \mathcal{F}) \)
And the integral version of that one

only allowing integers gives:

\[
\text{maximise } \sum_{s \in S} y_s \\
\text{subject to } \sum_{s \in F} y_s \leq 1, \quad \text{for all } F \in \mathcal{F} \\
y_s \in \{0, 1\}, \quad \text{for all } s \in S
\]

this gives the packing number \(\text{Pack}(S, \mathcal{F})\)

with the relation: \(\text{Pack}(S, \mathcal{F}) \leq \text{F-Pack}(S, \mathcal{F})\)
The packing number

we can interpret the packing number \( \text{Pack}(S, \mathcal{F}) \) as:

- the maximum size \( |T| \) of a subset \( T \subseteq S \) so that
  \[ |T \cap F| \leq 1, \text{ for all } F \in \mathcal{F} \]

- i.e.: the maximum size \( |T| \) of a subset \( T \subseteq S \) so that
  no two elements of \( T \) appear together in a set from \( \mathcal{F} \)

for a graph \( G \): \( \text{Pack}(V_G, S_G) \) is the maximum size of
a set of vertices with no two elements in a stable set

so \( \text{Pack}(V_G, S_G) \) is just the clique number
The status so far

- for any good set system \((S, \mathcal{F})\) we have

\[
\text{Pack}(S, \mathcal{F}) \leq \text{F-Pack}(S, \mathcal{F}) = \text{F-Cov}(S, \mathcal{F}) \leq \text{Cov}(S, \mathcal{F})
\]

- we will add one more parameter:

the \textbf{circular covering number} \text{C-Cov}(S, \mathcal{F})
The circular covering number

- map the elements of $S$ to a circle so that:
  - for every unit interval $[x, x + 1)$ along the circle, elements mapped into that interval form a set from $\mathcal{F}$

- circular covering number $C-Cov(S, \mathcal{F})$:
  minimum circumference of a circle for which this is possible
The right place for the circular covering number - I

- for a good set system: \( C \text{-Cov}(S, \mathcal{F}) \leq \text{Cov}(S, \mathcal{F}) \)

- take a disjoint cover \( F_1, \ldots, F_k \) of \( (S, \mathcal{F}) \)

- put the elements of each \( F_i \) together at unit distance around a circle with circumference \( k \):

  - all \( s \in F_1 \)
  - all \( s \in F_2 \)

- gives a circular cover with circumference \( k \)
for a good set system: \( F\text{-Cov}(S, \mathcal{F}) \leq C\text{-Cov}(S, \mathcal{F}) \)

- take a circular cover along a circle

- “move” the unit interval with “unit speed” round the circle

- for a set \( F \) that appears in the interval at some point:
  
  denote by \( x_F \) the “length of time” it appears
for a good set system: $F\text{-Cov}(S, \mathcal{F}) \leq C\text{-Cov}(S, \mathcal{F})$

- take a circular cover along some circle
- for a set $F$ that appears in the interval at some point:
  denote by $x_F$ the “length of time” it appears

- then for all $s \in S$: $\sum_{F \ni s} x_F = 1$

- and $\sum_{F \in \mathcal{F}} x_F = \text{circumference}$

- this gives a fractional cover with value the circumference
Inequalities, inequalities, and more inequalities

so now we know:

\[
\text{Pack} \leq \text{F-Pack} = \text{F-Cov} \leq \text{C-Cov} \leq \text{Cov}
\]

can we say for which good set systems we have equality for one of the inequalities?

- probably too hard ("too local")

what about those that satisfy an equality "through and through"?
Through and through = induced

- $(S, \mathcal{F})$ a good set system and $T \subseteq S$, then define:

$$\mathcal{F}_T = \{ F \cap T \mid F \in \mathcal{F} \} = \{ F \in \mathcal{F} \mid F \subseteq T \}$$

- then $(T, \mathcal{F}_T)$ is again a good set system
  - called an induced set system

- for a graph $G$ with $U \subseteq V_G$:

  $$(\mathcal{S}_G)_U$$ are the stable sets of the subgraph induced by $U$$
**Degrees of perfection**

- a good set system is \((A = B)\)-perfect:
  - the system and all its induced systems satisfy \(A = B\)

- note that we have **six** degrees of perfection

- by definition, **perfect graphs** are exactly those graphs \(G\) for which \((V_G, S_G)\) is \((\text{Pack} = \text{Cov})\)-perfect

  - that makes them **perfect** for all inequalities!

\[
\text{Pack} \leq \text{F-Cov} \leq \text{C-Cov} \leq \text{Cov}
\]
What about the other set systems?

- we know non-perfect graphs very well:

**Strong Perfect Graph Theorem** (Chudnovsky et al., 2006)

- $G$ not a perfect graph $\iff$ $G$ contains an induced copy:
  - of an odd cycle $C_{2k+1}$, $k \geq 2$, or
  - of the complement $\overline{C}_{2k+1}$ of an odd cycle, $k \geq 2$

Pack $\leq$ F-Cov $\leq$ C-Cov $\leq$ Cov
What about other “graphical” set systems?

- for an odd cycle $C_{2k+1}$, $k \geq 2$, it is easy to check:
  - $\text{Pack}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 2$
  - $\text{F-Cov}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = \text{C-Cov}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 2 + \frac{1}{k}$
  - $\text{Cov}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 3$

- similar things happen for the complement $\overline{C_{2k+1}}$ of an odd cycle, $k \geq 2$

$\text{Pack} \leq \text{F-Cov} \leq \text{C-Cov} \leq \text{Cov}$
Perfect graphs are very perfect

so:

- a good set system of the form \((V_G, S_G)\) is
  - (Pack = F-Cov) -perfect, or
  - (Pack = C-Cov) -perfect, or
  - (Pack = Cov) -perfect, or
  - (F-Cov = Cov) -perfect, or
  - (C-Cov = Cov) -perfect

\[\iff \quad G \text{ is perfect}\]

Pack \leq F-Cov \leq C-Cov \leq Cov
And what about non-graphical set systems?

- Suppose \((S, \mathcal{F})\) is a good set system such that
  - all minimal sets not in \(\mathcal{F}\) have size 2
    (smaller than 2 is not possible, as \(\mathcal{F}\) covers \(S\))

- Then form the graph \(G\) with \(V_G = S\) by setting
  \[s_1 s_2 \in E_G \iff \{s_1, s_2\} \notin \mathcal{F}\]

- Easy to check: \((S, \mathcal{F}) = (V_G, S_G)\)

\[\text{Pack} \leq \text{F-Cov} \leq \text{C-Cov} \leq \text{Cov}\]
And what about non-graphical set systems?

- \((S, \mathcal{F})\) is a non-graphical good set system if there is a subset \(T \subseteq S\) with \(|T| = k \geq 3\) so that:
  - \(T \notin \mathcal{F}\)
  - but every proper subset of \(T\) is in \(\mathcal{F}\)

For such a \(T\), the induced set system \((T, \mathcal{F}_T)\) satisfies:

- \(\text{Pack}(T, \mathcal{F}_T) = 1\)
- \(\text{F-Cov}(T, \mathcal{F}_T) = \text{C-Cov}(T, \mathcal{F}_T) = 1 + \frac{1}{k - 1}\)
- \(\text{Cov}(T, \mathcal{F}_T) = 2\)

\(\text{Pack} \leq \text{F-Cov} \leq \text{C-Cov} \leq \text{Cov}\)
Perfect graphs are really, really perfect!

so:

- a good set system \((S, \mathcal{F})\) is
  - \((\text{Pack} = \text{F-Cov})\)-perfect, or
  - \((\text{Pack} = \text{C-Cov})\)-perfect, or
  - \((\text{Pack} = \text{Cov})\)-perfect, or
  - \((\text{F-Cov} = \text{Cov})\)-perfect, or
  - \((\text{C-Cov} = \text{Cov})\)-perfect

\[\iff (S, \mathcal{F}) = (V_G, S_G)\] for some perfect graph \(G\)

\(\text{Pack} \leq \text{F-Cov} \leq \text{C-Cov} \leq \text{Cov}\)
All that is left to do . . .

what good set systems \((S, \mathcal{F})\) are
\((F\text{-Cov} = C\text{-Cov})\)-perfect?

well . . .

- stable sets of perfect graphs
- stable sets of odd cycles or complements of odd cycles
- loopless matroids \(\text{(vdH & Thomassé)}\)
- and a lot more . . .

\[ F\text{-Cov} \leq C\text{-Cov} \]
What the **** is a loopless matroid?

- A set system \((S, \mathcal{F})\) is a **loopless matroid** if
  - \((S, \mathcal{F})\) is good
  - For each \(F_1, F_2 \in \mathcal{F}\) with \(|F_1| > |F_2|\):
    - There is an \(s \in F_1 \setminus F_2\) so that \(F_2 \cup \{s\} \in \mathcal{F}\)

**Example**

- Take \(V\) a vector space and \(U\) a subset of \(V \setminus \{0\}\)
  - Then \((U, \mathcal{I}_U)\) is a loopless matroid
  - So: \(\text{F-Cov}(U, \mathcal{I}_U) = \text{C-Cov}(U, \mathcal{I}_U)\)

\(\text{F-Cov} \leq \text{C-Cov}\)
The “remaining” case

- good set systems that are \((F-\text{Cov} = C-\text{Cov})\)-perfect:
  - stable sets of perfect graphs
  - stable sets of odd cycles or complements of odd cycles
  - loopless matroids
  - disjoint unions of the above
  - and probably a lot more . . .

\[ F-\text{Cov} \leq C-\text{Cov} \]
What’s the difference?

- stable set systems and loopless matroids are very different animals:

  - a set system \((S, \mathcal{F})\) is both a stable set system and a loopless matroid

  \[ \iff \]

  \[(S, \mathcal{F}) = (V_G, S_G) \text{ with } G \text{ a disjoint union of cliques}\]

\[ F-\text{Cov} \leq C-\text{Cov} \]
**The “remaining” case**

**questions:**

- can we characterise

  \((F\text{-Cov} = C\text{-Cov})\)-perfect set systems?

- or at least the graphs \(G\) for which \((V_G, S_G)\) is

  \((F\text{-Cov} = C\text{-Cov})\)-perfect?

- what “natural” class of set systems

  contains both matroids and stable sets of perfect graphs?

\[F\text{-Cov} \leq C\text{-Cov}\]
And another open problem

- the Strong Perfect Graph Theorem “easily” gives:
  a good set system of the form \((V_G, S_G)\) is
    - \((\text{Pack} = \text{F-Cov})\)-perfect,
    - \((\text{Pack} = \text{C-Cov})\)-perfect,
    - \((\text{Pack} = \text{Cov})\)-perfect,
    - \((\text{F-Cov} = \text{Cov})\)-perfect, or
    - \((\text{C-Cov} = \text{Cov})\)-perfect
  \[\iff G \text{ is perfect}\]

- all cases but one were known before the SPGT

- find a proof without using the SPGT that
  non-perfect graphs are not \((\text{C-Cov} = \text{Cov})\)-perfect