Degrees of Perfection

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Some good terminology

- **set system** $(S, \mathcal{F})$: a finite set S
  with a collection $\mathcal{F}$ of subsets of S

- a set system is "**good**" if :
  - $\mathcal{F}$ is closed under taking subsets, and
  - $\mathcal{F}$ covers all of S
    (for all $s \in S$ there is an $F \in \mathcal{F}$ with $s \in F$)

Two important examples

- $G = (V_G, E_G)$ a graph
  - take $\mathcal{S}_G$ the collection of all stable sets
    (sets containing no adjacent pairs of vertices)
  - then $(V_G, \mathcal{S}_G)$ is a good set system

- V a vector space, and U a subset of $V \setminus \{\mathbf{0}\}$
  - take $\mathcal{I}_U$ the collection of all
    linearly independent subsets of U
  - then $(U, \mathcal{I}_U)$ is a good set system

Coverings

- a **covering** of $(S, \mathcal{F})$:

  a collection of sets from $\mathcal{F}$ whose union is S

- **covering number Cov**$(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$:

  the minimum number of elements in a covering

- for a graph G : $\mathrm{Cov}(V_G, \mathcal{S}_G)$ is the minimum number of stable sets needed to cover all vertices

  - so $\mathrm{Cov}(V_G, \mathcal{S}_G)$ is just the **chromatic number**

Let's make life more complicated

- the covering number is also the solution of the IP problem :

$$
\begin{array}{lll}
\text{minimise} & \displaystyle\sum_{F \in \mathcal{F}} x_F & \\
\text{subject to} & \displaystyle\sum_{F \ni s} x_F \geq 1, & \text{for all } s \in S \\
& x_F \in \{0, 1\}, & \text{for all } F \in \mathcal{F}
\end{array}
$$

The fractional version

- removing the integrality condition :

$$\text{minimise} \quad \sum_{F \in \mathcal{F}} x_F$$

$$\text{subject to} \quad \sum_{F \ni s} x_F \geq 1, \qquad \text{for all } s \in S$$

$$x_F \geq 0, \qquad \text{for all } F \in \mathcal{F}$$

- gives the **fractional covering number** **F-Cov**$(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$
  - and we obviously have : F-Cov$(S, \mathcal{F}) \leq$ Cov$(S, \mathcal{F})$

Rule 1 of Linear Programming : dualise

- the dual LP problem of the fractional covering number is :

$$\text{maximise} \quad \sum_{s \in S} y_s$$

$$\text{subject to} \quad \sum_{s \in F} y_s \leq 1, \qquad \text{for all } F \in \mathcal{F}$$

$$y_s \geq 0, \qquad \text{for all } s \in S$$

- this gives the **fractional packing number** **F-Pack**$(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$
  - and by LP-duality : $\text{F-Pack}(S, \mathcal{F}) = \text{F-Cov}(S, \mathcal{F})$

And the integral version of that one

- only allowing integers gives :

$$
\begin{array}{lll}
\text{maximise} & \displaystyle\sum_{s \in S} y_s & \\
\text{subject to} & \displaystyle\sum_{s \in F} y_s \leq 1, & \text{for all } F \in \mathcal{F} \\
 & y_s \in \{0, 1\}, & \text{for all } s \in S
\end{array}
$$

- this gives the **packing number** $\mathbf{Pack}(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$
  - with the relation : $\text{Pack}(S, \mathcal{F}) \leq \text{F-Pack}(S, \mathcal{F})$

The packing number

- we can interpret the **packing number** $\mathbf{Pack}(\boldsymbol{S}, \boldsymbol{\mathcal{F}})$ as:
  - the maximum size $|T|$ of a subset $T \subseteq S$ so that $|T \cap F| \leq 1$, for all $F \in \mathcal{F}$
  - i.e.: the maximum size $|T|$ of a subset $T \subseteq S$ so that no two elements of T appear together in a set from $\mathcal{F}$
- for a graph G: $\mathrm{Pack}(V_G, \mathcal{S}_G)$ is the maximum size of a set of vertices with no two elements in a stable set
  - so $\mathrm{Pack}(V_G, \mathcal{S}_G)$ is just the **clique number**

The status so far

- for any good set system $(S, \mathcal{F})$ we have

$$\text{Pack}(S,\mathcal{F}) \;\leq\; \text{F-Pack}(S,\mathcal{F}) \;=\; \text{F-Cov}(S,\mathcal{F}) \;\leq\; \text{Cov}(S,\mathcal{F})$$

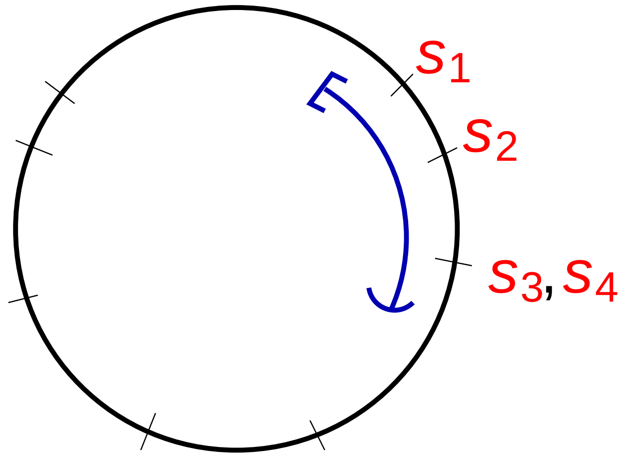
- we will add one more parameter :

  the **circular covering number** **C-Cov**$(S, \mathcal{F})$

The circular covering number

- map the elements of S to a circle so that :
  - for every unit interval $[x, x+1)$ along the circle

    elements mapped into that interval form a set from $\mathcal{F}$

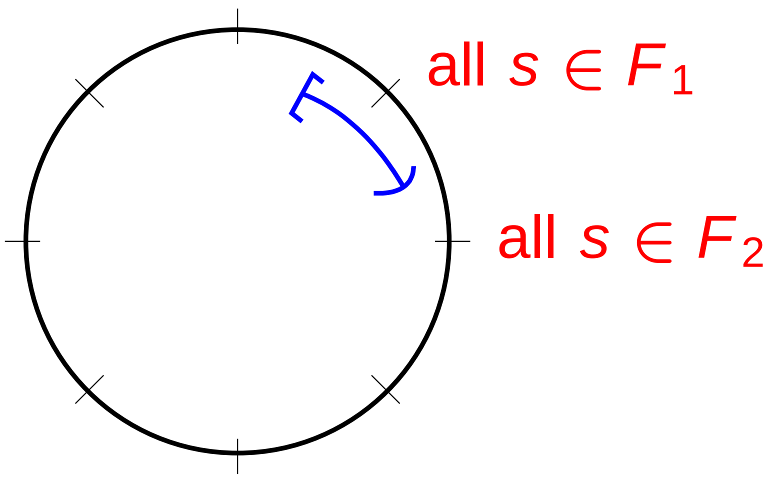


- **circular covering number** **C-Cov**$(S, \mathcal{F})$:

  minimum circumference of a circle for which this is possible

The right place for the circular covering number - I

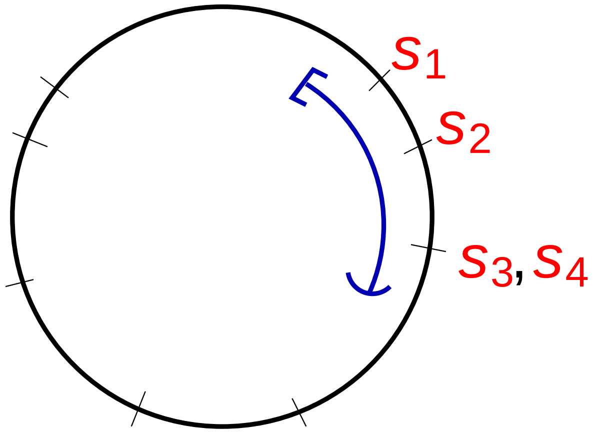
- for a good set system : $\text{C-Cov}(S, \mathcal{F}) \leq \text{Cov}(S, \mathcal{F})$
  - take a disjoint cover $F_1, \ldots, F_k$ of $(S, \mathcal{F})$
  - put the elements of each F_i together at unit distance around a circle with circumference k :



  - gives a circular cover with circumference k

The right place for the circular covering number - II

- for a good set system: $\text{F-Cov}(S, \mathcal{F}) \leq \text{C-Cov}(S, \mathcal{F})$
  - take a circular cover along a circle



  - "move" the unit interval with "unit speed" round the circle
  - for a set F that appears in the interval at some point:

    denote by x_F the "length of time" it appears

The right place for the circular covering number - II

- for a good set system: $\text{F-Cov}(S, \mathcal{F}) \leq \text{C-Cov}(S, \mathcal{F})$
  - take a circular cover along some circle
  - for a set F that appears in the interval at some point:

    denote by x_F the "length of time" it appears
  - then for all $s \in S$: $\sum_{F \ni s} x_F = 1$
  - and $\sum_{F \in \mathcal{F}} x_F = \text{circumference}$
  - this gives a fractional cover with value the circumference

Inequalities, inequalities, and more inequalities

- so now we know :

$$\text{Pack} \;\le\; \begin{array}{c}\text{F-Pack}\\ = \\ \text{F-Cov}\end{array} \;\le\; \text{C-Cov} \;\le\; \text{Cov}$$

- can we say for which good set systems we have equality for one of the inequalities ?
  - probably too hard (“too local”)
- what about those that satisfy an equality

  “through and through” ?

Through and through $=$ *induced*

- $(S, \mathcal{F})$ a good set system and $T \subseteq S$, then define:

$$\mathcal{F}_T = \{ F \cap T \mid F \in \mathcal{F} \} = \{ F \in \mathcal{F} \mid F \subseteq T \}$$

- then $(T, \mathcal{F}_T)$ is again a good set system
  - called an **induced** set system
- for a graph G with $U \subseteq V_G$:

  $(\mathcal{S}_G)_U$ are the stable sets of the subgraph induced by U

Degrees of perfection

- a good set system is $(\boldsymbol{A} = \boldsymbol{B})$**-perfect** :
  - the system and all its induced systems satisfy $A = B$

- note that we have **six** degrees of perfection

- by definition, perfect graphs are exactly those graphs G

  for which $(V_G, \mathcal{S}_G)$ is $(\text{Pack} = \text{Cov})$-perfect
  - that makes them perfect for all inequalities !

Pack $\leq$ F-Cov $\leq$ C-Cov $\leq$ Cov

What about the other set systems ?

- we know non-perfect graphs very well :

<u>**Strong Perfect Graph Theorem**</u> (Chudnovsky et al., 2006)

- G **not** a perfect graph $\iff$

  G contains an induced copy :

  - of an odd cycle C_{2k+1} , $k \geq 2$, or
  - of the complement $\overline{C_{2k+1}}$ of an odd cycle, $k \geq 2$

Pack $\leq$ F-Cov $\leq$ C-Cov $\leq$ Cov

What about other "graphical" set systems ?

- for an odd cycle C_{2k+1}, $k \geq 2$, it is easy to check :
  - $\mathrm{Pack}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 2$
  - $\text{F-Cov}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = \text{C-Cov}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 2 + \frac{1}{k}$
  - $\mathrm{Cov}(V_{C_{2k+1}}, \mathcal{S}_{C_{2k+1}}) = 3$

- similar things happen for

  the complement $\overline{C_{2k+1}}$ of an odd cycle, $k \geq 2$

Pack $\leq$ F-Cov $\leq$ C-Cov $\leq$ Cov

Perfect graphs are very perfect

so:

- a good set system of the form $(V_G, \mathcal{S}_G)$ is
  - (Pack $=$ F-Cov)-perfect, or
  - (Pack $=$ C-Cov)-perfect, or
  - (Pack $=$ Cov)-perfect, or
  - (F-Cov $=$ Cov)-perfect, or
  - (C-Cov $=$ Cov)-perfect

$\iff$ G is perfect

Pack $\leq$ F-Cov $\leq$ C-Cov $\leq$ Cov

And what about non-graphical set systems ?

- suppose $(S, \mathcal{F})$ is a good set system such that
  - all minimal sets not in $\mathcal{F}$ have size 2

    (smaller than 2 is not possible, as $\mathcal{F}$ covers S)

- then form the graph G with $V_G = S$ by setting

$$s_1 s_2 \in E_G \iff \{s_1, s_2\} \notin \mathcal{F}$$

- easy to check : $(S, \mathcal{F}) = (V_G, \mathcal{S}_G)$

And what about non-graphical set systems ?

- $(S, \mathcal{F})$ is a non-graphical good set system $\Longleftrightarrow$

  there is a subset $T \subseteq S$ with $|T| = k \geq 3$ so that :

  - $T \notin \mathcal{F}$
  - but every **proper** subset of T is in $\mathcal{F}$

- for such a T, the induced set system $(T, \mathcal{F}_T)$ satisfies :

  - $\text{Pack}(T, \mathcal{F}_T) = 1$
  - $\text{F-Cov}(T, \mathcal{F}_T) = \text{C-Cov}(T, \mathcal{F}_T) = 1 + \dfrac{1}{k-1}$
  - $\text{Cov}(T, \mathcal{F}_T) = 2$

Pack $\leq$ F-Cov $\leq$ C-Cov $\leq$ Cov

Perfect graphs are really, really perfect!

<u>**so**</u>:

- a good set system $(S, \mathcal{F})$ is
  - $(\text{Pack} = \text{F-Cov})$-perfect, or
  - $(\text{Pack} = \text{C-Cov})$-perfect, or
  - $(\text{Pack} = \text{Cov})$-perfect, or
  - $(\text{F-Cov} = \text{Cov})$-perfect, or
  - $(\text{C-Cov} = \text{Cov})$-perfect

$\iff$ $(S, \mathcal{F}) = (V_G, \mathcal{S}_G)$ for some perfect graph G

Pack $\leq$ F-Cov $\leq$ C-Cov $\leq$ Cov

All that is left to do . . .

- what good set systems $(S, \mathcal{F})$ are $(\text{F-Cov} = \text{C-Cov})$-perfect ?

- well . . .

  - stable sets of perfect graphs
  - stable sets of odd cycles or complements of odd cycles
  - loopless matroids (vdH & Thomassé)
  - and a lot more . . .

F-Cov $\leq$ C-Cov

What the $\ast\ast\ast\ast$ is a loopless matroid ?

- a set system $(S, \mathcal{F})$ is a **loopless matroid** if
  - $(S, \mathcal{F})$ is good
  - for each $F_1, F_2 \in \mathcal{F}$ with $|F_1| > |F_2|$:

    there is an $s \in F_1 \setminus F_2$ so that $F_2 \cup \{s\} \in \mathcal{F}$

example

- take V a vector space and U a subset of $V \setminus \{\mathbf{0}\}$
  - then $(U, \mathcal{I}_U)$ is a loopless matroid
  - so : $\text{F-Cov}(U, \mathcal{I}_U) = \text{C-Cov}(U, \mathcal{I}_U)$

F-Cov $\leq$ C-Cov

The "remaining" case

- good set systems that are (F-Cov $=$ C-Cov)-perfect:
  - stable sets of perfect graphs
  - stable sets of odd cycles or complements of odd cycles
  - loopless matroids
  - disjoint unions of the above
  - and probably a lot more . . .

F-Cov $\leq$ C-Cov

What's the difference ?

- stable set systems and loopless matroids are very different animals :
  - a set system $(S, \mathcal{F})$ is both

    a stable set system and a loopless matroid

    $\Longleftrightarrow$

    $(S, \mathcal{F}) = (V_G, \mathcal{S}_G)$ with G a disjoint union of cliques

The “remaining” case

questions :

- can we characterise

  $(\text{F-Cov} = \text{C-Cov})$ -perfect set systems ?

- or at least the graphs G for which $(V_G, \mathcal{S}_G)$ is

  $(\text{F-Cov} = \text{C-Cov})$ -perfect ?

- what “natural” class of set systems

  contains both matroids and stable sets of perfect graphs ?

And another open problem

- the Strong Perfect Graph Theorem "easily" gives:

  a good set system of the form $(V_G, \mathcal{S}_G)$ is

  - $(\text{Pack} = \text{F-Cov})$-perfect, $(\text{Pack} = \text{C-Cov})$-perfect, $(\text{Pack} = \text{Cov})$-perfect, $(\text{F-Cov} = \text{Cov})$-perfect, or $(\text{C-Cov} = \text{Cov})$-perfect

  $\iff$ G is perfect

- all cases but one were known before the SPGT

- find a proof without using the SPGT that

  non-perfect graphs are not $(\text{C-Cov} = \text{Cov})$-perfect