# Circular Arboricity of Graphs ( and Matroids )

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## *First : A more general problem*

- **set system** $(S, \mathcal{F})$ : a finite set $S$ with a collection $\mathcal{F}$ of subsets of $S$

- a set system is "**good**" if :
  - $\mathcal{F}$ is closed under taking subsets, and
  - $\mathcal{F}$ covers all of $S$ $\left( \bigcup_{F \in \mathcal{F}} F = S \right)$

- $(S, \mathcal{F})$ good $\implies$ for all $s \in S : \{s\} \in \mathcal{F}$

## *Coverings*

- a **covering** of $(S, \mathcal{F})$ :

  a collection of sets from $\mathcal{F}$ whose union is $S$

- **covering number** $\mathbf{Cov}(S, \mathcal{F})$ :

  the minimum number of elements in a covering

## Let's make things more complicated

- the covering number is also the solution of the IP problem :

$$\begin{array}{lll} \text{minimise} & \displaystyle\sum_{F \in \mathcal{F}} x_F & \\ \text{subject to} & \displaystyle\sum_{F \ni s} x_F \geq 1, & \text{for all } s \in S \\ & x_F \in \{0, 1\}, & \text{for all } F \in \mathcal{F} \end{array}$$

## *The fractional version*

- removing the integrality condition :

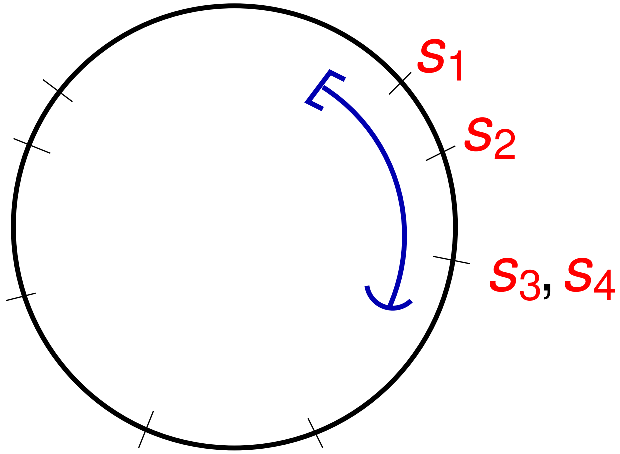
$$\begin{array}{lll} \text{minimise} & \displaystyle\sum_{F \in \mathcal{F}} x_F & \\ \text{subject to} & \displaystyle\sum_{F \ni s} x_F \ge 1, & \text{for all } s \in S \\ & x_F \ge 0, & \text{for all } F \in \mathcal{F} \end{array}$$

- gives the **fractional covering number F-Cov**$(S, \mathcal{F})$
  - and we obviously have : $\text{F-Cov}(S, \mathcal{F}) \le \text{Cov}(S, \mathcal{F})$

## *The circular covering number*

- map the elements of $S$ to a circle so that :
  - for every unit interval $[x, x+1)$ along the circle

    elements mapped into that interval form a set from $\mathcal{F}$

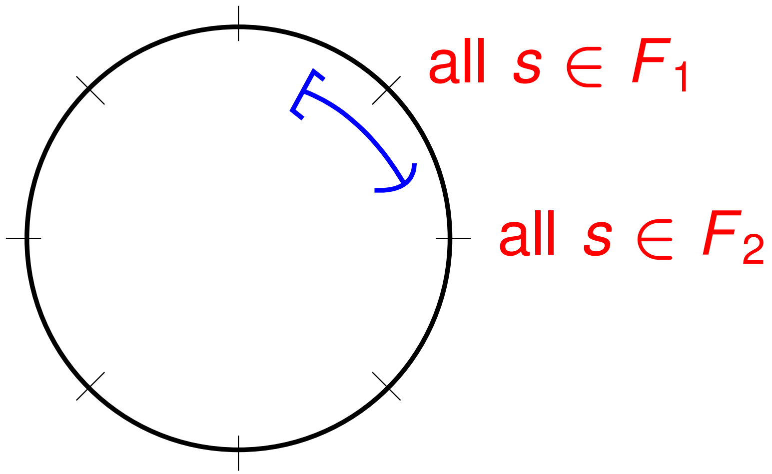



- **circular covering number C-Cov$(S, \mathcal{F})$** :

  minimum circumference of a circle for which this is possible

## *The right place for the circular covering number - I*

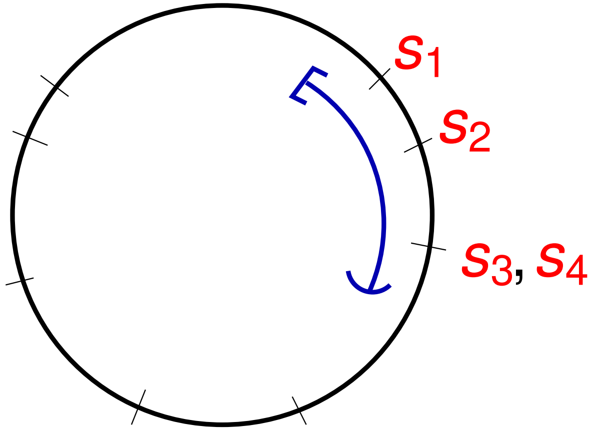
- for a good set system : $\text{C-Cov}(S, \mathcal{F}) \leq \text{Cov}(S, \mathcal{F})$
  - take a disjoint cover $F_1, \ldots, F_k$ of $(S, \mathcal{F})$
  - put the elements of each $F_i$ together at unit distance around a circle with circumference $k$ :




  - gives a circular cover with circumference $k$

## *The right place for the circular covering number - II*

- for a good set system : $\text{F-Cov}(S, \mathcal{F}) \leq \text{C-Cov}(S, \mathcal{F})$
  - take a circular cover along a circle



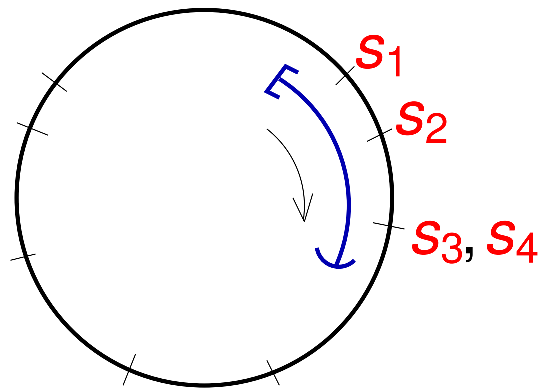

  - "move" the unit interval with "unit speed" round the circle
  - for a set $F$ that appears in the interval at some point :

    denote by $x_F$ the "length of time" it appears

## *The right place for the circular covering number - II*

- for a good set system : $\text{F-Cov}(S, \mathcal{F}) \leq \text{C-Cov}(S, \mathcal{F})$
  - take a circular cover along some circle
  - for a set $F$ that appears in the interval at some point :

    denote by $x_F$ the "length of time" it appears
  - then for all $s \in S$ : $\sum_{F \ni s} x_F = 1$
  - and $\sum_{F \in \mathcal{F}} x_F = \text{circumference}$



## *The right place for the circular covering number - II*

- for a good set system : F-Cov$(S, \mathcal{F}) \leq$ C-Cov$(S, \mathcal{F})$
  - take a circular cover along some circle
  - for a set $F$ that appears in the interval at some point :

    denote by $x_F$ the “length of time” it appears
  - then for all $s \in S$ : $\sum_{F \ni s} x_F = 1$
  - and $\sum_{F \in \mathcal{F}} x_F =$ circumference
  - this gives a fractional cover with value the circumference

# *Inequalities*

- so now we know :

$$\text{F-Cov} \leq \text{C-Cov} \leq \text{Cov}$$

- can we say for what good set systems we have equality for one of the inequalities ?

## *But first some examples*

- $G = (V_G, E_G)$ a graph
  - take $\mathcal{S}_G$ the collection of all stable sets in $V_G$

    ( sets containing no adjacent pairs of vertices )

- $\text{Cov}(V_G, \mathcal{S}_G)$ :

  minimum number of stable sets needed to cover $V_G$

**so** :

- $\text{Cov}(V_G, \mathcal{S}_G) = \chi(G)$ — chromatic number of $G$
- $\text{F-Cov}(V_G, \mathcal{S}_G) = \chi_f(G)$ — fractional chromatic nr.
- $\text{C-Cov}(V_G, \mathcal{S}_G) = \chi_c(G)$ — circular chromatic nr.

## *Another example*

- $G = (V_G, E_G)$ a graph, with multiple edges allowed
  - $\mathcal{A}_G$ : the collection of subsets of $E_G$ that induce an acyclic graph ( $=$ induce a forest )

- $\mathrm{Cov}(E_G, \mathcal{A}_G)$ : minimum number of forests needed to cover $E_G$

**so** :

- $\mathrm{Cov}(E_G, \mathcal{A}_G) = a(G)$ arboricity of $G$
- $\text{F-Cov}(E_G, \mathcal{A}_G) = a_f(G)$ fractional arboricity
- $\text{C-Cov}(E_G, \mathcal{A}_G) = a_c(G)$ circular arboricity

## *Back to the question*

- now we know for all good set systems :

  F-Cov $\leq$ C-Cov $\leq$ Cov

- when do we have equality ?
  - probably too hard ( “too local” )

- better question ( maybe ) :
  - when do we have equality for $(S, \mathcal{F})$

    and for all its induced set systems ?

# *Induced set systems*

- $(S, \mathcal{F})$ a good set system, then for $T \subseteq S$ define :

$$\mathcal{F}_T \;=\; \{\, F \cap T \mid F \in \mathcal{F} \,\} \;=\; \{\, F \in \mathcal{F} \mid F \subseteq T \,\}$$

- then $(T, \mathcal{F}_T)$ is the set system **induced by** $\boldsymbol{T}$

  ( it is automatically good )

- for a graph $G$ :
  - $U \subseteq V_G$ :   $(\mathcal{S}_G)_U$ are the stable sets of the subgraph induced by $U$
  - $D \subseteq E_G$ :   $(\mathcal{A}_G)_D$ are the acyclic edge sets of the subgraph induced by $D$

# *Perfect graphs*

- a graph $G$ is **perfect** if

  for all induced subgraphs $H$ of $G$: $\omega(H) = \chi(H)$

  ( $\boldsymbol{\omega(H)}$ : **clique number** of $H$ )

- for any graph $H$: $\omega(H) \leq \chi_f(H)$

- **so** :

  $G$ a perfect graph

  $\Longrightarrow$ for all $U \subseteq V_G$ :

  $$\text{F-Cov}(U, (\mathcal{S}_G)_U) = \text{Cov}(U, (\mathcal{S}_G)_U)$$

  and $$\text{C-Cov}(U, (\mathcal{S}_G)_U) = \text{Cov}(U, (\mathcal{S}_G)_U)$$

## *Perfect graphs are really special*

**but in fact** :

- $(S, \mathcal{F})$ any good set system

  then :

  $$\text{F-Cov}(T, \mathcal{F}_T) = \text{Cov}(T, \mathcal{F}_T) \text{ for all induced } (T, \mathcal{F}_T)$$

  $$\iff \quad (S, \mathcal{F}) = (V_G, \mathcal{S}_G) \text{ for some perfect graph } G$$

- $$\text{C-Cov}(T, \mathcal{F}_T) = \text{Cov}(T, \mathcal{F}_T) \text{ for all induced } (T, \mathcal{F}_T)$$

  $$\iff \quad (S, \mathcal{F}) = (V_G, \mathcal{S}_G) \text{ for some perfect graph } G$$

## *All that is left to do . . .*

- for what good set systems $(S, \mathcal{F})$ do we have :

  $\text{F-Cov}(T, \mathcal{F}_T) = \text{C-Cov}(T, \mathcal{F}_T)$ for all induced $(T, \mathcal{F}_T)$ ?

- well . . .
  - $(S, \mathcal{F}) = (V_G, \mathcal{S}_G)$ for a perfect graph $G$
  - $(S, \mathcal{F}) = (E_G, \mathcal{A}_G)$ for any graph $G$ ( vdH & Thomassé )

    ( actually, for any matroid )

  - we know a few more special cases
  - and a few constructions for new ones from old ones

## *Fractional and circular arboricity*

- for the remainder of the talk, we'll look at the result
  - for any graph $G$ : $a_f(G) = a_c(G)$ (vdH & Thomassé)

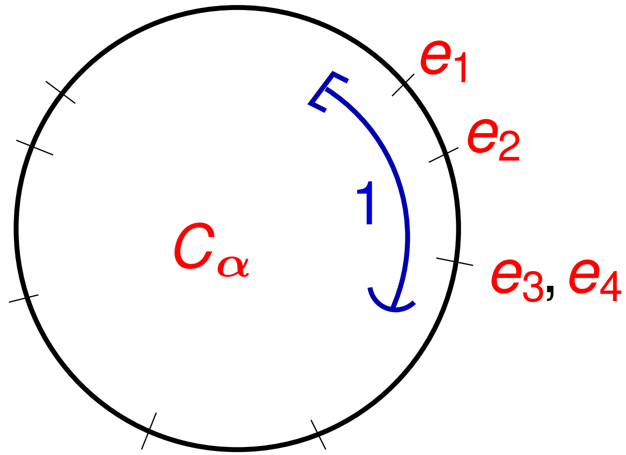
- some extra notation :
  - $\boldsymbol{C_\alpha}$ : circle with circumference $\alpha$ ($\alpha \in \mathbf{R}$, $\alpha > 0$)
  - $\mathbf{Z}_k$ : integers modulo $k$

    (think : $1, 2, \ldots, k$ with a circular ordering)

## *Circular arboricity again*

- we want to map the edges $E_G$ to the circle $C_\alpha$ so that :
  - for every unit interval $[x, x+1)$ along the circle

    edges mapped into that interval form a forest




- **circular arboricity $a_c(G)$** :

  minimum $\alpha$ for which this is possible

## *A bound on the circular arboricity*

- for every subgraph $H \subseteq G$ we must have :
  - a forest can have at most $|V_H| - 1$ edges from $H$
  - so every unit interval of $C_\alpha$ can have at most $|V_H| - 1$ edges from $H$
- so we must have : $a_c(G) \geq \max_{H \subseteq G} \frac{|E_H|}{|V_H| - 1}$
- **Conjecture** ( Gonçalves ) : $a_c(G) = \max_{H \subseteq G} \frac{|E_H|}{|V_H| - 1}$

## *Integral and fractional arboricity*

**Theorem** ( Nash-Williams, 1964; Edmonds, 1964 )

- $K \geq \max_{H \subseteq G} \left\lceil \frac{|E_H|}{|V_H| - 1} \right\rceil$ ( where $K \in \mathbf{N}$ )

  $\Longrightarrow$ $E_G$ can be partitioned into $K$ forests

- this means : $a_c(G) \leq a(G) = \max_{H \subseteq G} \left\lceil \frac{|E_H|}{|V_H| - 1} \right\rceil$

- **folklore** :

  for the fractional arboricity : $a_f(G) = \max_{H \subseteq G} \frac{|E_H|}{|V_H| - 1}$

## *Quick proof of the fractional arboricity*

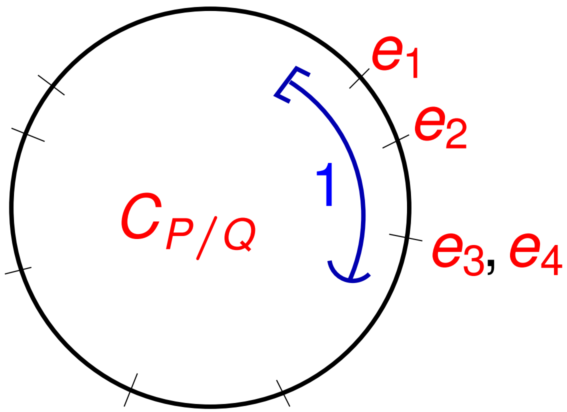
- suppose $\max_{H \subseteq G} \frac{|E_H|}{|V_H| - 1} = \frac{P}{Q}$
- form $G^Q$ by replacing each edge by $Q$ parallel edges
- then : $\max_{H \subseteq G^Q} \left\lceil \frac{|E_H|}{|V_H| - 1} \right\rceil = \max_{H \subseteq G^Q} \frac{|E_H|}{|V_H| - 1} = P$
- Nash-Williams; Edmonds :

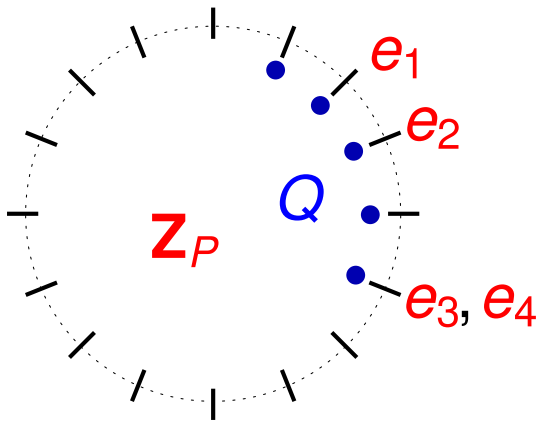
  edges of $G^Q$ can be covered by $P$ forests
- so : $G$ has $P$ forests covering each edge at least $Q$ times
- set $x_F = 1/Q$ for these forests $\square$

## *Back to what we want to prove*

- want to prove : $a_c(G) = \max_{H \subseteq G} \dfrac{|E_H|}{|V_H| - 1} = \dfrac{P}{Q}$

- instead of "real" circle $C_{P/Q}$ and unit intervals, we can :
  - consider the "discrete" circle $\mathbf{Z}_P$
  - and have "intervals" of $Q$ consecutive integers

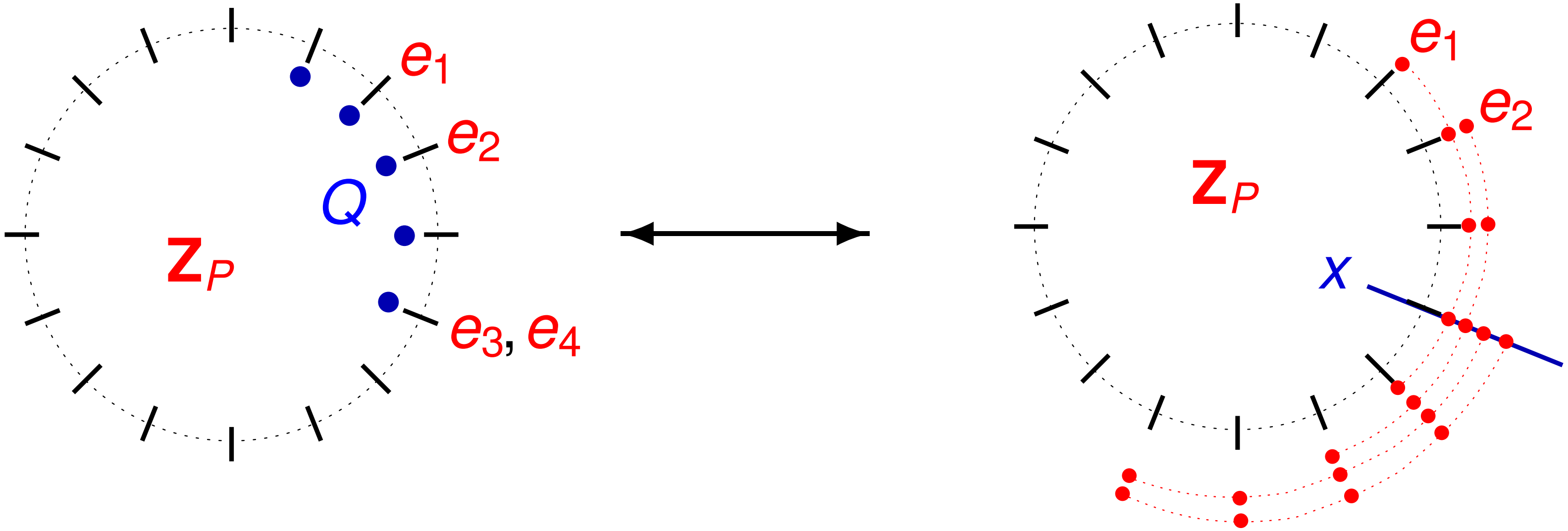


## *Changing our point of view*

- we want to :
  - map the edges into $\mathbf{Z}_P$
  - so that : edges mapped
    into any set of $Q$ consecutive points form a forest
- this is equivalent to :
  - map each edge to $Q$ consecutive points of $\mathbf{Z}_P$
  - so that for $x \in \mathbf{Z}_P$ : edges covering $x$ form a forest

## *Changing our point of view*

- 


- crucial idea :
  generalise this, by allowing different edges
  to be mapped to a different number of consecutive points
  - in other words : give each edge $e$ some weight $w(e)$

# *The general theorem*

**Theorem**

- given : $K \in \mathbf{N}$ and edge weights $w : E_G \to \mathbf{N}$
- suppose : for all $H \subseteq G$ : $K \geq \dfrac{\sum_{e \in E_H} w(e)}{|V_H| - 1}$
- then :
  - we can map each $e$ to $w(e)$ consecutive points of $\mathbf{Z}_K$
  - so that for all $x \in \mathbf{Z}_K$ : edges covering $x$ form a forest

**Corollary** :

$K = P$ and $w \equiv Q$ gives : $a_c(G) = \dfrac{P}{Q} = \max\limits_{H \subseteq G} \dfrac{|E_H|}{|V_H| - 1}$

## *Some ideas from the proof*

- proof by induction on $\sum w(e)$
- choose an edge $e_1$ and replace $w(e_1)$ by $w(e_1) - 1$
  - find a suitable mapping to $\mathbf{Z}_K$ with this reduced weight
- say $e_1$ gets mapped to the sequence $x_0, \ldots, x_1 - 1$
  - map an extra copy of $e_1$ to position $x_1$
  - this may introduce a cycle at position $x_1$
- there is an edge $e_2$ in this cycle not mapped to $x_1 - 1$
- say $e_2$ gets mapped to the sequence $x_1, \ldots, x_2 - 1$

## *Some ideas from the proof*

- say $e_2$ gets mapped to the sequence $x_1, \ldots, x_2 - 1$
  - remove the copy of $e_2$ from $x_1$
    - this will break the cycle at $x_1$ !!
  - map a new copy of $e_2$ to position $x_2$
  - this may introduce a cycle at position $x_2$
- there is an edge $e_3$ in this cycle not mapped to $x_2 - 1$
- say $e_3$ gets mapped to the sequence $x_2, \ldots, x_3 - 1$

## *Some ideas from the proof*

- say $e_3$ gets mapped to the sequence $x_2, \ldots, x_3 - 1$
  - remove the copy of $e_3$ from $x_2$
    - this will break the cycle at $x_2$
  - map a new copy of $e_3$ to position $x_3$
  - this may introduce a cycle at position $x_3$
- there is an edge $e_4$ in this cycle not mapped to $x_3 - 1$
- ad infinitum . . . . . . . . .

**NOT !**

## *Another corollary*

**Conjecture** (Kajitani et al., 1988)

- if : $\max\limits_{H \subseteq G} \dfrac{|E_H|}{|V_H| - 1} = \dfrac{|E_G|}{|V_G| - 1}$ ( $G$ is **uniformly dense** )
-
  - there exists a **circular ordering** of $E_G$
  - so that each $|V_G| - 1$ consecutive edges form a spanning tree
- known to be true for
  - a few special cases of graphs
  - graphs consisting of **2** edge-disjoint spanning trees (that case even open for matroids)

## *The other corollary*

**Theorem**

- if :
  - $$\max_{H \subseteq G} \frac{|E_H|}{|V_H| - 1} = \frac{|E_G|}{|V_G| - 1}$$
  - and : $|V_G| - 1$ and $|E_G|$ are co-prime
- then :
  - there exists a **circular ordering** of $E_G$
  - so that each $|V_G| - 1$ consecutive edges form a spanning tree

## *Some open problems - I*

- before the Strong Perfect Graph Theorem we knew :
  - $G$ not perfect

    $\Longrightarrow$ there is an $H \subseteq G$ so that $\chi_f(H) \neq \chi(H)$

- the Strong Perfect Graph Theorem makes it trivial to prove :
  - $G$ not perfect

    $\Longrightarrow$ there is an $H \subseteq G$ so that $\chi_c(H) \neq \chi(H)$

  - but can we prove this without it ?

# *Some open problems - II*

- can we characterise the graphs $G$
  - for which $\chi_f(H) = \chi_c(H)$, for all $H \subseteq G$ ?

## *Some open problems - III*

- we know two large classes of good set systems $(S, \mathcal{F})$ for which :
  - F-Cov$(T, \mathcal{F}_T) =$ C-Cov$(T, \mathcal{F}_T)$ for all induced $(T, \mathcal{F}_T)$
- namely :
  - $(V_G, \mathcal{S}_G)$ for perfect graphs $G$
  - $(E_M, \mathcal{I}_M)$ for loopless matroids $M$

    ( with $(E_G, \mathcal{A}_G)$ for all all graphs $G$ as special cases )
- what "natural" class of set systems contains

  both matroids and stable sets of perfect graphs ?