Graph Colouring with Distances

JAN VAN DEN HEUVEL

UPMC, Paris, 7 April 2011

Department of Mathematics
London School of Economics and Political Science

The basics of graph colouring

- **vertex-colouring** with $\boldsymbol{k}$ colours :
  adjacent vertices must receive different colours

- **chromatic number** $\chi(\boldsymbol{G})$:
  minimum k so that a vertex-colouring exists

- **list-colouring** : as vertex-colouring,
  but each vertex v has its own list $L(v)$ of colours

- **choice number** $\mathbf{ch}(\boldsymbol{G})$:
  minimum k so that if all $|L(v)| \geq k$,
  then a proper list vertex-colouring exists

Another way to look at vertex-colouring

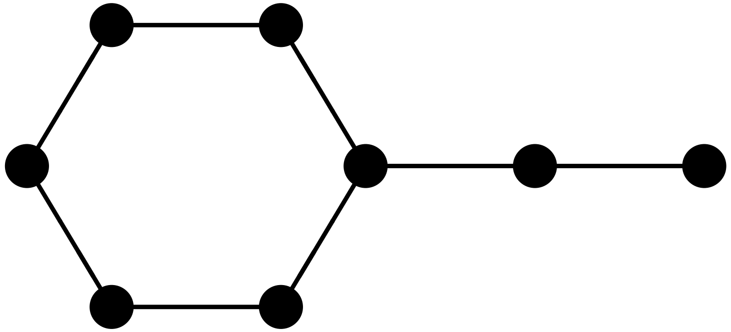
- vertex-colouring :

  vertices at distance one must receive different colours

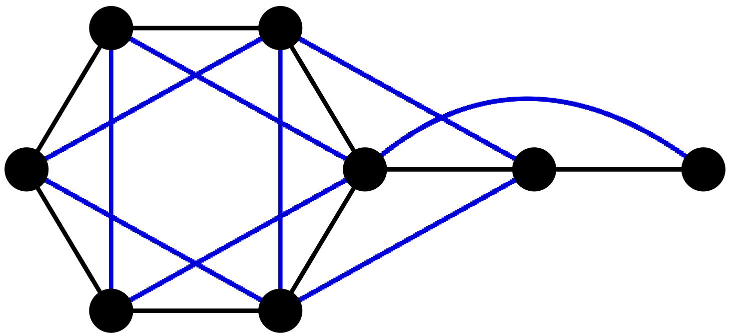
- now suppose we want vertices at larger distances (say, up to distance D) to receive different colours as well

- can be modelled using the D **-th power** G^D **of a graph** :
  - same vertex set as G
  - edges between vertices with distance at most D in G

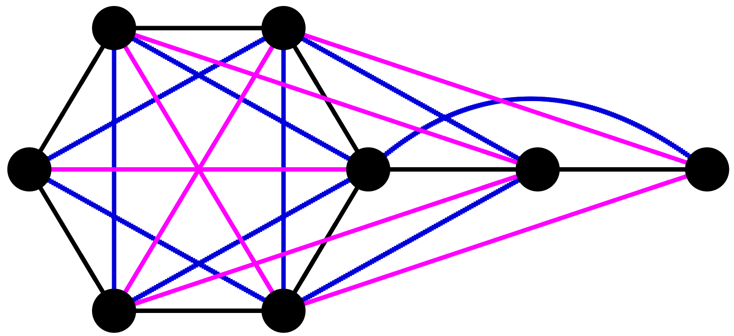
Powers of a graph



G

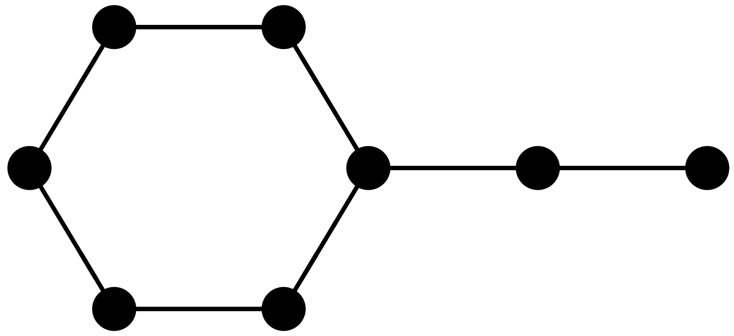


G^2

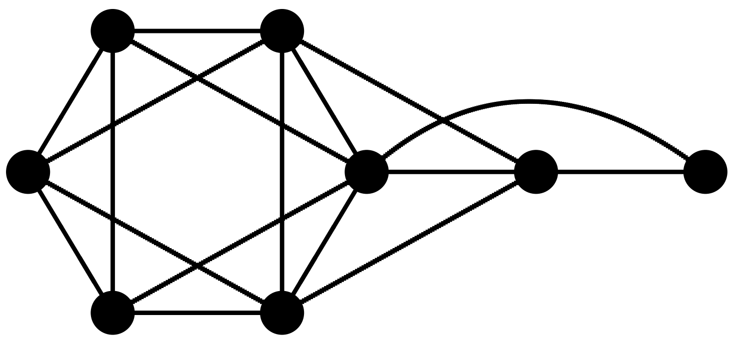


G^3

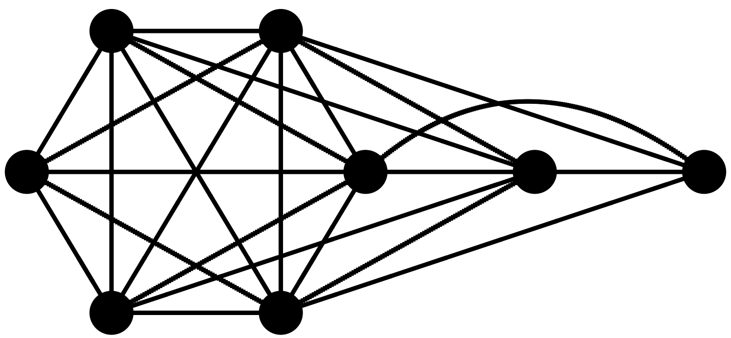
Powers of a graph



G



G^2



G^3

A first conjecture / problem

- powers of graphs seem to have more structure than graphs in general

<u>**List-Square-Colouring Conjecture**</u> (Kostochka & Woodall, 2001)

- for any graph G : $\mathrm{ch}(G^2) = \chi(G^2)$

- if true, then $\mathrm{ch}(G^D) = \chi(G^D)$ for all even D

  (since $G^{2d} = (G^d)^2$)

- what about $\mathrm{ch}(G^D) = \chi(G^D)$ for odd D ?

Colouring powers of a graph

easy fact

- $$\Delta(G^D) \leq \sum_{i=0}^{D-1} \Delta(G)\,(\Delta(G)-1)^i = O(\Delta(G)^D)$$

  ($\Delta = \Delta(G)$: maximum degree of G)

  - **so** : $\chi(G^D) \leq O(\Delta(G)^D)$

- but for very few graphs you would expect to need that many colours

The square of d-degenerate graphs

- G is $\boldsymbol{d}$ **-degenerate** : every subgraph of G has a vertex of degree at most d

- **easy, but not trivial** :

  G d -degenerate $\Longrightarrow$ G^2 is $((2d-1)\,\Delta)$ -degenerate

so :

- G planar $\Longrightarrow$ $\chi(G^2) \leq 9\,\Delta + 1$

The square of planar graphs

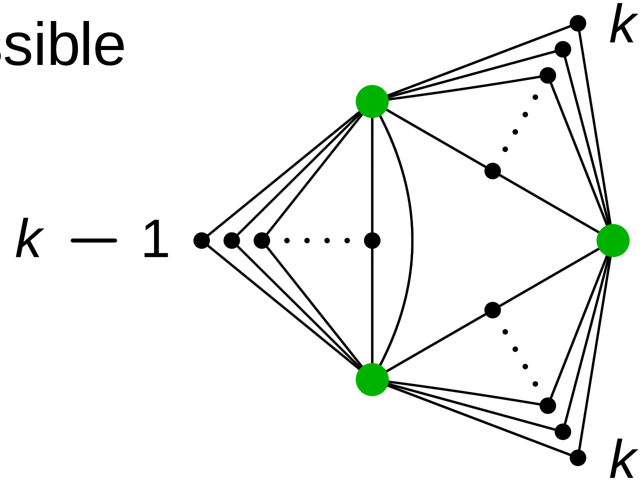
Conjecture (Wegner, 1977)

- G planar

$$\Longrightarrow \quad \chi(G^2) \le \begin{cases} 7, & \text{if } \Delta = 3 \\ \Delta + 5, & \text{if } 4 \le \Delta \le 7 \\ \left\lfloor {}^{3}\!/_{2}\, \Delta \right\rfloor + 1, & \text{if } \Delta \ge 8 \end{cases}$$

- bounds would be best possible

case $\Delta = 2k \ge 8$:



Towards Wegner's Conjecture

G planar $\Longrightarrow$

- $\chi(G^2) \leq 8\,\Delta - 22$ (Jonas, PhD, 1993)
- $\chi(G^2) \leq 3\,\Delta + 5$ (Wong, MSc, 1996)
- $\chi(G^2) \leq 2\,\Delta + 25$ (vdH & McGuinness, 2003)
- $\chi(G^2) \leq {}^{9}\!/_{5}\,\Delta + 1$ (for $\Delta \geq 47$)
  (Borodin, Broersma, Glebov & vdH, 2001)
- $\chi(G^2) \leq {}^{5}\!/_{3}\,\Delta + 24$ (for $\Delta \geq 241$)
  (Molloy & Salavatipour, 2005)

Towards Wegner's Conjecture

Theorem (Havet, vdH, McDiarmid & Reed, 2008+)

- G planar $\Longrightarrow$ $\chi(G^2) \leq \left({}^3/_2 + o(1)\right) \Delta \quad (\Delta \to \infty)$

- - we actually prove the list-colouring version
  - and for much larger classes of graphs :

Theorem

- G graph, $K_{3,k}$ -minor free for some fixed k

$$\Longrightarrow \quad \mathrm{ch}(G^2) \leq \left({}^3/_2 + o(1)\right) \Delta$$

Even more general results ?

Property

- G graph, H -minor free for some fixed graph H

  $\Longrightarrow \quad ch(G^2) \leq C_H \, \boldsymbol{\Delta} + O(1)$, for some constant C_H

Question

- given H, what is the smallest possible C_H ?

e.g.

- for $H = K_{3,k}$ we know $C_{K_{3,k}} = {}^3\!/_2$
- for $H = K_5$ we have $2 \leq C_{K_5} \leq 9$
- for $H = K_{4,4}$ we have $C_{K_{4,4}} \geq {}^7\!/_3$

The clique number

Corollary

- G graph, $K_{3,k}$ -minor free for some fixed k

$$\Longrightarrow \quad \omega(G^2) \ \le \ \left({}^3\!/_2 + o(1)\right) \Delta$$

can be partially improved to

Theorem (Amini, Esperet & vdH, 2009+)

- G embeddable on a fixed surface S

$$\Longrightarrow \quad \omega(G^2) \ \le \ {}^3\!/_2 \, \Delta + O(1)$$

Theorem (Cohen & vdH, 2011+)

- G planar, $\Delta(G) \ge 41 \quad \Longrightarrow \quad \omega(G^2) \ \le \ \left\lfloor {}^3\!/_2 \, \Delta \right\rfloor + 1$

Sketch of the proof of square of planar graph

uses induction on the number of vertices

- **2-neighbour** : vertex at distance at most two
- $\boldsymbol{d^2(v)}$: number of 2-neighbours of v

  $=$ number of neighbours of v in G^2

- we would like to remove a vertex v with $d^2(v) \leq {}^3\!/_2\,\Delta$
  - but that can change distances in $G - v$
- contraction to a neighbour u will solve the distance problem
  - but may increase maximum degree if $d(u) + d(v) > \Delta$
- **so** : easy induction possible if there is an edge uv

  with $d(u) + d(v) \leq \Delta$ and $d^2(v) \leq {}^3\!/_2\,\Delta$

When easy induction is not possible

S, **small** vertices : degree at most some constant C

B, **big** vertices : degree more than C

H, **huge** vertices : degree at least $\frac{1}{2}\,\boldsymbol{\Delta}$

- small vertices v have a least two big neighbours
  (otherwise contractible to neighbour and $d^2(v) \le {}^3/_2\,\boldsymbol{\Delta}$)

- a planar graph has fewer than $3\,|V|$ edges
  and fewer than $2\,|V|$ edges if it is bipartite

  so :

  - all but $O(|V|/C)$ vertices are small
  - fewer than $2\,|B|$ vertices in $V \setminus B$
    have more than two neighbours in B

When easy induction is not possible

S , **small** vertices : degree at most some constant C

B , **big** vertices : degree more than C

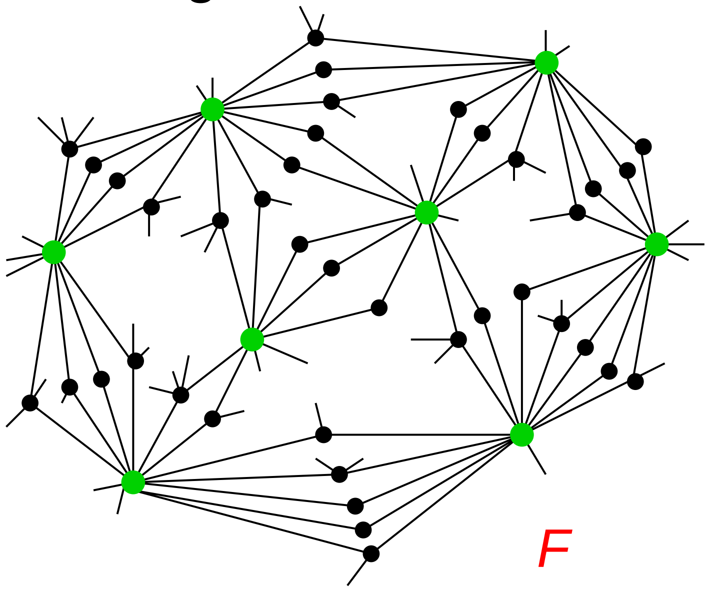
H , **huge** vertices : degree at least $\frac{1}{2}\,\Delta$

we can conclude :

- "most" vertices are small
- and these have exactly two huge neighbours

The structure so far

- there is a large subgraph ***F*** of *G* looking like :



- green vertices ***X***
  have degree at least $\frac{1}{2}\,\boldsymbol{\Delta}$
- black vertices ***Y***
  have degree at most *C*
- all other neighbours of *Y* -vertices are also small

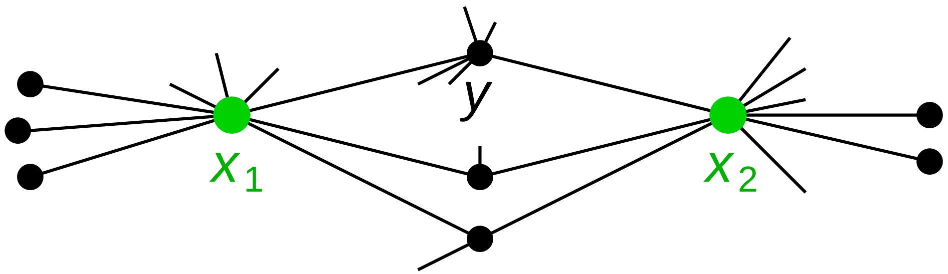
inside *F* there is a subgraph ***H*** which satisfies additionally :

- only "few" edges from *X* to rest of *G*
- *H* satisfies "some edge density condition"

The other induction step

- remove from G the Y-vertices in H (by contraction)
- colour the smaller graph (can be done by induction)
- in the original graph G:

  what to do with the uncoloured Y-vertices?



- 2-neighbours of y already coloured (i.e., outside Y):
  - at most $(d_G(x_1) - d_H(x_1)) + (d_G(x_2) - d_H(x_2))$

    2-neighbours outside Y via x_1, x_2
  - at most C^2 other 2-neighbours outside Y

Transferring to edge-colouring

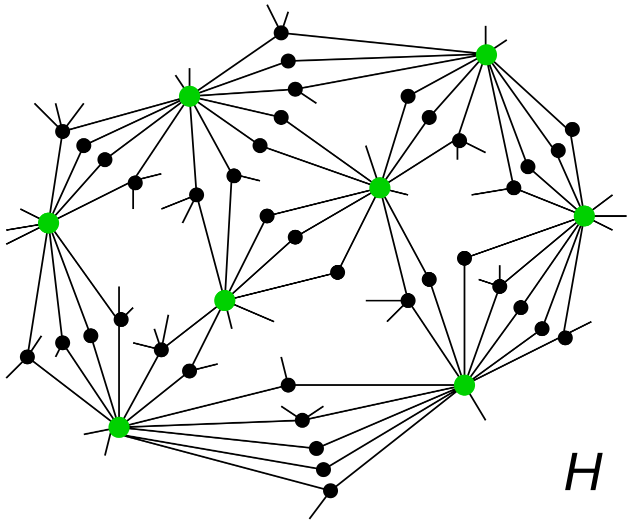
- so a vertex y from Y has at least

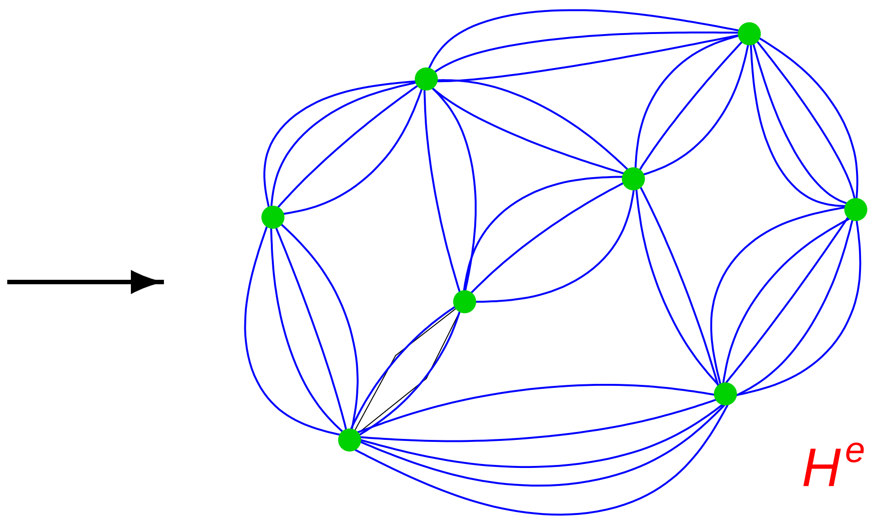
$$(3/2+\varepsilon)\,\Delta - (d_G(x_1) - d_H(x_1)) - (d_G(x_2) - d_H(x_2)) - C^2$$

  colours still available

- colouring Y is "almost" like

  list-colouring edges of the multigraph $\boldsymbol{H^e}$:





Edge-colouring multigraphs

- $\chi'(G)$: chromatic index of multigraph G
- $ch'(G)$: list chromatic index of multigraph G
- $\chi'_f(G)$: fractional chromatic index of multigraph G

Theorem (Kahn, 1996, 2000)

- multigraph G with Δ large enough

$$\Longrightarrow \quad ch'(G) \approx \chi'(G) \approx \chi'_f(G)$$

- in fact, Kahn's proofs provide something much more general
  - allowing lists of unequal size

Kahn's result

Theorem (Kahn, 2000)

for $0 < \delta < 1$, $\alpha > 0$, there exists $\mathbf{\Delta}_{\delta,\alpha}$ so that if $\mathbf{\Delta} \geq \mathbf{\Delta}_{\delta,\alpha}$:

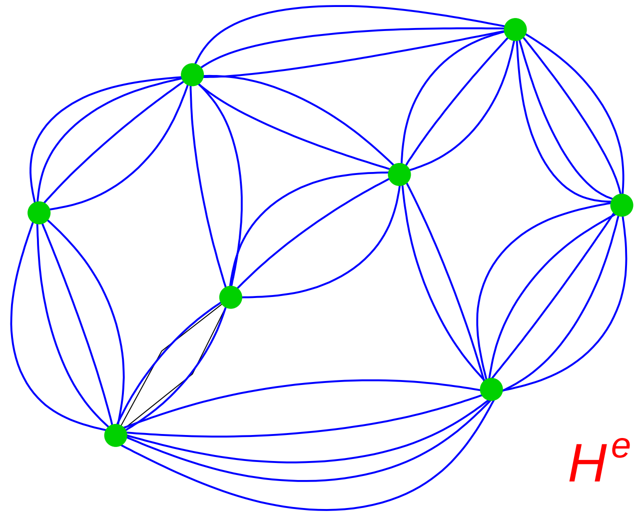
- G a multigraph with maximum degree at most $\mathbf{\Delta}$
- each edge e has a list $L(e)$ of colours so that
  - for all edges e : $|L(e)| \geq \alpha\,\mathbf{\Delta}$
  - for all vertices v : $\displaystyle\sum_{e \ni v} |L(e)|^{-1} \leq 1 \cdot (1-\delta)$
  - for all $K \subseteq G$ with $|V(K)| \geq 3$ odd:

$$\sum_{e \in E(K)} |L(e)|^{-1} \leq \tfrac{1}{2}(|V(K)|-1) \cdot (1-\delta)$$

- then there exists a proper colouring of the edges of G so that each edge gets colours from its own list

Kahn's approach for our case

- we have a multigraph H^e :



- so that each edge $e = x_1 x_2$ has a list $L(e)$ of at least

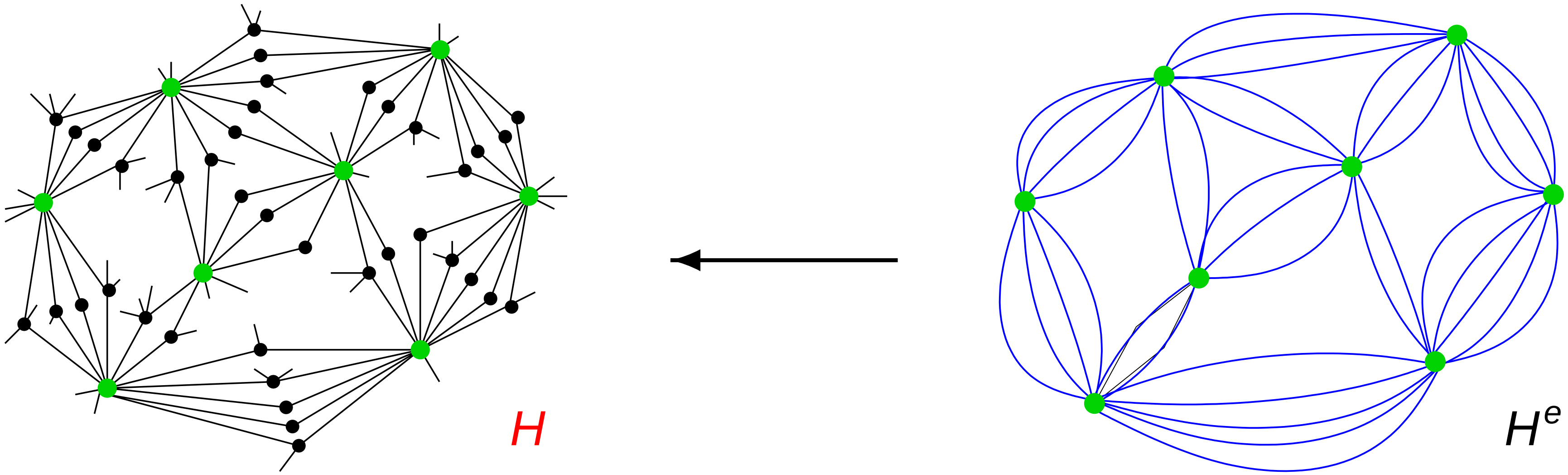
$$\left(3/2+\varepsilon\right)\Delta - \left(d_G(x_1) - d_H(x_1)\right) - \left(d_G(x_2) - d_H(x_2)\right) - C^2$$

colours

- and H^e satisfies "some edge density condition"

Extending Kahn's approach

- those density conditions guarantee that Kahn's conditions are satisfied for H^e
- $\Longrightarrow$ we can edge-colour H^e



- $\Longrightarrow$ we can colour the Y-vertices in H, choosing from the left-over colours for each Y-vertex
- also : we can deal with the "almost" list-edge colouring

What about distances larger than 2 ?

Theorem (Agnarsson & Halldórsson, 2003)

- G planar $\implies$ $\chi(G^D) \le c_D\,\Delta^{\lfloor D/2 \rfloor}$

- best possible : take Δ -regular tree with radius $\lfloor \frac{1}{2} D \rfloor$

in fact, their proof gives something much more general :

Theorem

- G k -degenerate $\implies$ $\chi(G^D) \le c_{k,D}\,\Delta^{\lfloor D/2 \rfloor}$

Main ideas of a simple proof

- G is ***m*-orientable**: G has an orientation in which every vertex has outdegree at most m

  - G is k-degenerate $\Longrightarrow$ G is k-orientable

  - G is m-orientable $\Longrightarrow$ G is $2m$-degenerate

    $\Longrightarrow$ $\chi(G) \leq 2m+1$

Theorem

- G is m-orientable $\Longrightarrow$ G^D is $c_{m,D}\,\Delta^{\lfloor D/2 \rfloor}$-orientable

Main ideas of a simple proof

- fix an orientation $\vec{G}$ of G with maximum outdegree m, and fix $D \geq 1$
- let uv be an edge in G^D
  - so there is a uv-path $u = x_0, x_1, \ldots, x_\ell = v$ of length $\ell \leq D$
- orient uv in G^D according to the majority of the orientation of the edges in that uv-path (when going from u to v) (arbitrarily if a tie)

Main ideas of a simple proof

- so outdegree in oriented G^D of a vertex u is at most:
  - the number of uv-paths of length $\ell \le D$ in G
    with at least $\left\lceil \frac{1}{2}\ell \right\rceil$ edges oriented $x_i \to x_{i+1}$ in $\vec{G}$
- and the number of such paths is at most:

$$\sum_{\ell=1}^{D} \sum_{i=\lceil \ell/2 \rceil}^{\ell} \binom{\ell}{i} \cdot m^{i} \cdot \Delta^{\ell-i}$$

$$= \sum_{\ell=1}^{D} \sum_{j=0}^{\lfloor \ell/2 \rfloor} \binom{\ell}{j} \cdot m^{\ell-j} \cdot \Delta^{j} \le c_{m,D}\, \Delta^{\lfloor D/2 \rfloor}$$

Colouring the cube of planar graphs

- so now we know there is some constant c_3 so that:

  G planar $\Longrightarrow$ $\chi(G^3) \leq c_3\,\Delta + O(1)$

  - but what is the best c_3 ?
    - we only know: $4 \leq c_3 \leq 68$

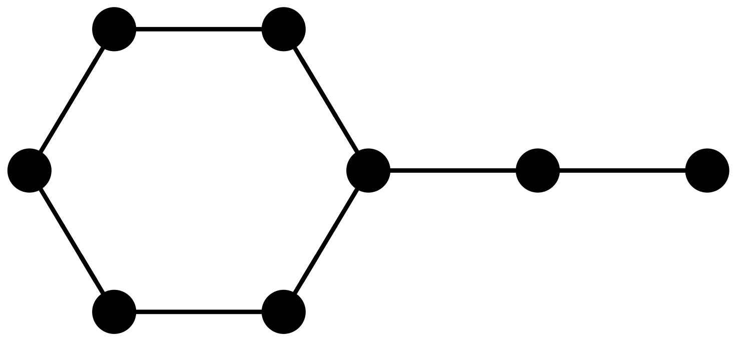
- and what about distance $D > 3$?

A variant with exact distances

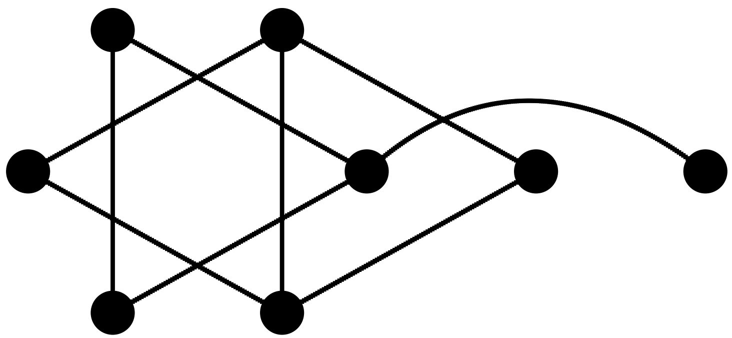
- suppose we only want vertices at distance exactly D to have different colours

- can be modelled using the **exact distance graph $G^{=D}$** of G :
  - same vertex set as G
  - edges between vertices with distance exactly D in G

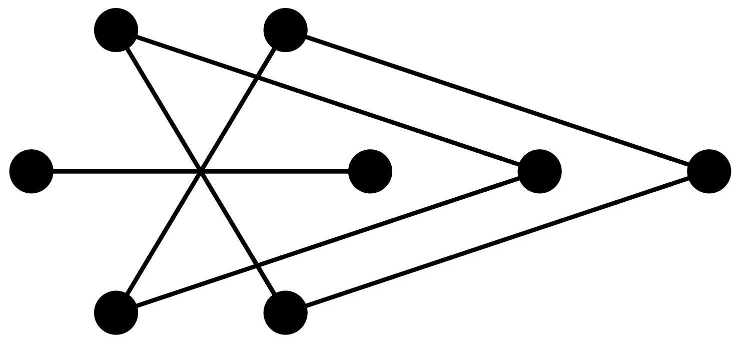
Exact distance graphs



G



$G^{=2}$



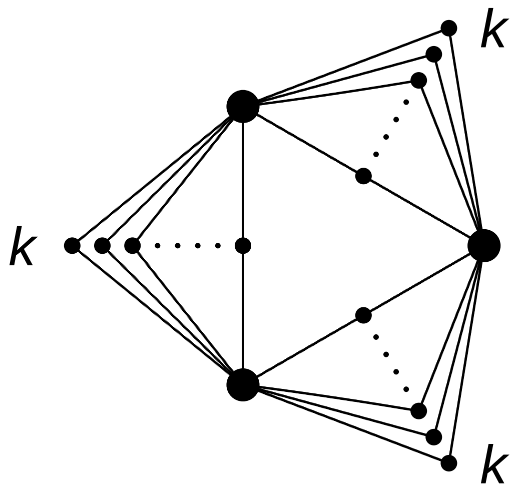
$G^{=3}$

Colouring exact distance graphs of planar graphs

- obviously :

  G planar $\Longrightarrow$ $\chi(G^{=D}) \le \chi(G^D) \le O(\Delta^{\lfloor D/2 \rfloor})$

- and for $D = 2$ we can have $\chi(G^{=2}) = {}^3/_2\,\Delta$:



- in fact, for all <u>even</u> D , the bound seems to be ${}^3/_2\,\Delta^{D/2}$

Colouring exact distance graphs

- for odd D, the situation is very different:

Theorem (Nešetřil & Ossona de Mendez, 2008)

- $\mathcal{K}$ a graph class with "bounded expansion", D odd
  - then there exists a constant $c_{\mathcal{K},D}$ so that:

$$G \in \mathcal{K} \quad \Longrightarrow \quad \chi(G^{=D}) \leq c_{\mathcal{K},D}$$

- a proper minor-closed class is of bounded expansion
  - hence planar graphs are of bounded expansion

Colouring exact distance graphs

- the result is best possible in many senses :
  - not true for even D
  - not true for k -degenerate graphs :
    - consider $S_{n,D}$: complete graph K_n with edges replaced by paths of length D
    - $S_{n,D}$ is 2-degenerate, but $\chi((S_{n,D})^{=D}) = n$
  - not true if " u, v have distance exactly D " replaced by "there is a uv -path of length D "
    - consider wheel W_n with n spokes

The exact cube of planar graphs

- so now we know : G planar $\implies$ $\chi(G^{=3}) \leq c'_3$
  - short proof ?
  - what can we say about c'_3 ?
    - best known bounds : $6 \leq c'_3 \leq 10^{10^{10}}$
- G triangle-free planar $\implies$ $\chi(G^{=3}) \leq 16$

  (consequence of result in (Naserasr, 2007))
- in general : what can we say

  about the structure of $G^{=3}$ for planar G ?