Fractional Colouring and Pre-colouring Extension of Graphs

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The basics of graph colouring

- **vertex-colouring** with k colours :
  adjacent vertices must receive different colours

- **chromatic number** $\chi(G)$:
  minimum k so that a vertex-colouring exists

general question :

- what can we say if some vertices are already pre-coloured ?
- in particular : can $\chi(G)$ colours still be enough ?

Pre-colouring questions

next best questions :

- how many extra colours may be needed ?
- and what conditions on the pre-coloured vertices can make life easier ?

Question (Thomassen, 1997)

- G planar,

  $W \subseteq V(G)$, a set of vertices so that

  distance between any two vertices in W is at least 100

  - can any 5-colouring of W

    be extended to a 5-colouring of G ?

The first answer

- $\mathrm{dist}(W)$: minimum distance between any two vertices in W

Theorem (Albertson, 1998)

- G any graph with chromatic number χ

  $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq 4$

  $\Longrightarrow$ any $(\chi+1)$-colouring of W

  can be extended to a $(\chi+1)$-colouring of G

Some more answers

Theorem (Albertson, 1998)

- G planar graph

  $W \subseteq V(G)$ with $\text{dist}(W) \geq 3$

  $\Longrightarrow$ any 6-colouring of W

  can be extended to a 6-colouring of G

Theorem

- G any graph with chromatic number χ

  $W \subseteq V(G)$ with $\text{dist}(W) \geq 3$

  $\Longrightarrow$ any $(\chi+\chi)$-colouring of W

  can be extended to a $(\chi+\chi)$-colouring of G

A different kind of colouring

- **fractional K-colouring** of graph G ($K \in \mathbf{R}$, $K \geq 0$) :
  - every vertex $v \in V$ is assigned a subset $\phi(v) \subseteq [0, K]$ so that :
    - every subset $\phi(v)$ has 'measure' 1
    - and $uv \in E(G) \implies \phi(u) \cap \phi(v) = \varnothing$

- **fractional chromatic number** $\chi_F(G)$:

$$= \inf \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

$$= \min \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

A different kind of colouring

- **note** : we always have $\chi_F(G) \le \chi(G)$
  - but the difference can be arbitrarily large

-
  - $\chi_F(G) = 1 \iff G$ has no edges
  - $\chi_F(G) = 2 \iff G$ has edges and is bipartite
  - for all rational $K \ge 2$: there exist G with $\chi_F(G) = K$

Pre-colouring in the fractional world

- so now suppose that for some vertices $W \subseteq V(G)$, the vertices in W are already pre-coloured :
  - vertices $w \in W$ have been given some set $\phi(w)$ of measure 1
- when can this be extended to a fractional colouring of the whole graph G?
- in general we should expect to require more than $\chi_F(G)$ colours

The set-up of the problem

- - G a graph with fractional chromatic number $\chi_F \geq 2$
  - $D \geq 3$ an integer
  - $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$
- - the vertices $w \in W$

    are pre-coloured with $\phi(w) \subseteq [0, \chi_F + \alpha]$
    - for some real $\alpha \geq 0$
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$
- how large should α be to make sure this can be done ?

A major part of the answer

Theorem (Král', Krnc, Kupec, Lužar & Volec, 2011)

- extension is always possible, provided α is at least :
  - $$\frac{\sqrt{(\lfloor D/4\rfloor\,\chi_F-1)^2+4\,\lfloor D/4\rfloor\,(\chi_F-1)}-\lfloor D/4\rfloor\,\chi_F+1}{2\,\lfloor D/4\rfloor},$$ if $D\equiv 0 \bmod 4$
  - $$\frac{\chi_F-1}{\lfloor D/4\rfloor\,\chi_F},$$ if $D\equiv 1 \bmod 4$
  - $$\frac{\sqrt{(\lfloor D/4\rfloor\,\chi_F)^2+4\,\lfloor D/4\rfloor\,(\chi_F-1)}-\lfloor D/4\rfloor\,\chi_F}{2\,\lfloor D/4\rfloor},$$ if $D\equiv 2 \bmod 4$
  - $$\frac{\chi_F-1}{\lfloor D/4\rfloor\,\chi_F+\chi_F-1},$$ if $D\equiv 3 \bmod 4$

A major part of the answer

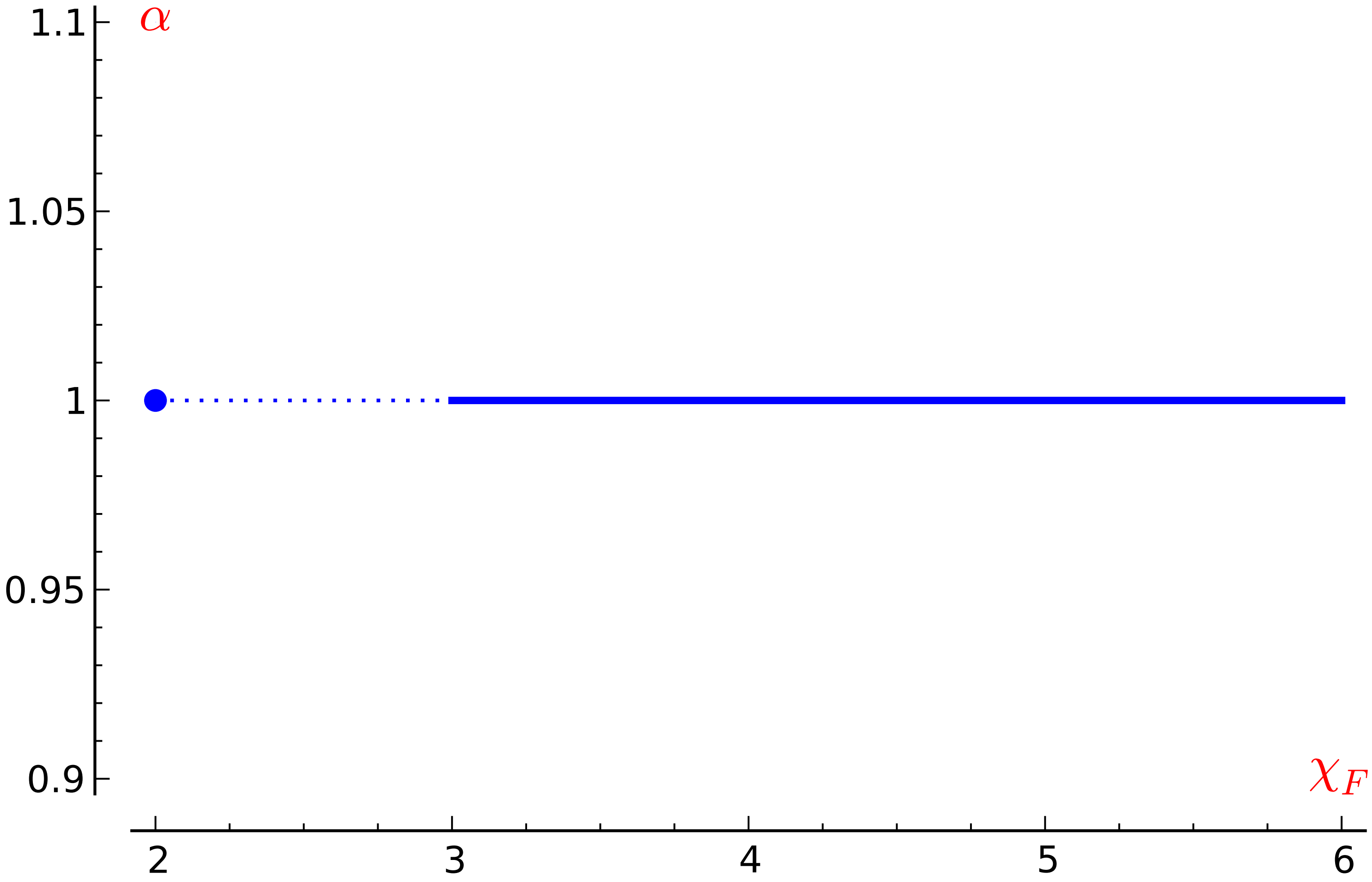
Theorem (Král', Krnc, Kupec, Lužar & Volec, 2011)

- moreover, these bounds on α are best possible,
  - for all $D \geq 3$
  - and if $\chi_F = 2$ or $\chi_F \geq 3$

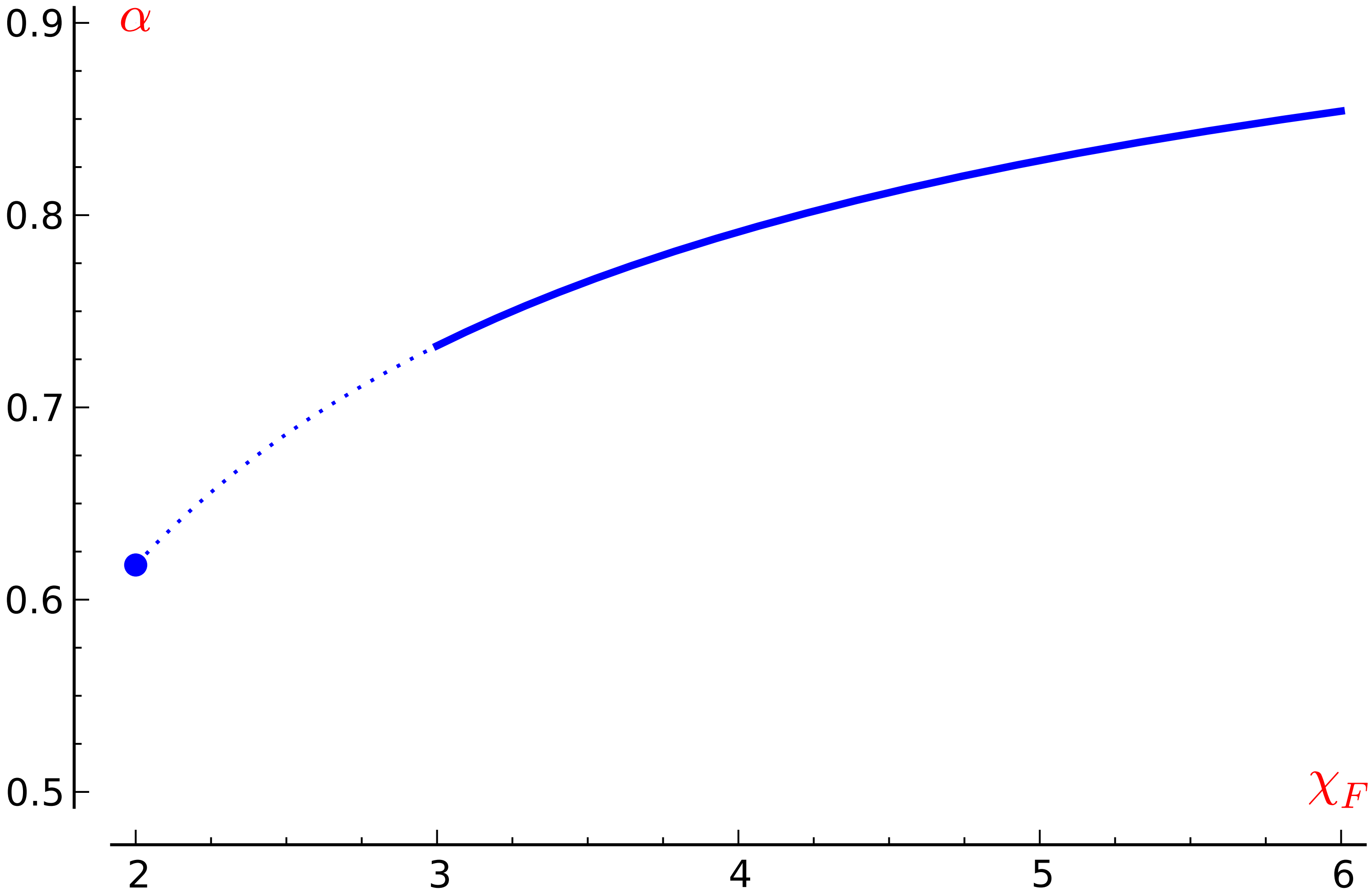
A major part of the answer

- **in other words** :
  - for all integers $D \geq 3$,
    all rational numbers $\chi_F \in \{2\} \cup [3, \infty)$,
    and all $\alpha \geq 0$ failing the bound for that D and χ_F
  - there is a graph G with fractional chromatic number χ_F,
    a subset $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$,
    and a fractional pre-colouring $\phi(w) \subseteq [0, \chi_F + \alpha]$
    for $w \in W$
  - so that ϕ cannot be extended to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$ only

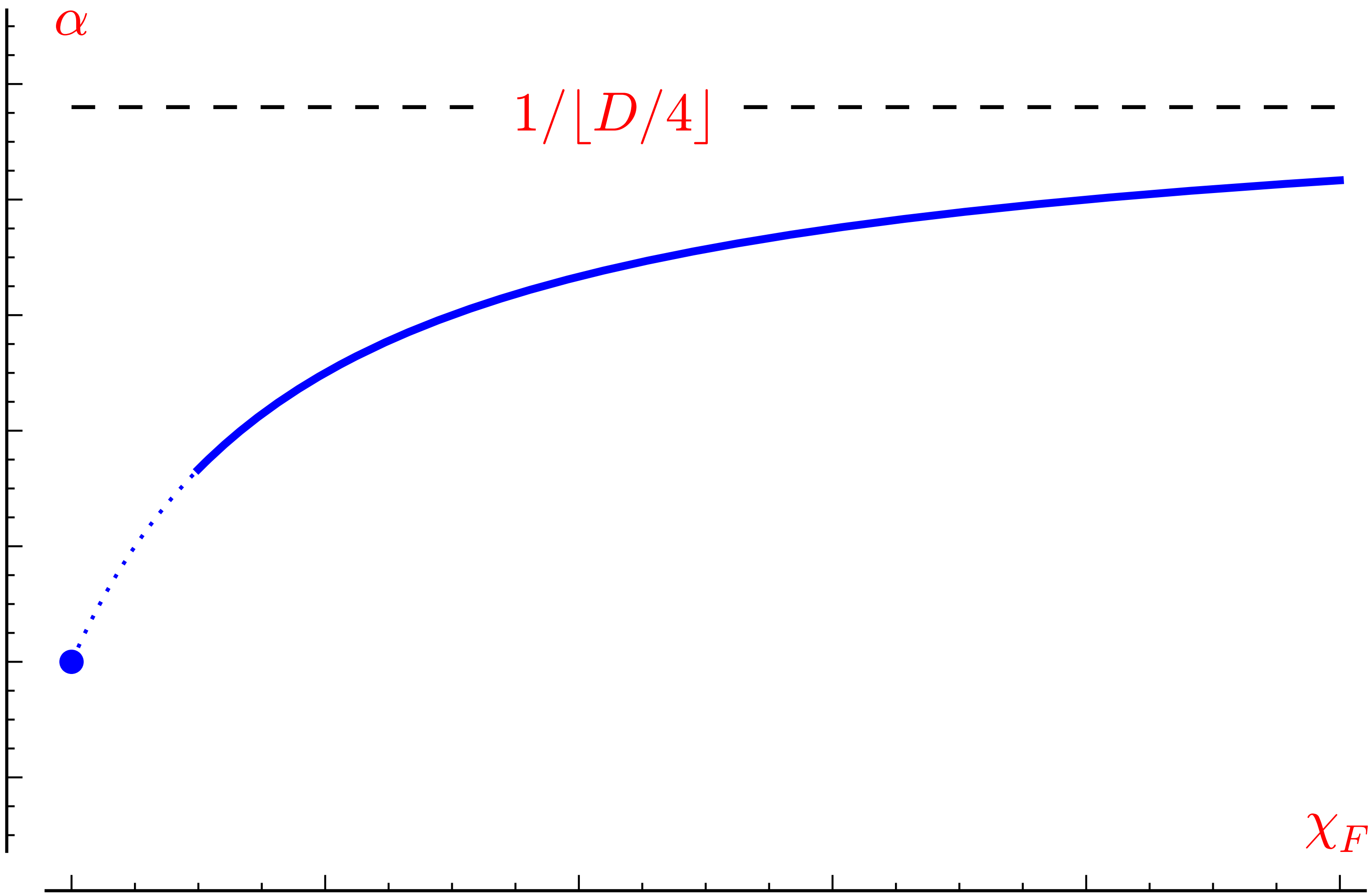
The picture for $D = 3$



The picture for $D = 4$



The picture for general $D \geq 4$



Almost the complete answer

- so we know the full answer for all $D \geq 3$,
  and for $\chi_F = 2$ or $\chi_F \geq 3$

- so what happens in the gap $2 < \chi_F < 3$?

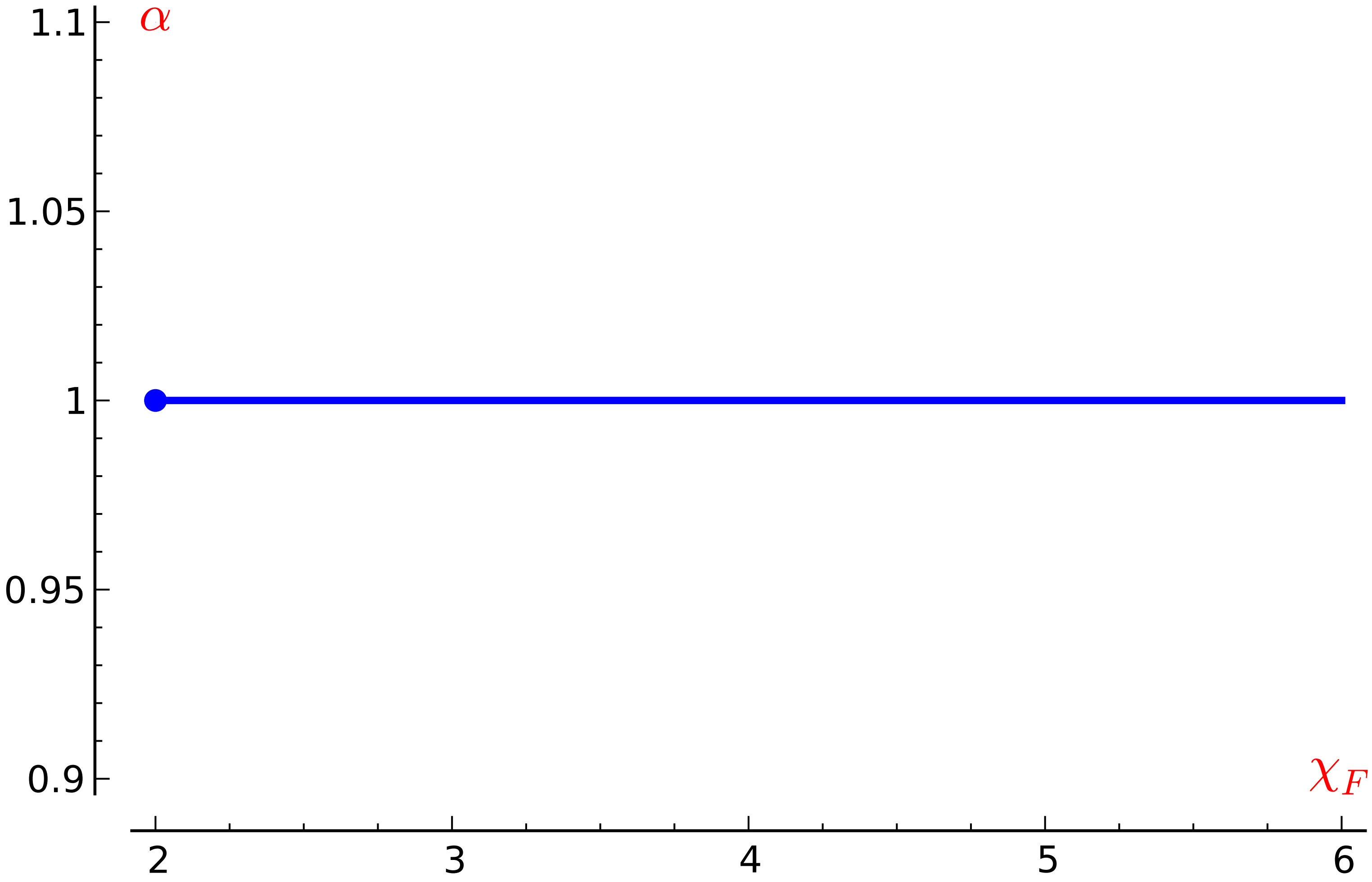
Some more answers

- the problem again :
  - we have some $W \subseteq V(G)$ with $\text{dist}(W) \geq D$
  - the vertices $w \in W$ are pre-coloured
    with $\phi(w) \subseteq [0, \chi_F + \alpha]$ of 'measure' 1
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$

Theorem

- for $D = 3$ we need : $\alpha \geq 1$
- and this bound is best possible for all $\chi_F \geq 2$

The full picture for $D = 3$

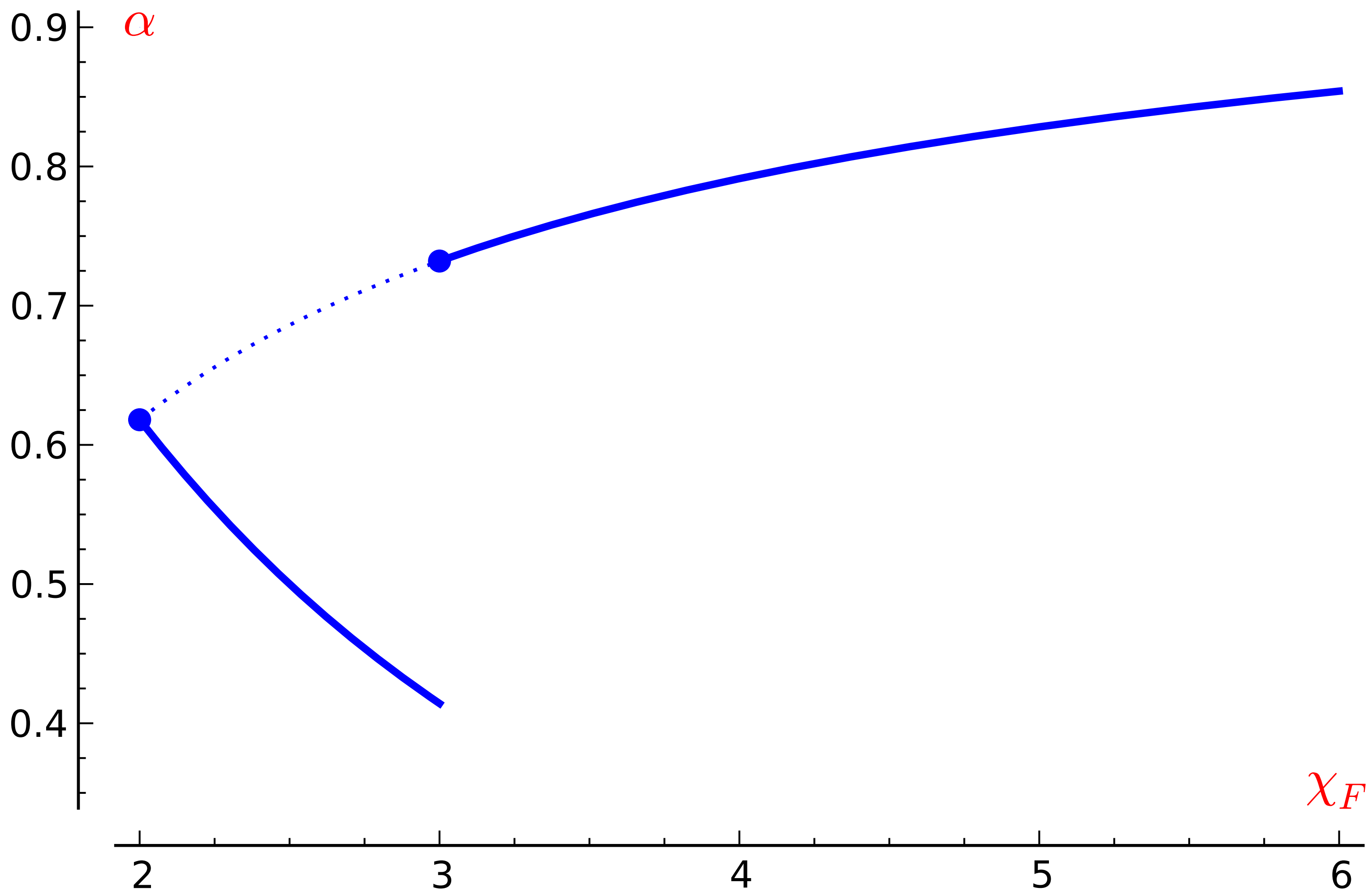


The answer for $D = 4$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 4$ we need :
  - $\alpha \geq \dfrac{\sqrt{(\chi_F - 1)^2 + 4\,(\chi_F - 1)} - \chi_F + 1}{2}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{\sqrt{(\chi_F - 1)^2 + 4} - \chi_F + 1}{2}$, for $2 \leq \chi_F < 3$
- and these bounds are best possible

The full picture for $D = 4$

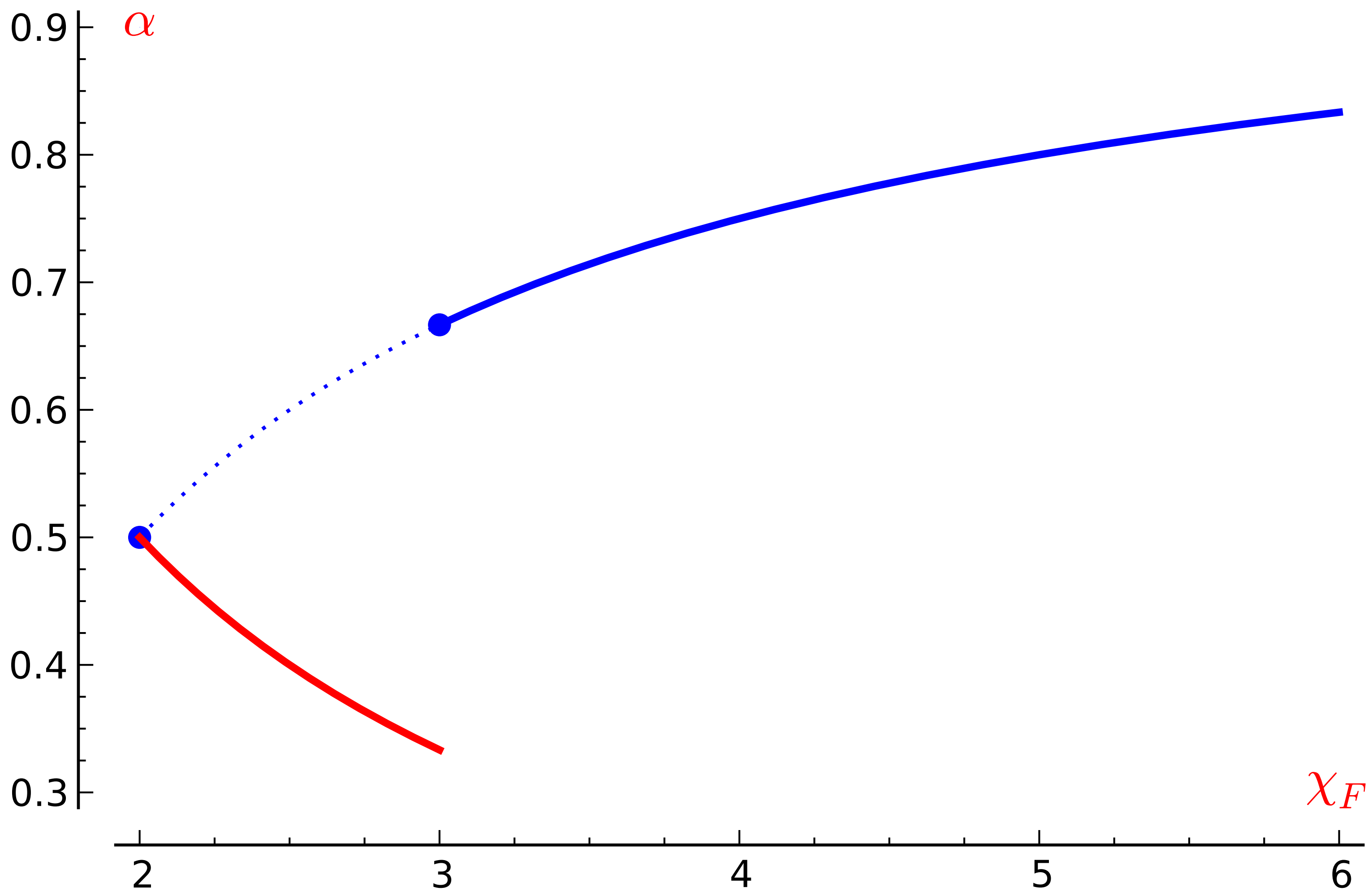


Almost the answer for $D = 5$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 5$ we need :
  - $\alpha \geq \dfrac{\chi_F - 1}{\chi_F}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{1}{\chi_F}$, for $2 \leq \chi_F < 3$
- but we don't know if the bound for $2 \leq \chi_F < 3$ is best possible

Almost the full picture for $D = 5$

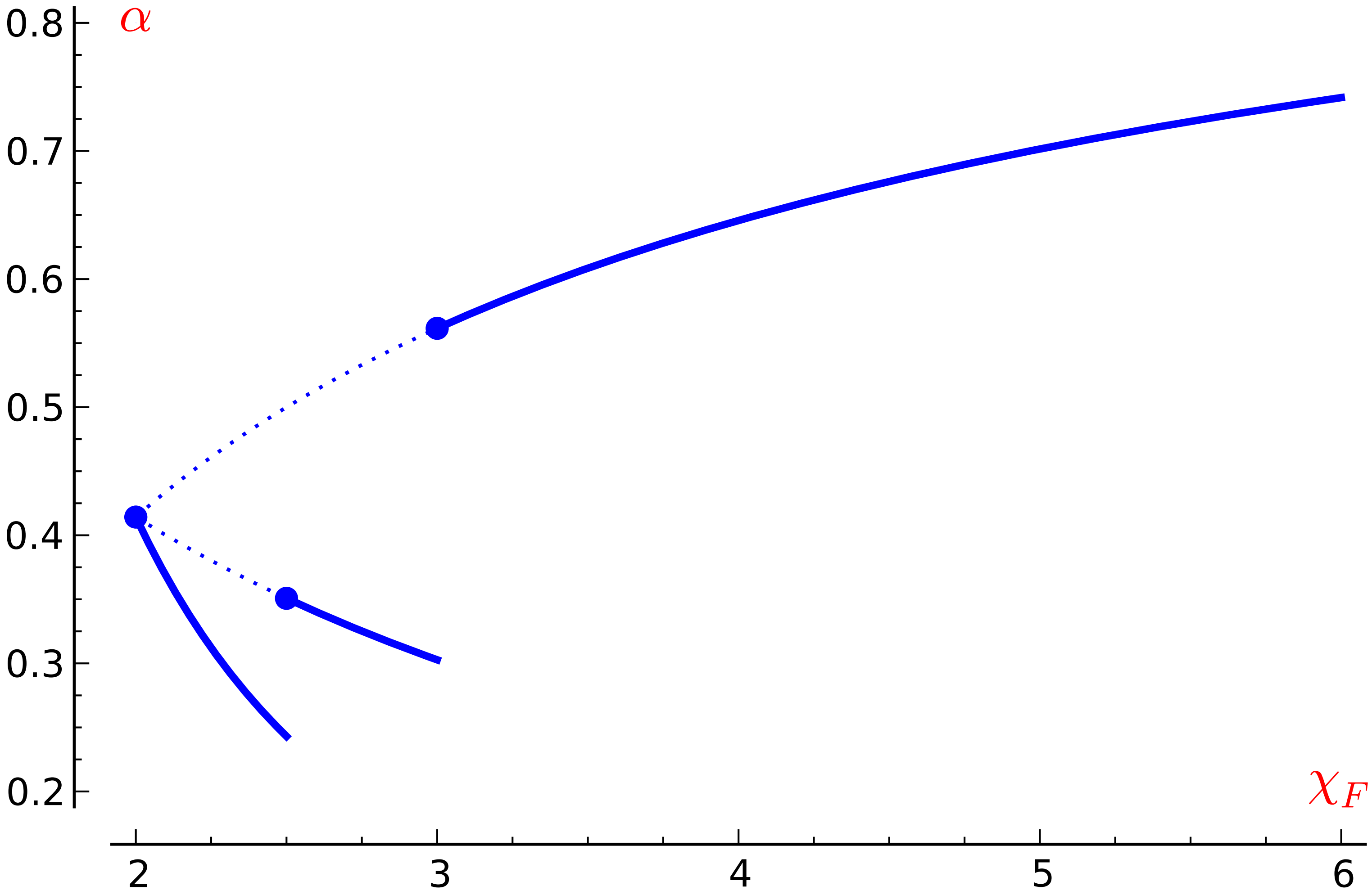


The answer for $D = 6$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 6$ we need :
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4\,(\chi_F - 1)} - \chi_F}{2}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4} - \chi_F}{2}$, for $2\frac{1}{2} \leq \chi_F < 3$
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4/(\chi_F - 1)} - \chi_F}{2}$, for $2 \leq \chi_F < 2\frac{1}{2}$
- and these bounds are best possible

The full picture for $D = 6$



And for $D \geq 7$

- for $D \geq 7$ we have no further precise results for $2 < \chi_F < 3$
  - but all indications are that it gets more and more complicated when D gets larger

- to understand the strange behaviour for $2 < \chi_F < 3$ we need to have a look at some aspects of the proof
  - and for that we need to have a further look at fractional colouring first

Fractional colouring again

- fractional K-colouring of graph G :
  - assignment of subsets $\phi(v) \subseteq [0, K]$ to $v \in V$ so that :
    - every subset $\phi(v)$ has 'measure' 1
    - and $uv \in E(G) \implies \phi(u) \cap \phi(v) = \varnothing$

Alternative definition for fractional colouring

- notation: $[m]$: the set $\{1, \ldots, m\}$

  $\binom{[m]}{q}$: the collection of q-subsets of $[m]$

- **(m, q)-colouring** of graph G $(1 \le q \le m)$:
  - every $v \in V$ is assigned a subset $\psi(v) \in \binom{[m]}{q}$, so that:
    - $uv \in E(G) \implies \psi(u) \cap \psi(v) = \varnothing$

- and then: $\chi_F(G) = \min\left\{ \dfrac{m}{q} \;\middle|\; G \text{ has an } (m, q)\text{-colouring} \right\}$

And another definition

- **Kneser graph** $\boldsymbol{Kn(m,q)}$:
  - vertices : all of $\binom{[m]}{q}$, edge $uv \iff u \cap v = \varnothing$
- G has an (m,q)-colouring

  $\iff$ there is a homomorphism $G \longrightarrow Kn(m,q)$
- $\chi_F(G) = \chi_F \iff$ there exist m, q with $\chi_F = \dfrac{m}{q}$,

  so that there is a homomorphism $G \longrightarrow Kn(m,q)$
  - we can interpret this as just a labelling of the vertices of G, using labels from $\binom{[m]}{q}$

Fractional colouring and Kneser graphs

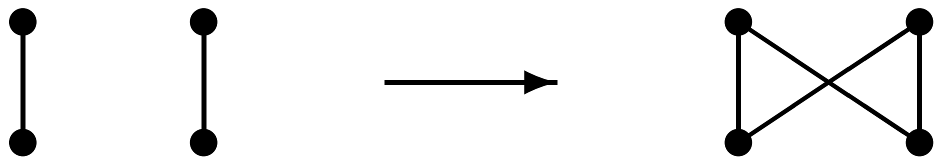
- so to understand fractional colouring,
  we can use Kneser graphs
- but we want to deal with pre-colouring
  of vertex sets with a minimum distance D
  - for that we need to build more complicated graphs
- in the rest of this talk we only look at the case D is even

Armed Kneser graphs

- given some m, q, D even, and an integer L
  - we start with a single $Kn(m,q)$ as a **core**
  - out of the core we grow L disjoint **arms**,

    each consisting of $D/2$ disjoint copies of $Kn(m,q)$
  - we link two consecutive copies of $Kn(m,q)$ in each arm as follows :
    - u_1 in copy 1 and v_2 in copy 2 :

$$u_1 \sim v_2 \quad \Longleftrightarrow \quad uv \text{ is an edge in } Kn(m,q)$$



Armed Kneser graphs

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$$u_1 \sim v_2 \quad \Longleftrightarrow \quad uv \text{ is an edge in } Kn(m, q)$$

- call the result the **armed Kneser graph** $\boldsymbol{a\text{-}Kn(m, q, D, L)}$

Using armed Kneser graphs

- - now suppose we have a graph G with $\chi_F(G) = \chi_F$

    so : $G \longrightarrow Kn(m, q)$, for some m, q with $\dfrac{m}{q} = \chi_F$

  - and a set $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$

- take the armed Kneser graph $a\text{-}Kn(m, q, D, |W|)$

  with $|W|$ arms

- we will map G to this armed Kneser graph

  - using the labels given by the homomorphism

    $$G \longrightarrow Kn(m, q)$$

Using armed Kneser graphs

- each $w \in W$ gets its own arm
  - map w in the copy of $Kn(m,q)$ at the end of its arm
  - map the neighbours of w in G

    in the copy of $Kn(m,q)$ one step closer to the core
  - map the neighbours of the neighbours of w in G

    in the next copy (closer to the core) of $Kn(m,q)$
  - etc.
- map all vertices at distance at least $D/2$ from w in G

  in the core of the armed Kneser graph

Using armed Kneser graphs

- this mapping of G in the armed Kneser graph satisfies :
  - images of elements of W have distance D
  - a pre-colouring of W gives a pre-colouring of the images of W
  - a fractional colouring of the armed Kneser graph can be mapped back to a fractional colouring of G

in other words :

- all aspects of fractional pre-colouring extensions of graphs are determined by fractional pre-colouring extensions of armed Kneser graphs !

Pre-colouring extensions of armed Kneser graphs

- - suppose we have an armed Kneser graph

    $a\text{-}Kn(m, q, D, L)$

  - and one pre-coloured vertex in the end of each arm

- in a fractional colouring extending that pre-colouring :
  - the core must ‘accommodate’ all arms
    - so will have to be given some ‘average’ colouring
  - so along the arms, the colouring extension must connect
    - the pre-coloured vertex in the end
    - with some ‘average’ colouring of the core

Colouring along an arm of an armed Kneser graph

- so consider an arm of an armed Kneser graph
  - with one pre-coloured vertex w' in its end
- the colouring along the arm is mostly determined by :
  - w' itself, in the end copy of $Kn(m, q)$
  - then by vertices in the 2 nd copy of $Kn(m, q)$ that are neighbours of w'
  - and then by vertices in the 3 rd copy of $Kn(m, q)$ that are neighbours of neighbours of w'
  - etc.

Now things get interesting

- w' is a vertex in $Kn(m, q)$, i.e., a q-subset of $[m]$
- its neighbours are the q-subsets of $[m]$ disjoint from w'
  - those neighbours together form a subgraph that is isomorphic to the Kneser graph $Kn(m-q, q)$
- for $\chi_F = \dfrac{m}{q} \geq 3$ we have
$$\chi_F\big(Kn(m-q, q)\big) = \frac{m-q}{q} = \chi_F - 1$$
- for $2 \leq \chi_F = \dfrac{m}{q} < 3$,
$Kn(m-q, q)$ has just isolated vertices
  - hence in those cases : $\chi_F\big(Kn(m-q, q)\big) = 1$

Now things get interesting

- so for $\chi_F = \geq 3$ we have

$$\chi_F\big(\text{set of neighbours of } w'\big) = \chi_F - 1$$

- while for $2 \leq \chi_F < 3$ we have

$$\chi_F\big(\text{set of neighbours of } w'\big) = 1$$

- this causes the difference between the two cases when $D = 4$

  (then the armed Kneser graph has arms of length 2, with the set of neighbours of w' in the middle)

And things get even more interesting

- next consider the set of neighbours of neighbours of w'
- for $\chi_F = \dfrac{m}{q} \geq 3$, this is the whole Kneser graph $Kn(m,q)$
- for $2 \leq \chi_F = \dfrac{m}{q} < 2\frac{1}{2}$,

  this is again a collection of isolated vertices
- but for $2\frac{1}{2} \leq \chi_F = \dfrac{m}{q} < 3$, it gets complicated
  - the structure is not a Kneser graph
  - its structure can vary even in cases where $\dfrac{m}{q} = \dfrac{m'}{q'}$

To summarise these findings

- when colouring along an arm of an an armed Kneser graph :
  - for $\chi_F = \dfrac{m}{q} \geq 3$, we are always dealing with structures that are Kneser graphs itself
  - for $2 \leq \chi_F = \dfrac{m}{q} < 3$, we have to consider structures that are not Kneser graphs
- we just seem to lack an understanding of the internal structure of Kneser graphs to deal with those latter cases in general