Extending Fractional Pre-colourings

JAN VAN DEN HEUVEL

joint work with: DAN KRÁL', MARTIN KUPEC, JEAN-SÉBASTIEN SERENI & JAN VOLEC

Department of Mathematics
London School of Economics and Political Science

The basics of graph colouring

- **vertex-colouring** with k colours :
  adjacent vertices must receive different colours

- **chromatic number** $\chi(G)$:
  minimum k so that a vertex-colouring exists

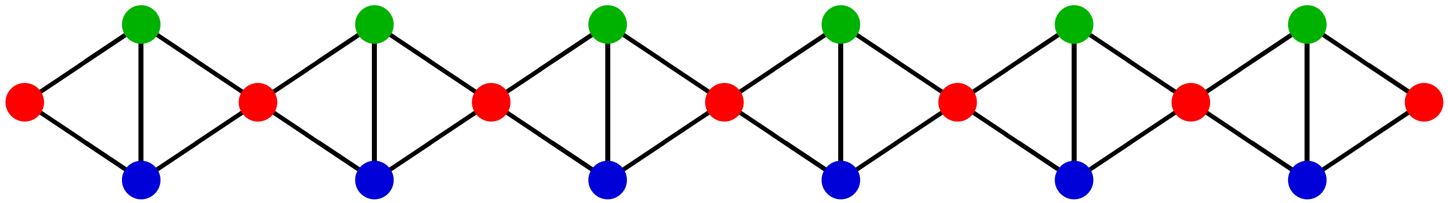
general question :

- what can we say if some vertices are already pre-coloured ?

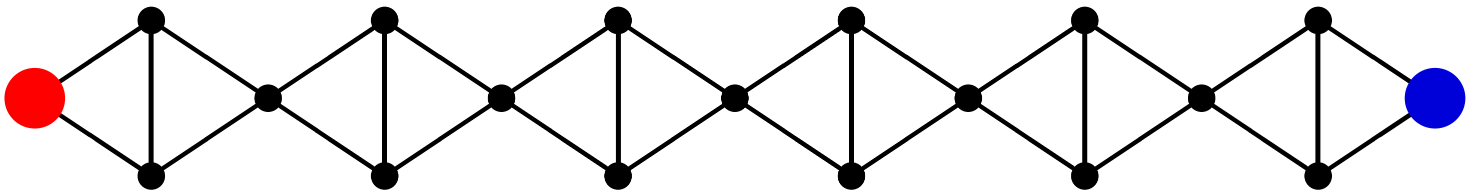
- in particular : will $\chi(G)$ colours still be enough ?

Not much chance

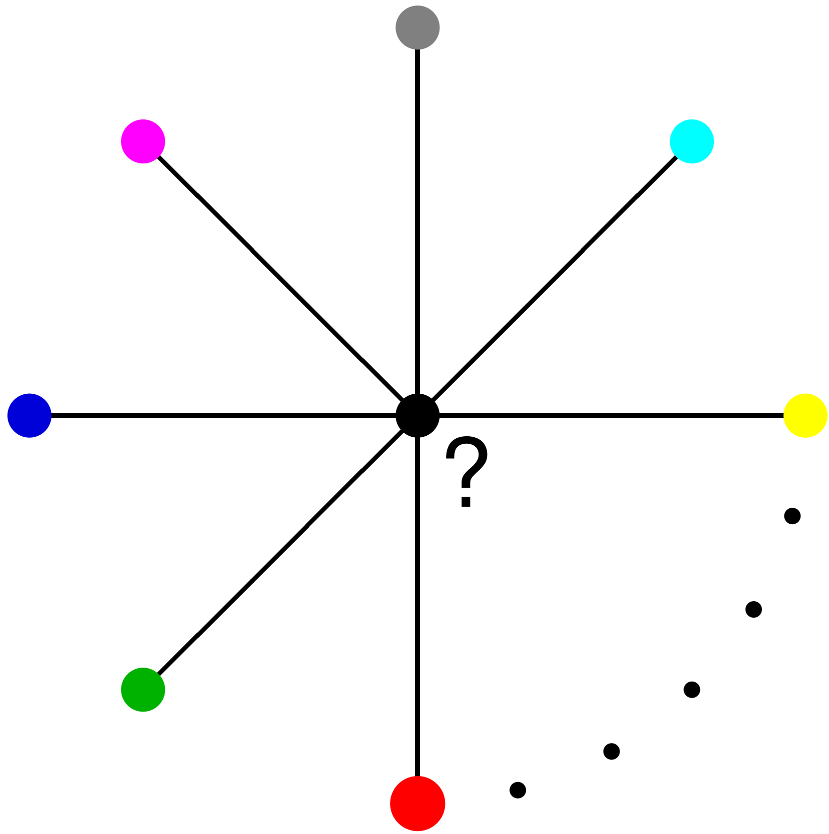
- **3-colourable graph** :



- but it can't be done with 3 colours if we start :



Not even if we have lots of extra colours



Pre-colouring questions

next best questions :

- what condition on pre-coloured vertices makes life easier ?
- and how many extra colours are needed then ?

- dist(P) : minimum distance between any two vertices in P

Question (Thomassen, 1997)

- G planar

  $P \subseteq V(G)$ a set of vertices with dist(P) at least 100

  - can any 5-colouring of P

    be extended to a 5-colouring of G ?

The first answer

Theorem (Albertson, 1998)

- G any graph, chromatic number χ

  $P \subseteq V(G)$ with $\text{dist}(P) \geq 4$

  $\Longrightarrow$ any $(\chi+1)$-colouring of P

  can be extended to a $(\chi+1)$-colouring of G

Some more answers

Theorem (Albertson, 1998)

- G planar graph

  $P \subseteq V(G)$ with $\text{dist}(P) \geq 3$

  $\Longrightarrow$ any 6-colouring of P

  can be extended to a 6-colouring of G

Theorem

- G any graph, chromatic number χ

  $P \subseteq V(G)$ with $\text{dist}(P) \geq 3$

  $\Longrightarrow$ any $(\chi+\chi)$-colouring of P

  can be extended to a $(\chi+\chi)$-colouring of G

A different kind of colouring

- **fractional K-colouring** of graph G ($K \in \mathbf{R}$, $K \geq 0$):
  - every vertex $v \in V$ is assigned a subset $\phi(v) \subseteq [0, K]$ so that:
    - every subset $\phi(v)$ has 'measure' 1
    - and $uv \in E(G) \implies \phi(u) \cap \phi(v) = \varnothing$

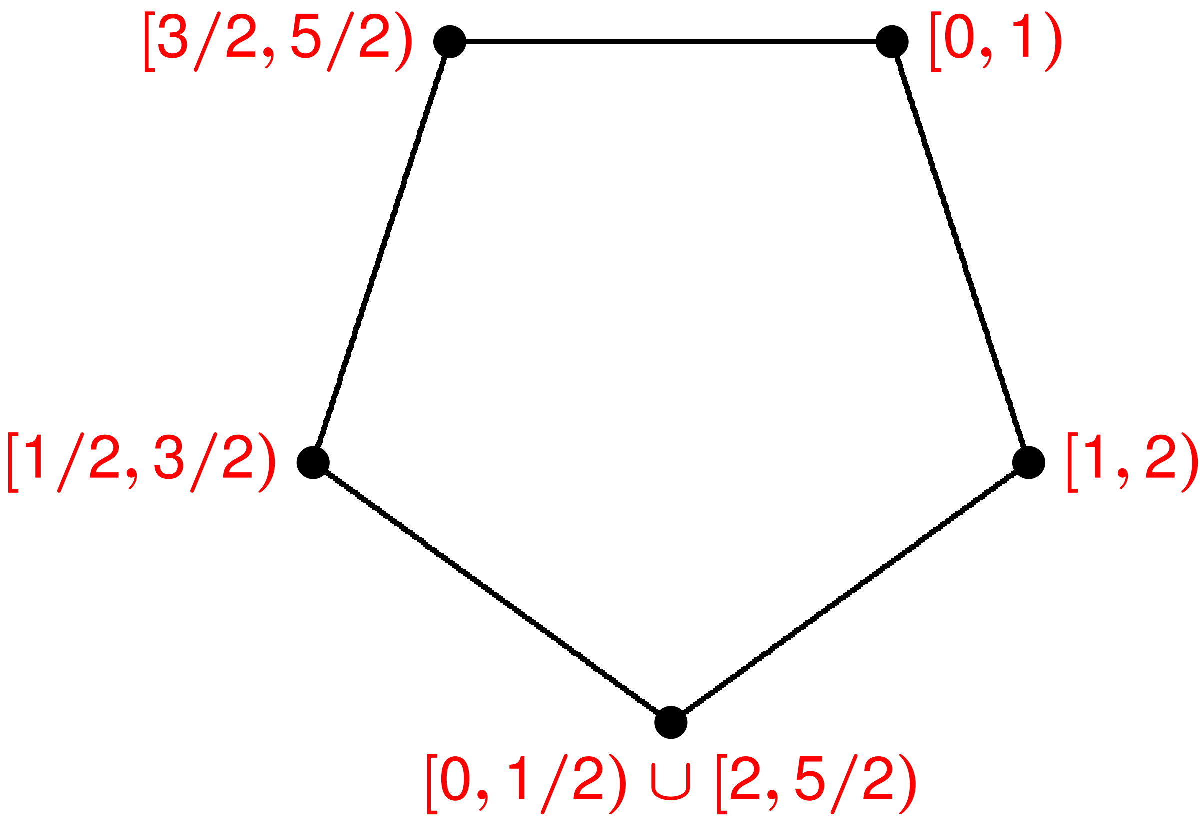
- **fractional chromatic number** $\chi_F(G)$:

$$= \inf \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

$$= \min \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

A different kind of colouring

- fractional $5/2$-colouring of C_5 :



A different kind of colouring

- **note** : we always have $\chi_F(G) \le \chi(G)$

- we have $\chi_F(G) \ge 2$ (except if G has no edges)
  - and every rational number $\chi_F \ge 2$ is possible

Pre-colouring in the fractional world

- so now suppose that for some vertices $P \subseteq V(G)$, the vertices in P are already pre-coloured :
  - vertices $p \in P$ have been given some set $\phi(p)$ of measure 1
- when can this be extended to a fractional colouring of the whole graph G ?
- in general we should expect to require more than $\chi_F(G)$ colours

The set-up of the problem

- - G a graph, fractional chromatic number $\chi_F \geq 2$
  - $D \geq 3$ an integer
  - $P \subseteq V(G)$ with $\mathrm{dist}(P) \geq D$
- - the vertices $p \in P$

    are pre-coloured with $\phi(p) \subseteq [0, \chi_F + \alpha]$
    - for some real $\alpha \geq 0$
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$
- how large should α be to make sure this can be done ?

A major part of the answer

Theorem (Král', Krnc, Kupec, Lužar & Volec, 2011)

- extension is always possible, provided α is at least :
  - $$\frac{\sqrt{(\lfloor D/4 \rfloor \chi_F - 1)^2 + 4 \lfloor D/4 \rfloor (\chi_F - 1)} - \lfloor D/4 \rfloor \chi_F + 1}{2 \lfloor D/4 \rfloor},$$ if $D \equiv 0 \bmod 4$
  - $$\frac{\chi_F - 1}{\lfloor D/4 \rfloor \chi_F},$$ if $D \equiv 1 \bmod 4$
  - $$\frac{\sqrt{(\lfloor D/4 \rfloor \chi_F)^2 + 4 \lfloor D/4 \rfloor (\chi_F - 1)} - \lfloor D/4 \rfloor \chi_F}{2 \lfloor D/4 \rfloor},$$ if $D \equiv 2 \bmod 4$
  - $$\frac{\chi_F - 1}{\lfloor D/4 \rfloor \chi_F + \chi_F - 1},$$ if $D \equiv 3 \bmod 4$

A major part of the answer

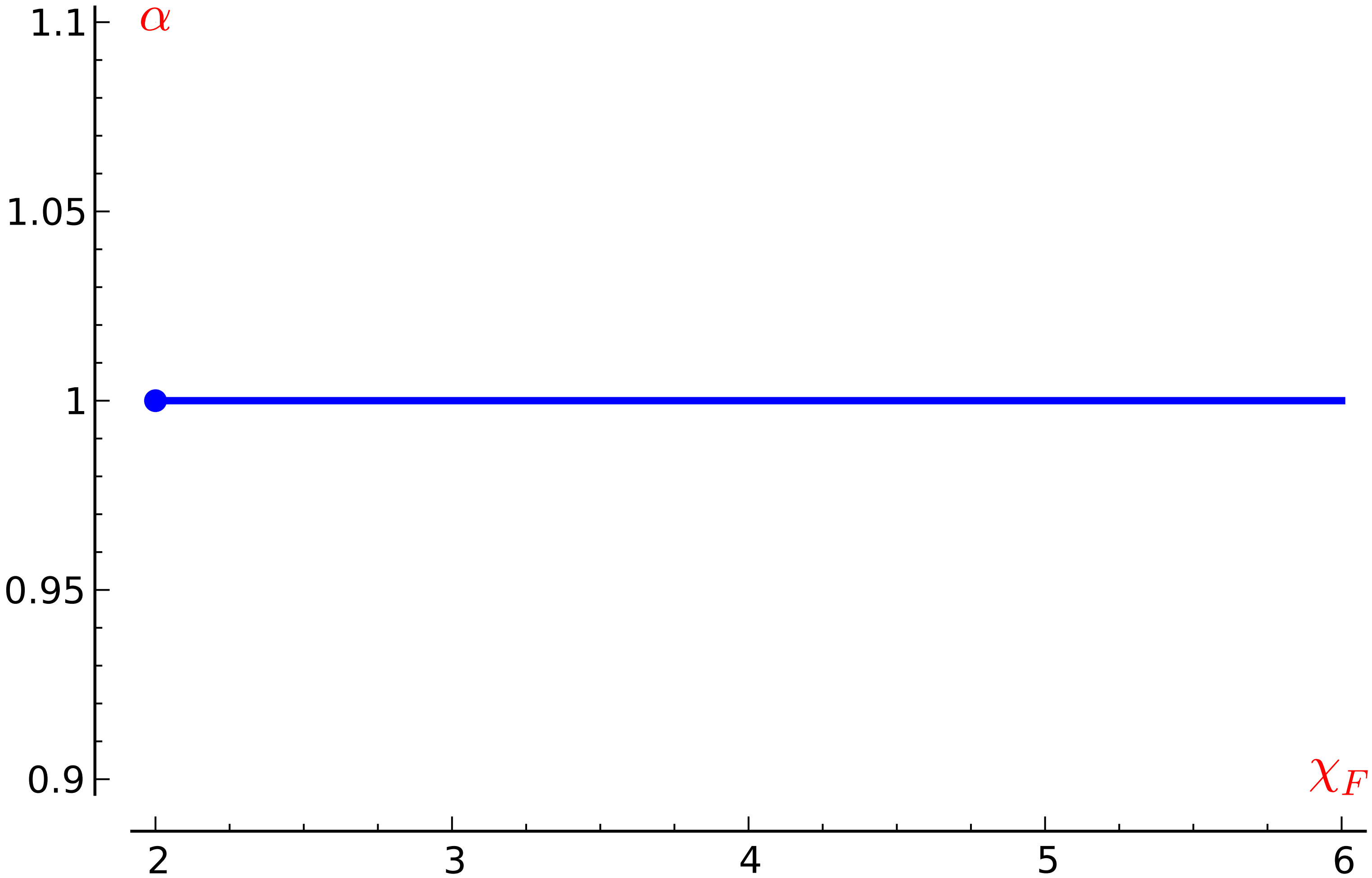
Theorem (Král', Krnc, Kupec, Lužar & Volec, 2011)

- moreover, these bounds on α are best possible,
  - if $D = 3$: for all $\chi_F \geq 2$
  - if $D \geq 4$: for $\chi_F = 2$ or $\chi_F \geq 3$

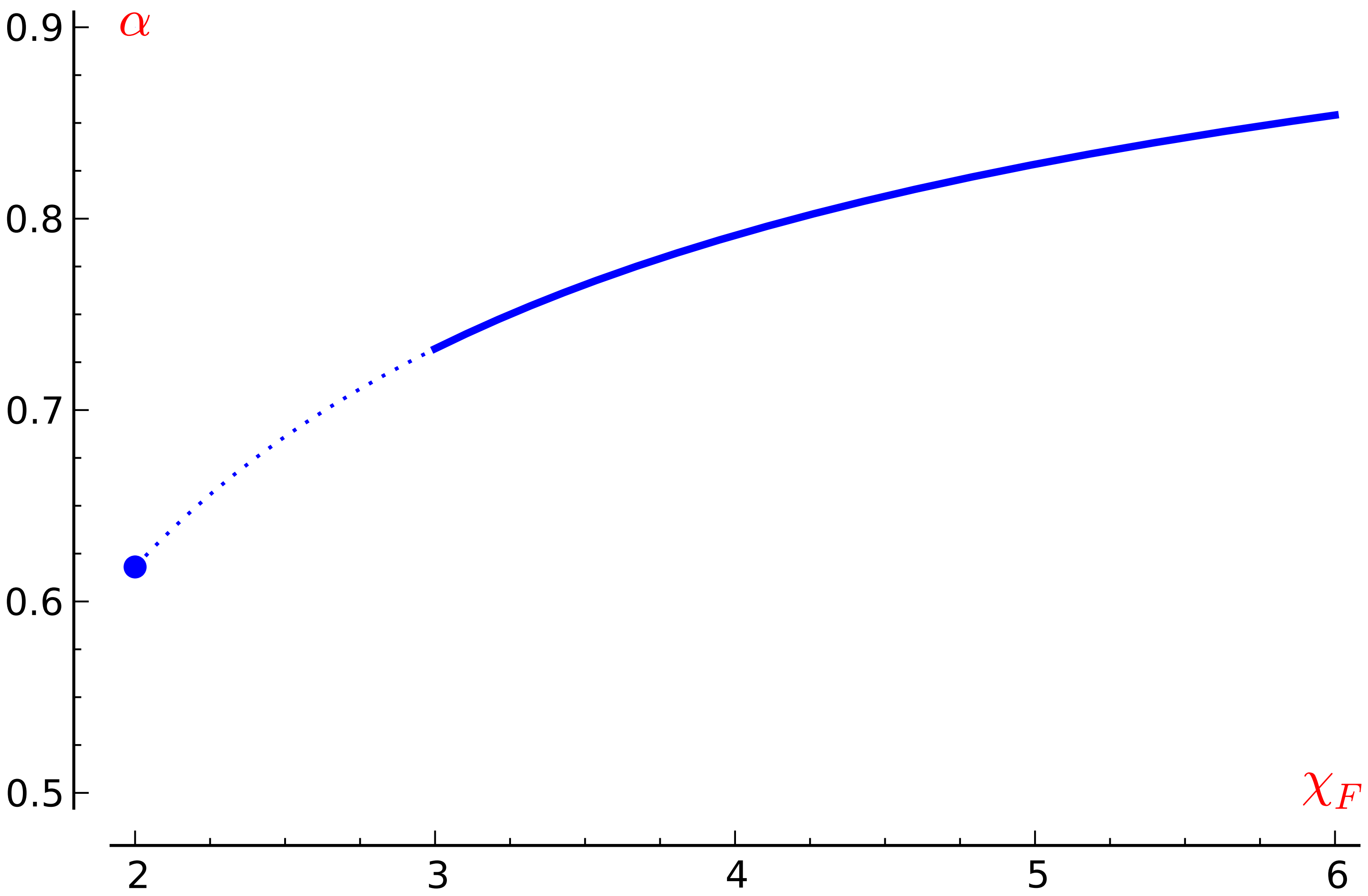
A major part of the answer

- **in other words** :
  - for all integers $D \geq 4$,
    all rational numbers $\chi_F \in \{2\} \cup [3, \infty)$,
    and all $\alpha \geq 0$ failing the bound for that D and χ_F
  - there is a graph G with fractional chromatic number χ_F,
    a subset $P \subseteq V(G)$ with $\text{dist}(P) \geq D$,
    and a fractional pre-colouring $\phi(p) \subseteq [0, \chi_F + \alpha]$
    for $p \in P$
  - so that ϕ cannot be extended to a fractional colouring
    of the whole G, using colours from $[0, \chi_F + \alpha]$ only

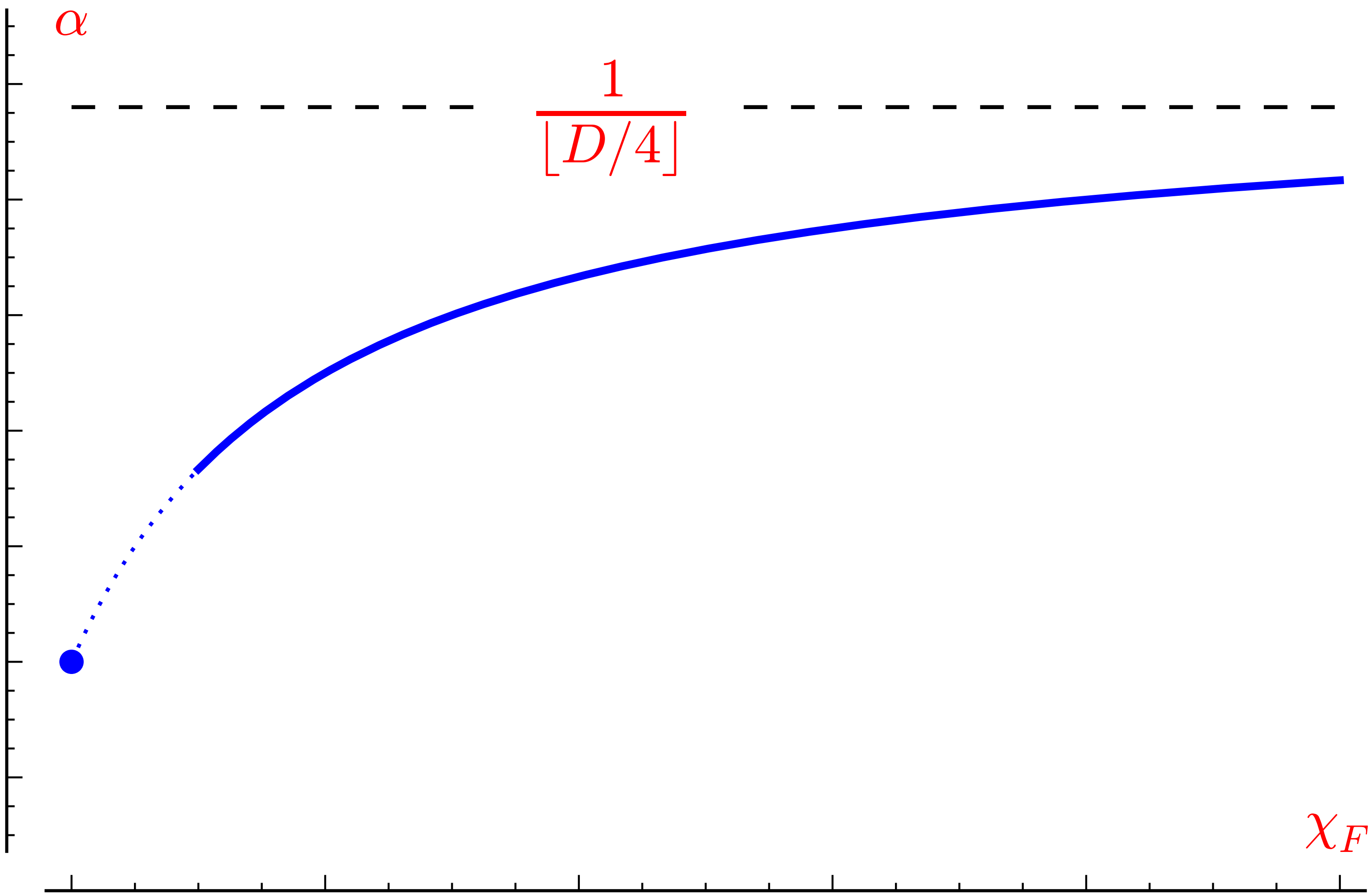
The picture for $D = 3$



The picture for $D = 4$



The picture for general $D \geq 4$



Almost the complete answer

- so we know the complete answer for all $D \geq 4$, and for $\chi_F = 2$ or $\chi_F \geq 3$

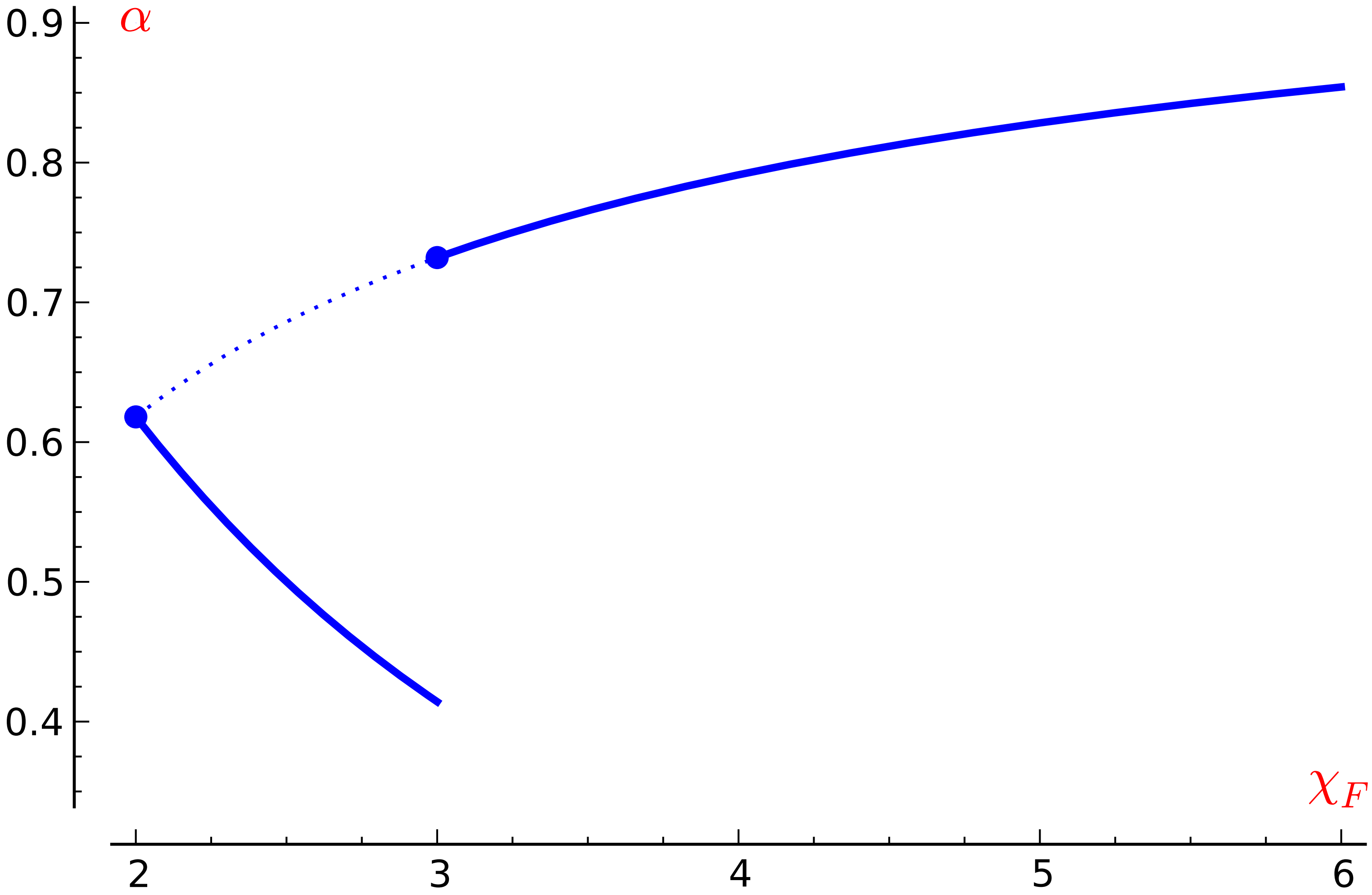
- so what happens in the gap $2 < \chi_F < 3$?

The complete answer for $D = 4$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 4$ we need :
  - $\alpha \geq \dfrac{\sqrt{(\chi_F - 1)^2 + 4\,(\chi_F - 1)} - \chi_F + 1}{2}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{\sqrt{(\chi_F - 1)^2 + 4} - \chi_F + 1}{2}$, for $2 \leq \chi_F < 3$
- and these bounds are best possible

The complete picture for $D = 4$

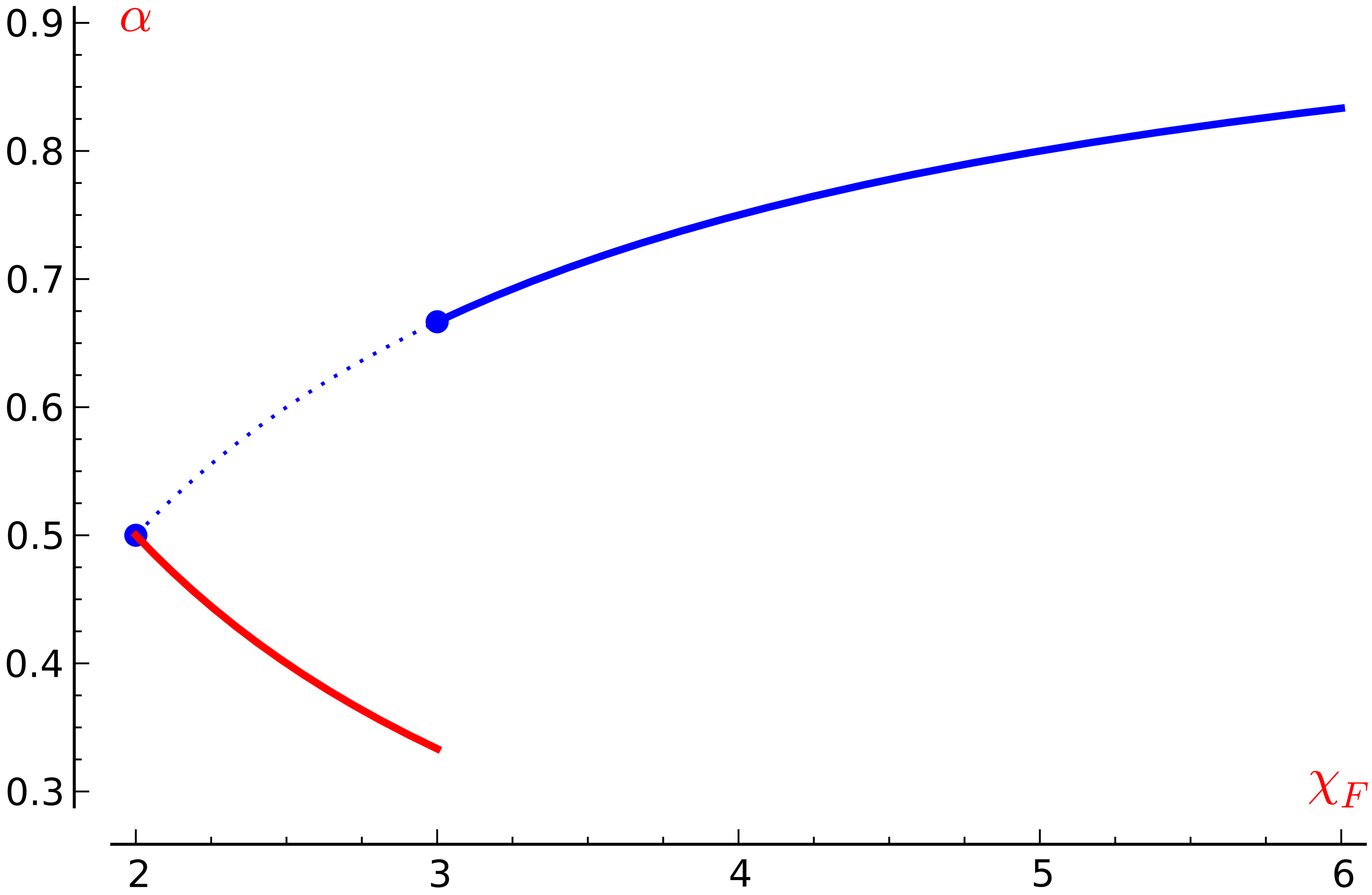


Almost the complete answer for $D = 5$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 5$ we need :
  - $\alpha \geq \dfrac{\chi_F - 1}{\chi_F}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{1}{\chi_F}$, for $2 \leq \chi_F < 3$
- but we don't know if the bound for $2 \leq \chi_F < 3$ is best possible

Almost the complete picture for $D = 5$

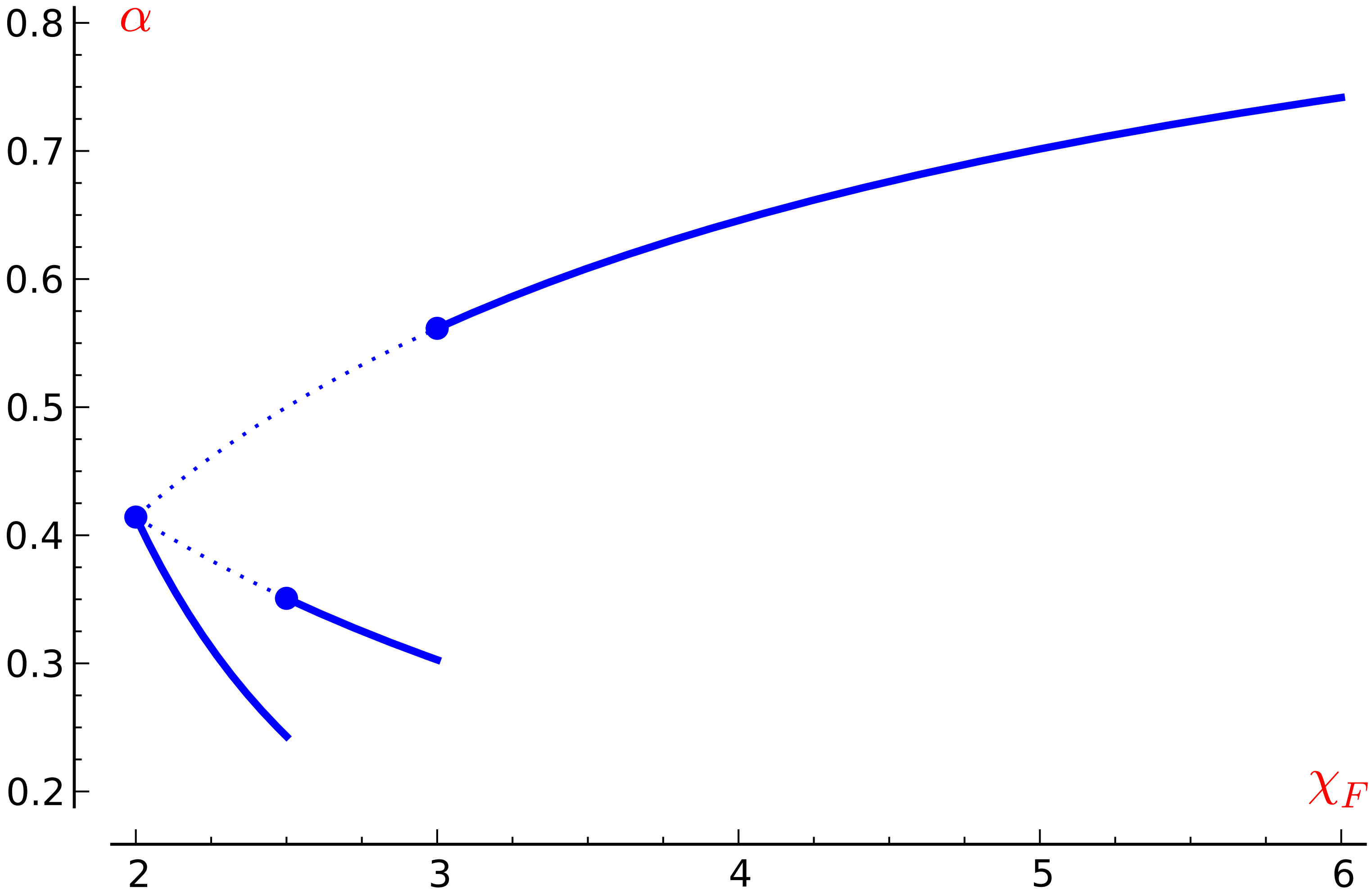


The complete answer for $D = 6$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 6$ we need :
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4\,(\chi_F - 1)} - \chi_F}{2}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4} - \chi_F}{2}$, for $2\frac{1}{2} \leq \chi_F < 3$
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4/(\chi_F - 1)} - \chi_F}{2}$, for $2 \leq \chi_F < 2\frac{1}{2}$
- and these bounds are best possible

The complete picture for $D = 6$



And for $D \geq 7$

- for $D \geq 7$ we have no further precise results for $2 < \chi_F < 3$
  - but all indications are that it gets more and more complicated when D gets larger

Alternative definition for fractional colouring

- **Kneser graph** $\boldsymbol{Kn(m,q)}$:
  - vertices : the collection of q-subsets of $\{1,\ldots,m\}$
  - uv an edge : $u \cap v = \varnothing$

- **note** : $\chi_F(Kn(m,q)) = \begin{cases} m/q, & \text{if } m \geq 2\,q \\ 1, & \text{if } q \leq m < 2\,q \end{cases}$

- **and** : $\chi_F(G) = \chi_F$

  $\iff$ there is a homomorphism $G \longrightarrow Kn(m,q)$

  for some m,q with $\chi_F = m/q$

Fractional colouring and Kneser graphs

- so to understand fractional colouring,
  we can use Kneser graphs
- but we want to deal with pre-colouring
  of vertex sets with a minimum distance D
  - for that we need to build more complicated graphs
  - by ‘gluing’ Kneser graphs together

Pre-colouring involving Kneser graphs

- so consider a Kneser graph
  - with one pre-coloured vertex v
- to extend this to a colouring of the whole graph :
  - next consider the vertices in $Kn(m, q)$ that are neighbours of v
  - then the neighbours of the neighbours of v
  - etc.

Now things get interesting

- v is a vertex in $Kn(m, q)$, i.e., a q-subset of $\{1, \ldots, m\}$
- its neighbours are the q-subsets that are disjoint from v
  - those neighbours together form a subgraph
    that is isomorphic to the Kneser graph $Kn(m - q, q)$
- so for $\chi_F = \dfrac{m}{q} \geq 3$:
  $$\chi_F(\text{set of neighbours of } v) = \frac{m-q}{q} = \chi_F - 1$$
- while for $2 \leq \chi_F = \dfrac{m}{q} < 3$:
  $$\chi_F(\text{set of neighbours of } v) = \chi_F(\text{isolated vertices}) = 1$$

And things get even more interesting

- next consider the set of neighbours of neighbours of v
- for $\chi_F = \dfrac{m}{q} \geq 3$, this is the whole Kneser graph $Kn(m,q)$
- for $2 \leq \chi_F = \dfrac{m}{q} < 2\frac{1}{2}$,

  this is again a collection of isolated vertices
- but for $2\frac{1}{2} \leq \chi_F = \dfrac{m}{q} < 3$, it gets complicated
  - the structure is not a Kneser graph
  - its structure can vary even in cases where $\dfrac{m}{q} = \dfrac{m'}{q'}$

To summarise these findings

- when extending a fractional pre-colouring of graphs that are formed by 'gluing' together Kneser graphs :
  - for $\chi_F = \dfrac{m}{q} \geq 3$, we are always dealing with structures that are Kneser graphs itself
  - for $2 \leq \chi_F = \dfrac{m}{q} < 3$, we have to consider structures that are not Kneser graphs
- we just seem to lack an understanding of the internal structure of Kneser graphs to deal with those latter cases in general