Fire Containment in Planar Graphs

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Fire in a graph
Fire in a graph – it starts at some vertex
Fire in a graph – and then spreads to its neighbours
Fire in a graph – and to their neighbours
Fire in a graph – and their neighbours . . .
Fire in a graph – until it’s all gone
Firefighters to the rescue!

- suppose we have some **firefighters**
- one firefighter can:
  - at any time step move to any vertex
  - protect that vertex, which will **stay protected for ever** afterwards
The start of the fire again
Two firefighters at work
The fire spreads a bit
The firefighters continue their work
The fire spreads ...
One more move from a firefighter
The danger is over …
The Firefighter Problem

- given some input information
  such as: the graph,
  the vertex where the fire starts,
  the number of firefighters per step, etc.

- possible aims:
  - maximise the number of saved vertices
  - minimise the time until the fire is under control
  - minimise the number of firefighters needed to protect a given number of vertices
A typical problem: the solo firefighter

- **Input:** graph $G$, vertex $v$, integer $K$
  
  **Question:** if a fire starts at $v$, can a single firefighter save at least $K$ vertices?

- this problem is **NP-complete** (MacGillivray & Wang, 2003)
  
  - even restricted to **trees with maximum degree 3**
A small side-step: infinite grids

Theorem

(Wang & Moeller, 2002; Fogarty, 2003; Develin & Hartke, 2007)

- for a $d$-dimensional grid $\mathbb{Z}^d$ we have
  - $d = 1$ or $d \geq 3 \implies$ one fire can be contained by $2d - 1$ firefighters (in 2 steps)
  - $d = 2 \implies$ one fire can be contained by 2 firefighters (in 8 steps)
  - all numbers are best possible
The square grid with three firefighters
The square grid with three firefighters
The square grid with three firefighters
The square grid with three firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with **two** firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters
The square grid with two firefighters – done!
suppose we only know the graph is from some graph class (say, the graph is planar)

and we want to save “most” of the graph

if we always want to do this, we may need many firefighters
Fire containment in graph classes

- what
  if we want to save “most” of the graph, “most of the time”? 

- suppose we have \( k \) firefighters
  - \( \rho_k(G, v) : \) proportion of vertices of \( G \) that can be saved with \( k \) firefighters if the fire starts in vertex \( v \)
  - \( \rho_k(G) : \) expected value of \( \rho_k(G, v) \) if \( v \) is chosen uniformly at random
    \[
    \rho_k(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \rho_k(G, v)
    \]
what we really want to answer for some graph class $G$:

what is the minimum $k$ such that $\inf_{G \in G} \rho_k(G) > 0$?

in other words:

what is the minimum $k$ such that for every $G \in G$:

- for a positive fraction of the vertices in $G$, if a fire starts in one of these vertices, we can save a positive fraction of $G$, using at most $k$ firefighters at each step

let's call that the firefighter number $\text{ff}(G)$ of the class $G$
The survival rate of some graph classes

**Theorem**  (Cai, Cheng, Verbin & Zhou, 2009+)

- $G$ outerplanar, $n$ vertices $\implies \rho_1(G) \geq 1 - O(\log n / n)$

- *outerplanar*: can be drawn in the plane with all vertices on the outside face
The survival rate of some graph classes

Theorem (Cai, Cheng, Verbin & Zhou, 2009+)

- $G$ outerplanar, $n$ vertices $\implies \rho_1(G) \geq 1 - O(\log n / n)$

- **outerplanar**: can be drawn in the plane with all vertices on the outside face

- hence for **outerplanar graphs** $OP : \text{ff}(OP) = 1$

- note that **trees** are outerplanar
The survival rate of some graph classes

**Theorem**  (Wang, Finbow & Wang, 2010)

- \(|E(G)| \leq \ell |V(G)|\), for some integer \(\ell\)

\[\Rightarrow \rho_{2\ell-1}(G) \geq \frac{2}{5}\ell\]

**Corollary**

- \(G\) planar \(\Rightarrow\) \(|E(G)| < 3|V(G)|\)

\[\Rightarrow \rho_5(G) \geq \frac{2}{15}\]

- hence for planar graphs \(\mathcal{P}\): \(ff(\mathcal{P}) \leq 5\)

**Question**

- what is the firefighter number \(ff(\mathcal{P})\) of planar graphs?
The survival rate of planar graphs

Observation

- $\text{ff}(\mathcal{P}) \geq 2$:
The survival rate of planar graphs

Observation
- $ff(P) \geq 2$

Theorem
- $ff(P) \leq 4$

- In fact, we can prove: $ff(P) \leq "3 + \varepsilon"$
  - We need 4 firefighters in the first step only, for each following step we need only 3 firefighters
The survival rate of triangle-free planar graphs

**Theorem**

- If $G$ is triangle-free planar, then $\rho_2(G) \geq 1/238320$
- So for triangle-free planar graphs $\mathcal{P}_4$: $ff(\mathcal{P}_4) \leq 2$
- This is best possible:

- So: $ff(\mathcal{P}_4) = 2$
Some ideas from the proofs

- the proofs consist of two main steps:
  - find a collection of “defendable configurations”
    - a subgraph with a given vertex \( v \)
      so that if the fire starts in \( v \),
      only a finite number of vertices will be lost
  - show that there is a constant \( \alpha > 0 \),
    so that every graph \( G \) in the class
    has at least \( \alpha \cdot |V(G)| \) defendable configurations
    - uses the discharging method,
      but in a non-standard way
Defendable configurations for 2 firefighters
And another defendable configuration for 2 firefighters
The discharging method

- traditionally used to show that any planar graph:
  - contains at least one subgraph from a set of “good” configurations

- we needed to modify it to show that any planar graph:
  - contains many subgraphs (in fact, linearly many) from the set of defendable configurations
    - our techniques are not that different from the traditional method
    - but we need to be much more precise
The firefighter number of planar graphs

- so now we know: $2 \leq ff(P) \leq 4$

- next step: prove that $ff(P) \leq 3$
  - i.e., get rid of the extra firefighter needed in the first step
  - maybe possible to extend our proof, using some more careful (“messier”) analysis

- and then, can we go to: $ff(P) = 2$?
  - very likely to need new ideas
The challenge of $\mathcal{f}(P) \leq 2$

- sometimes the firefighters need to act locally
But sometimes acting *locally* doesn’t work
But sometimes acting *locally* doesn’t work

- looks like a sensible start ...
But sometimes acting *locally* doesn’t work

- looks like a sensible start …
But sometimes acting *locally* doesn’t work

- continue with that strategy ...
But sometimes acting *locally* doesn’t work

but what happens on the boundary ... ?
Sometimes we must think *globally*
Sometimes we must think *globally*
Sometimes we must think **globally**
Sometimes we must think *globally*
Sometimes we must think **globally**
But what to do in general?
The firefighter number of planar graphs

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- and then, can we go to: $ff(P) = 2$?
  - very likely to need new ideas
  - and is it even true . . . ?