Fire Containment in Planar Graphs

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joint work with :

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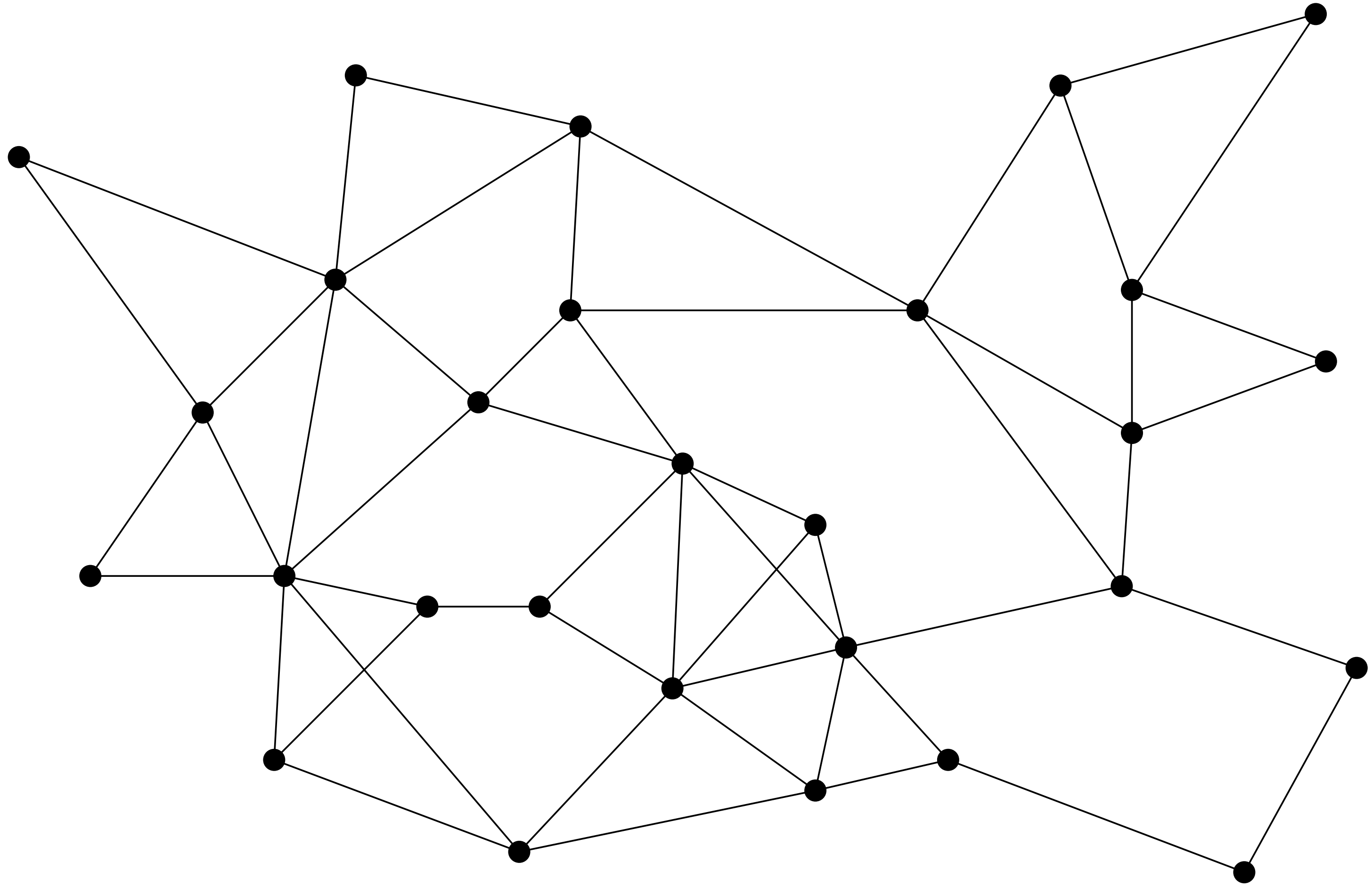
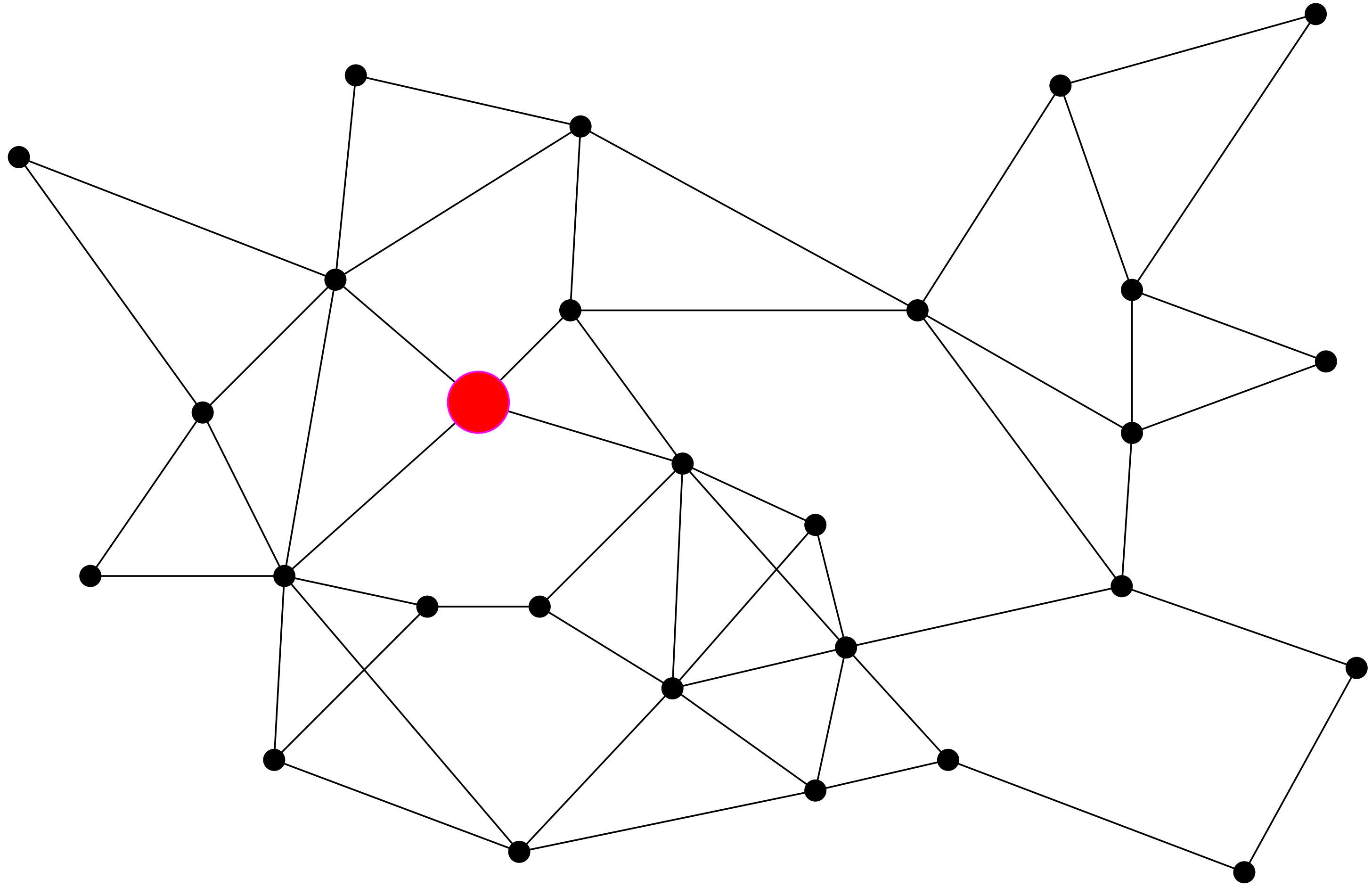
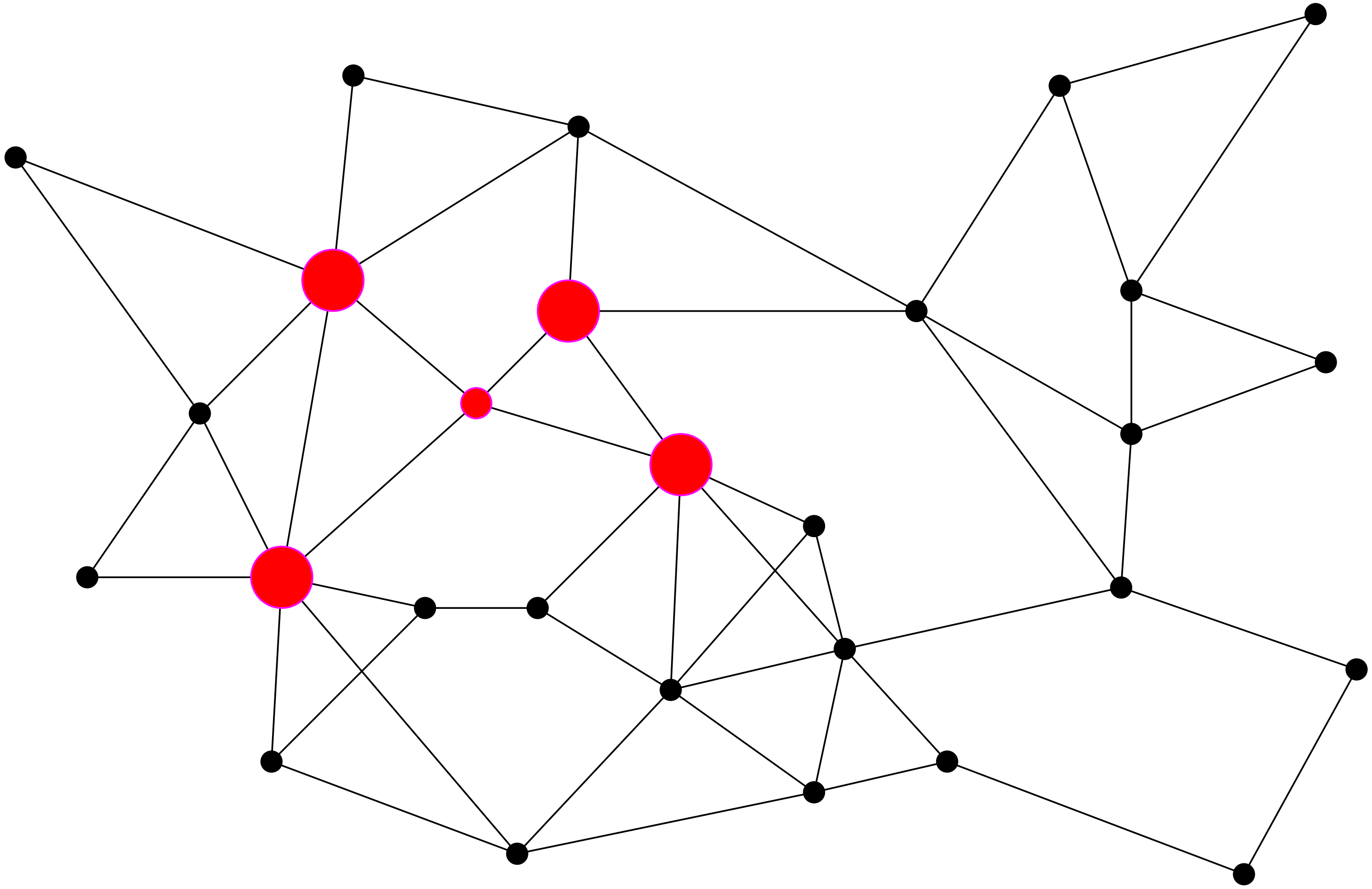
Fire in a graph



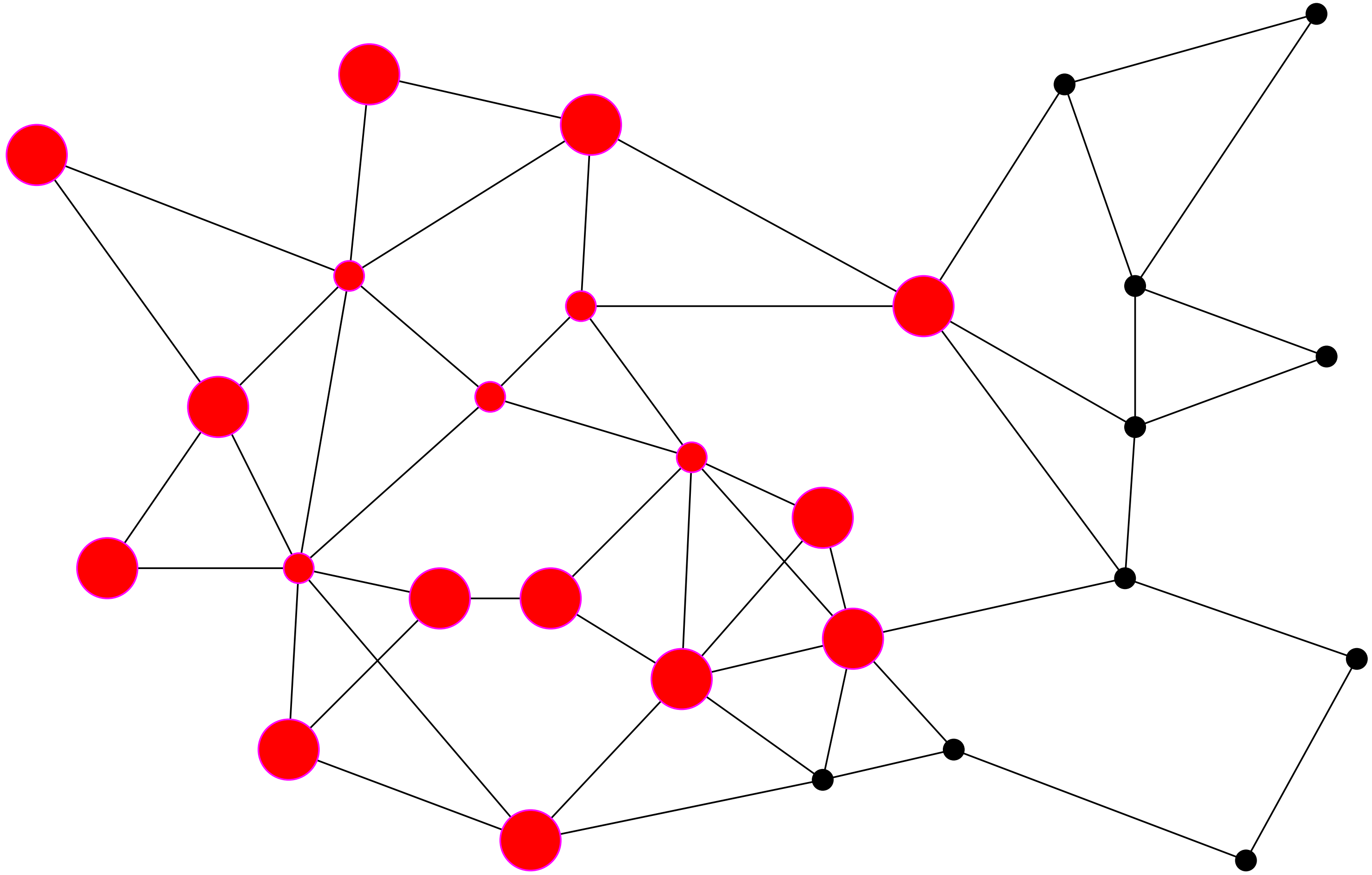
Fire in a graph – it starts at some vertex



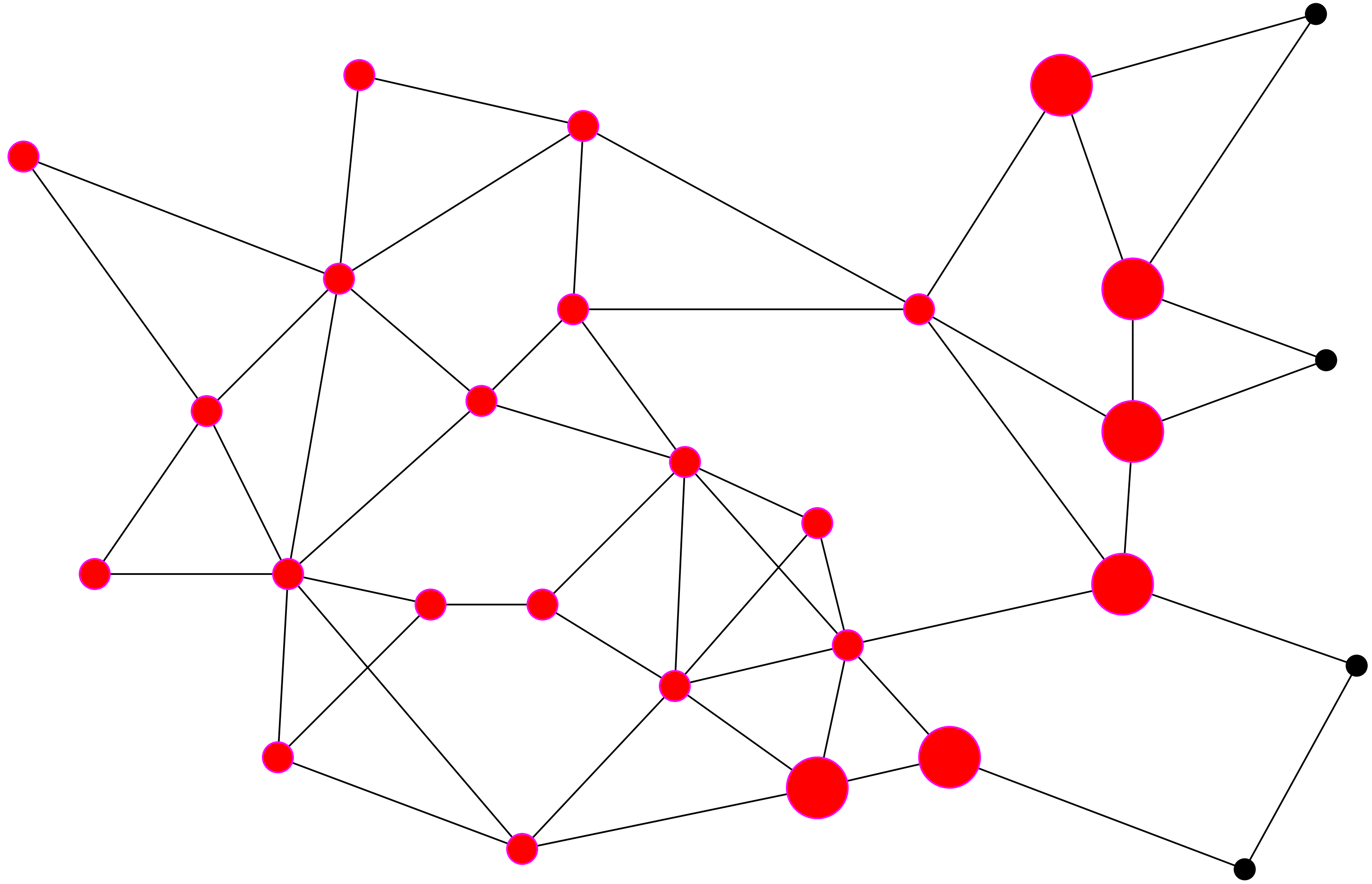
Fire in a graph – and then spreads to its neighbours



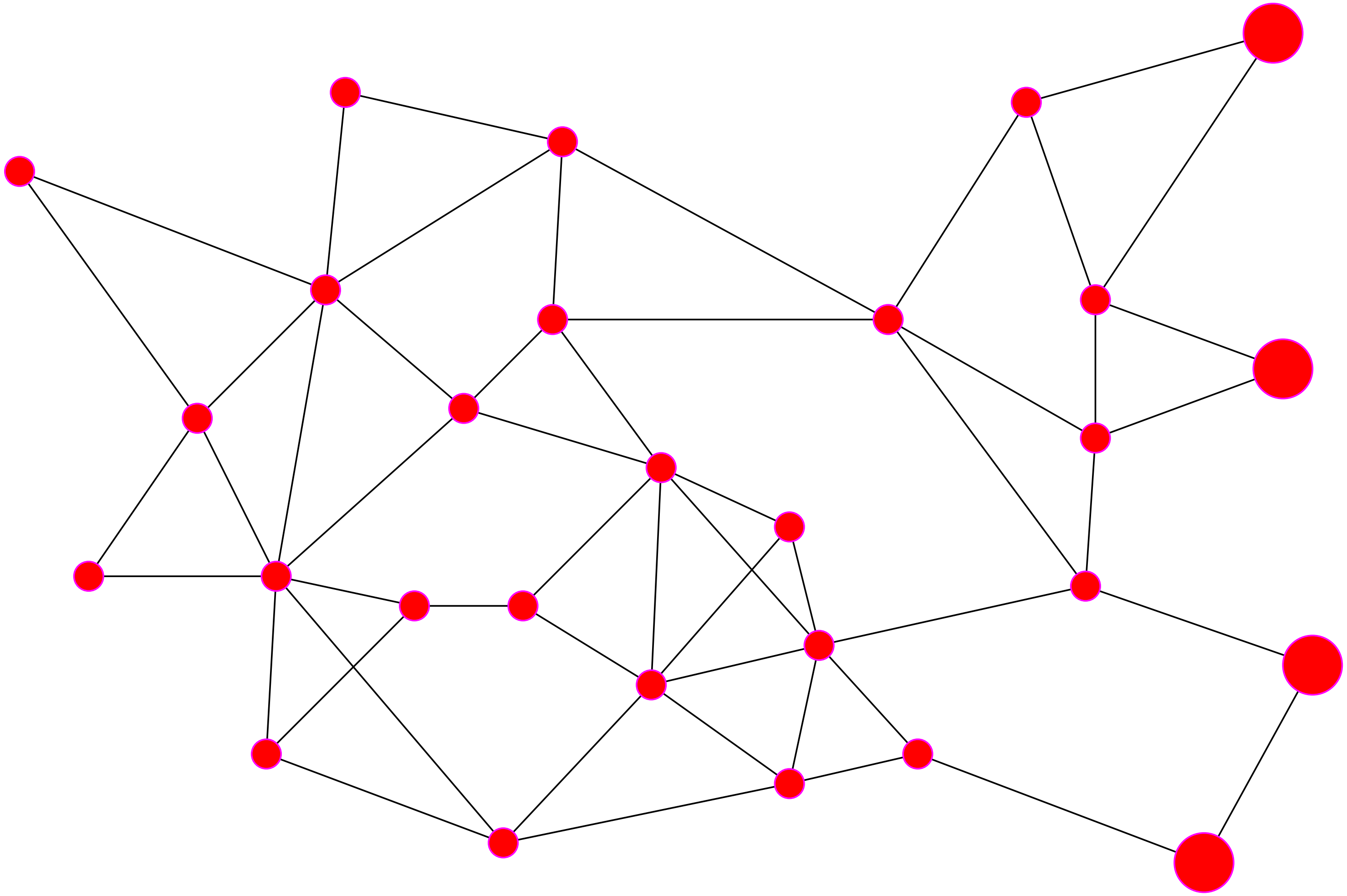
Fire in a graph – and to their neighbours



Fire in a graph – and their neighbours . . .



Fire in a graph – until it's all gone

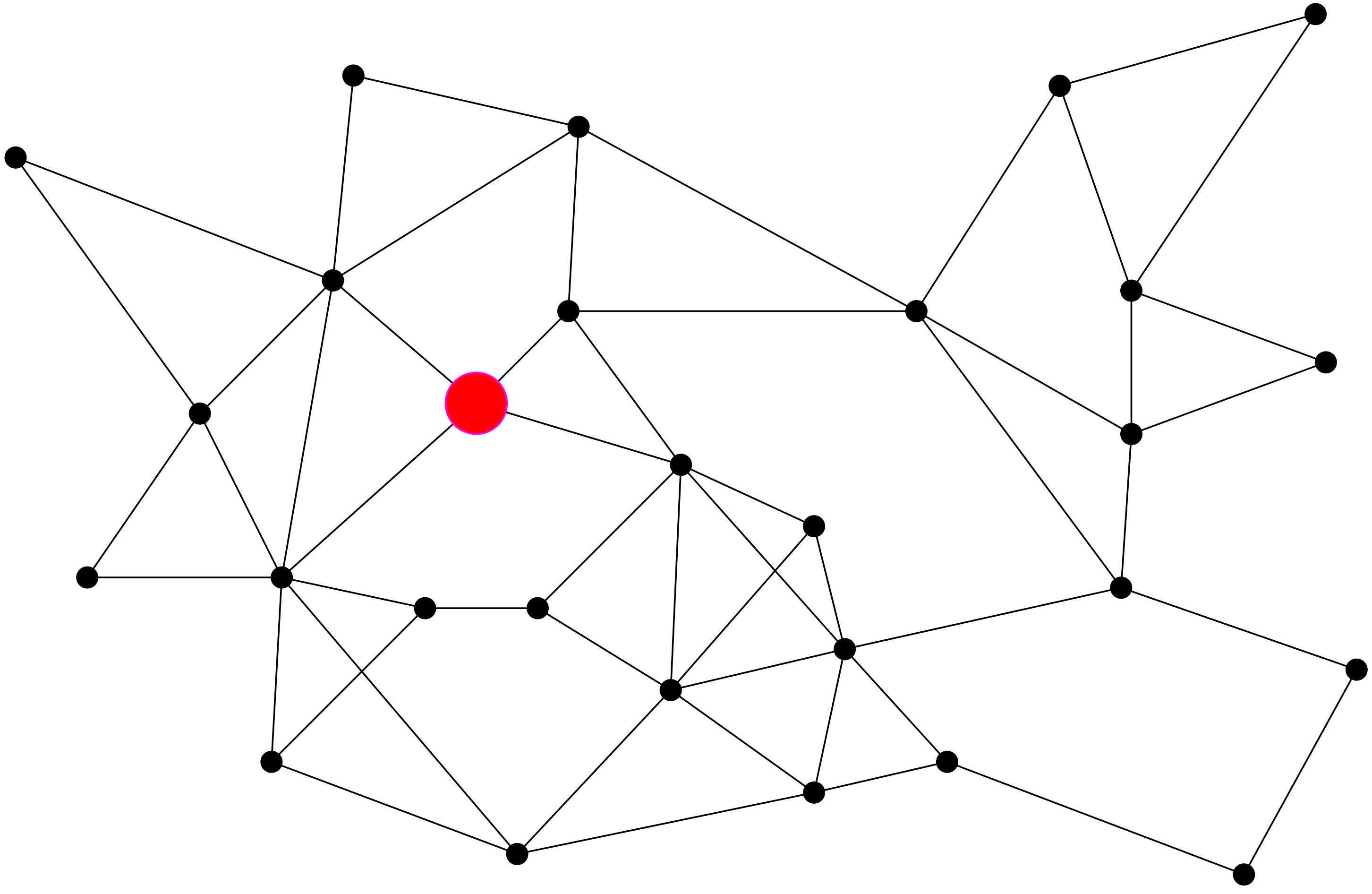


Firefighters to the rescue !

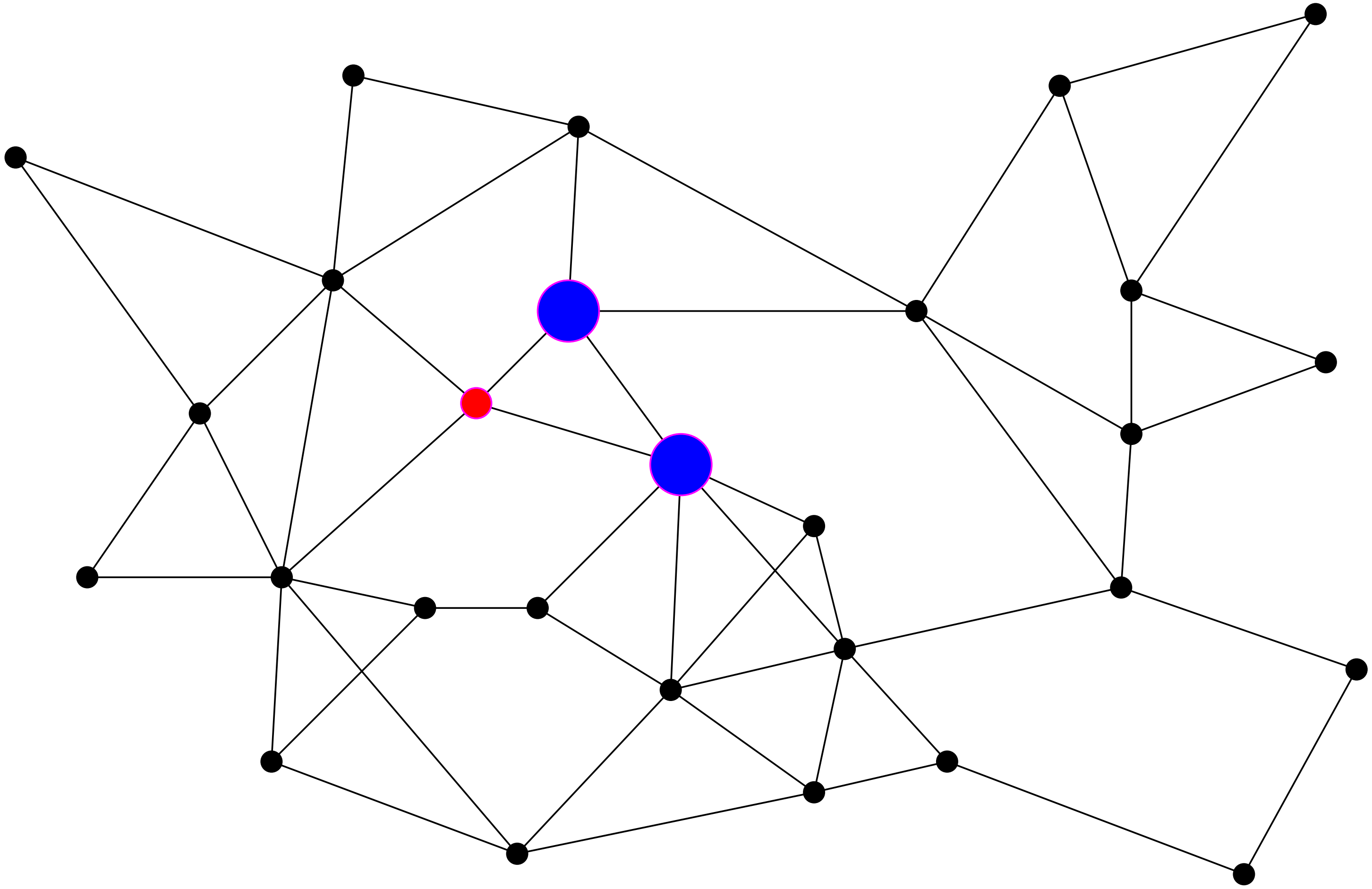
- suppose we have some **firefighters**
- one firefighter can :
  - at any time step move to any vertex
  - protect that vertex,

    which will stay protected for ever afterwards

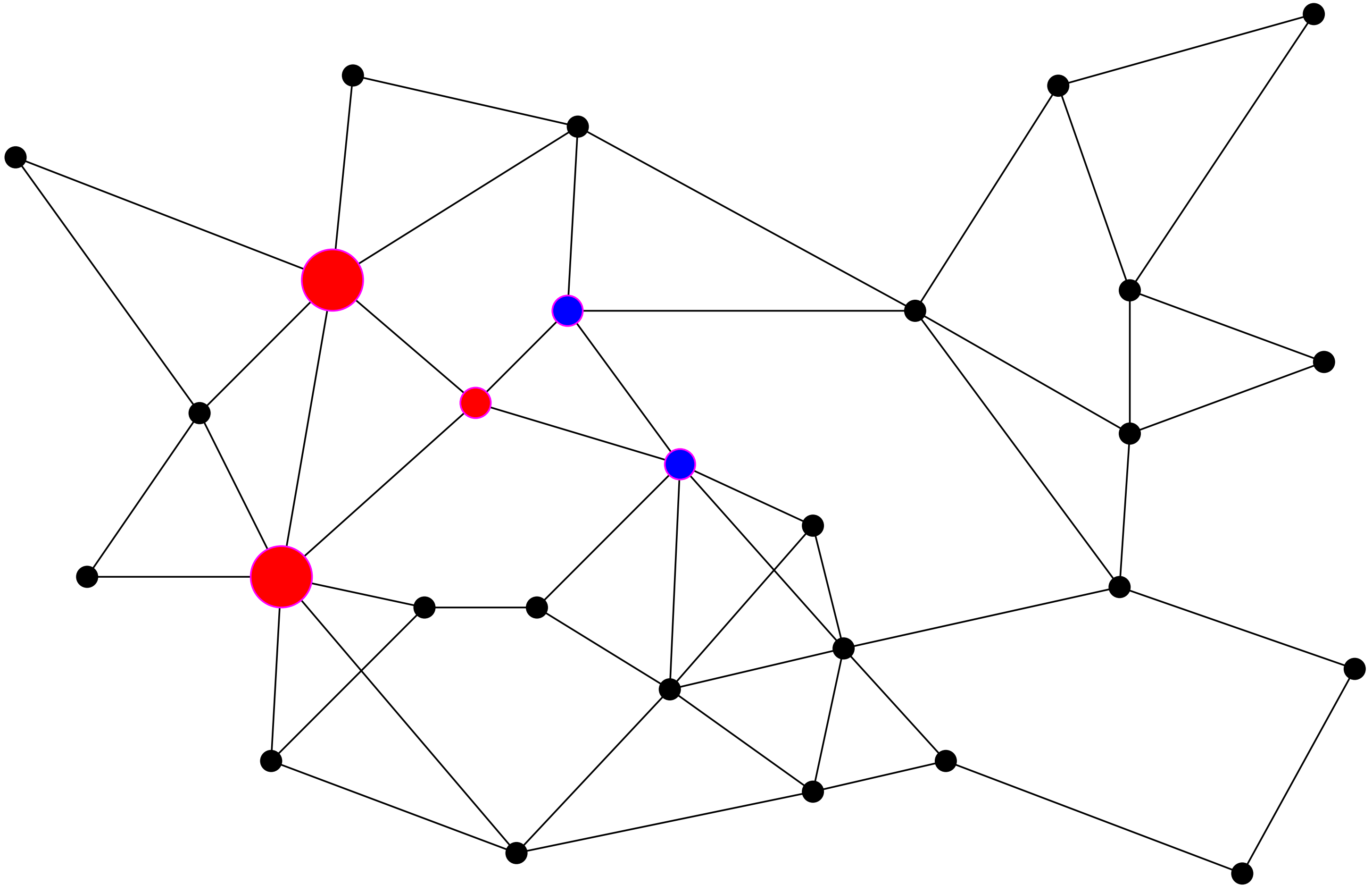
The start of the fire again



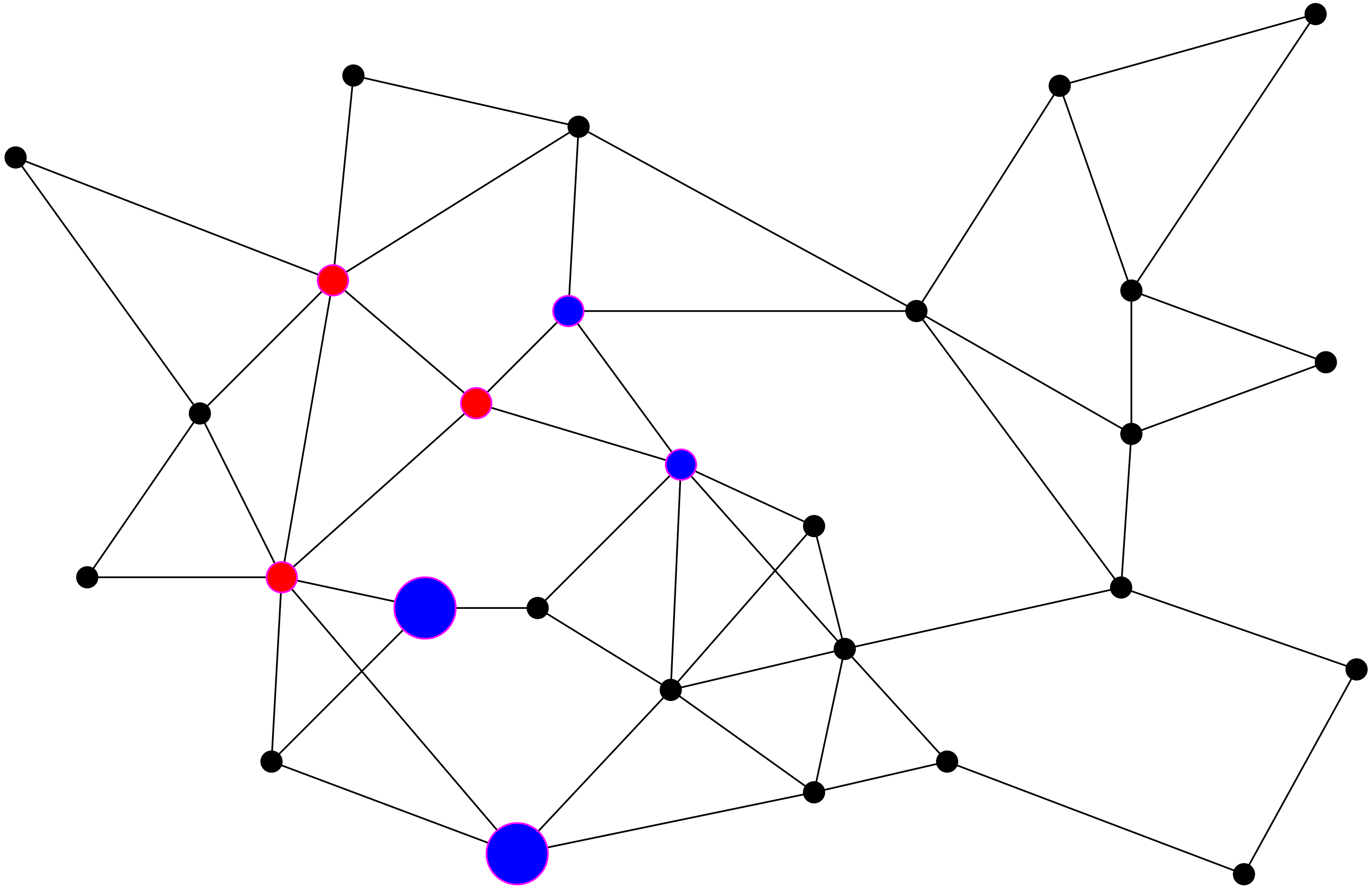
Two firefighters at work



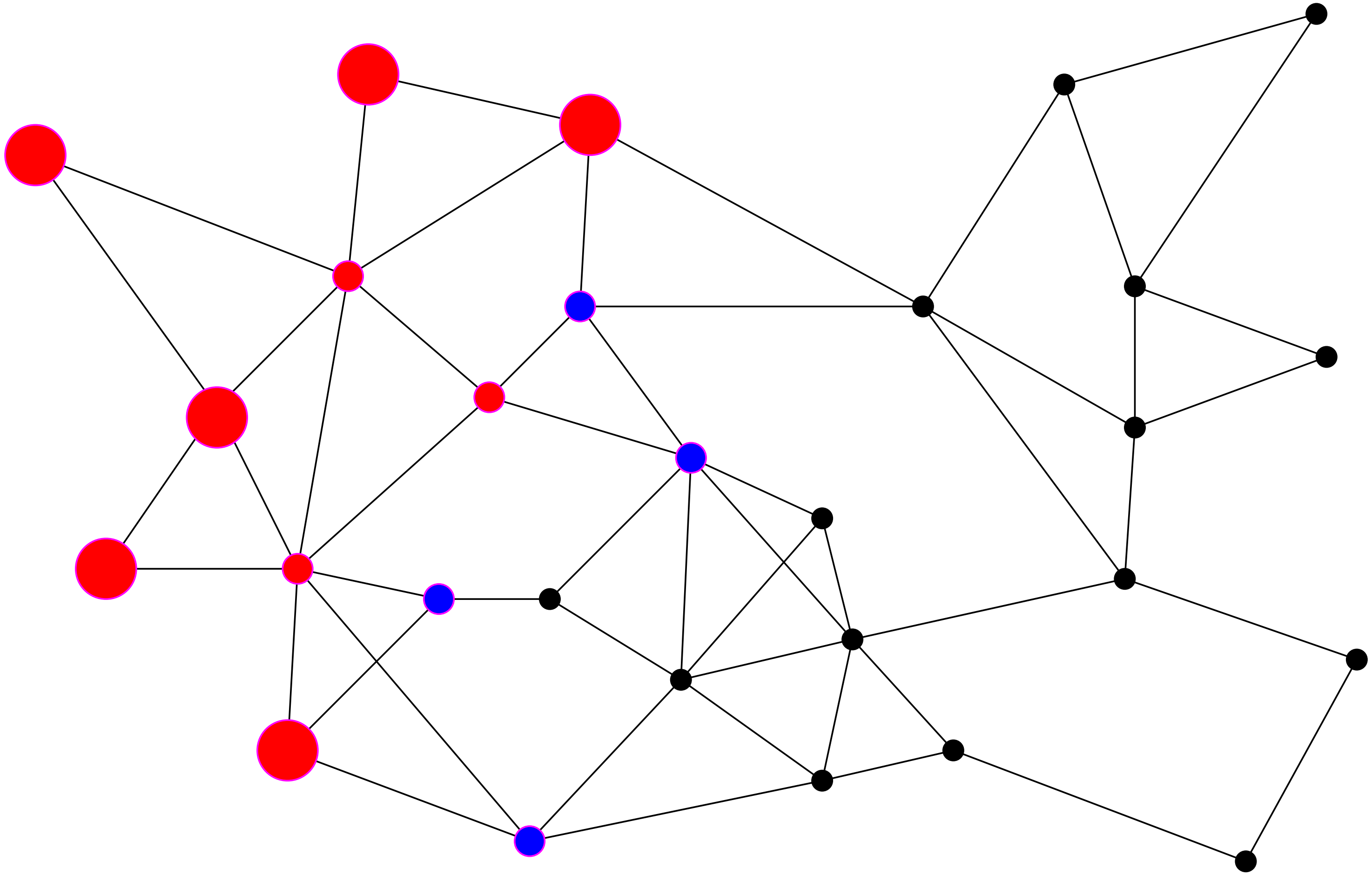
The fire spreads a bit



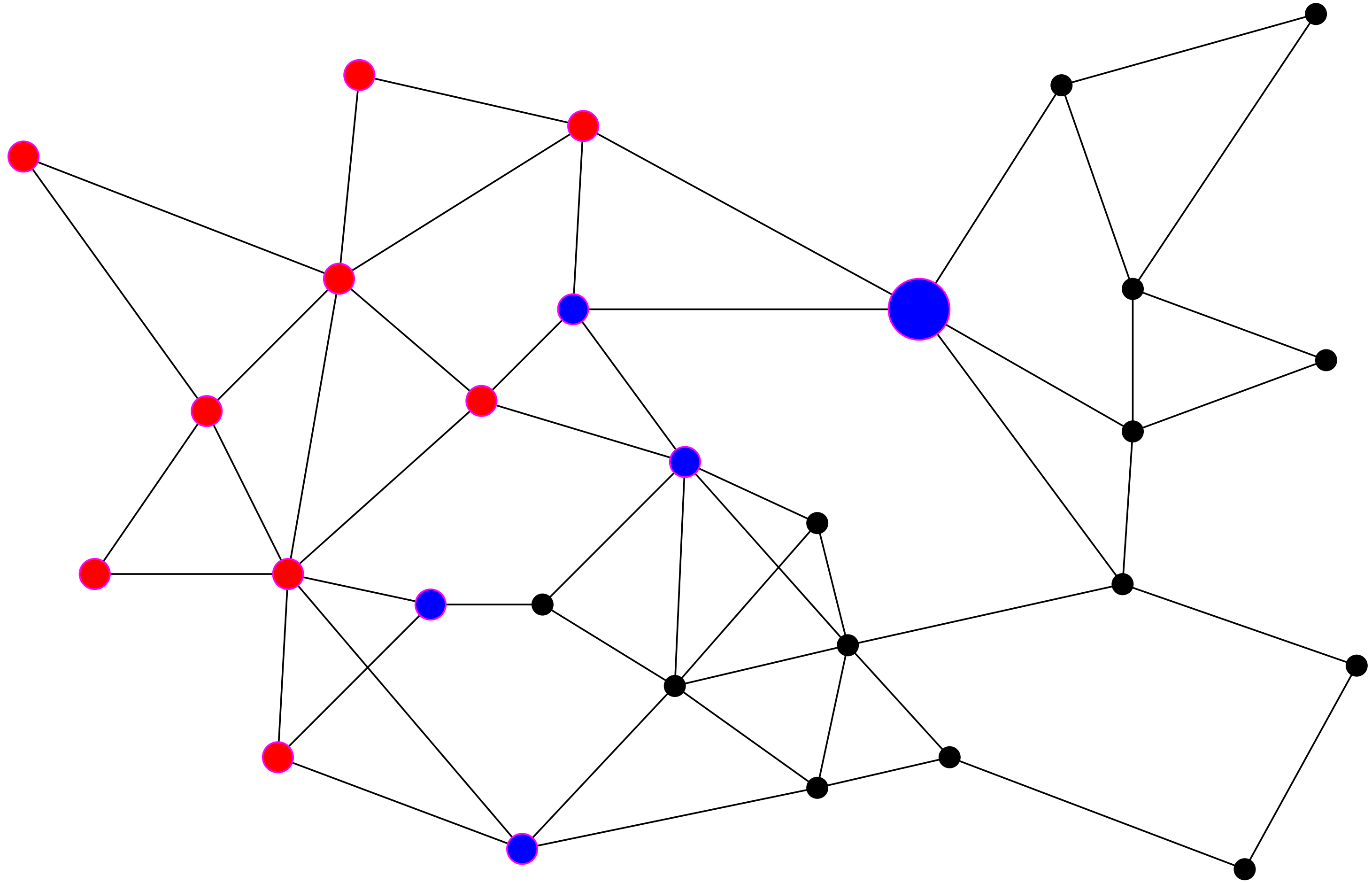
The firefighters continue their work



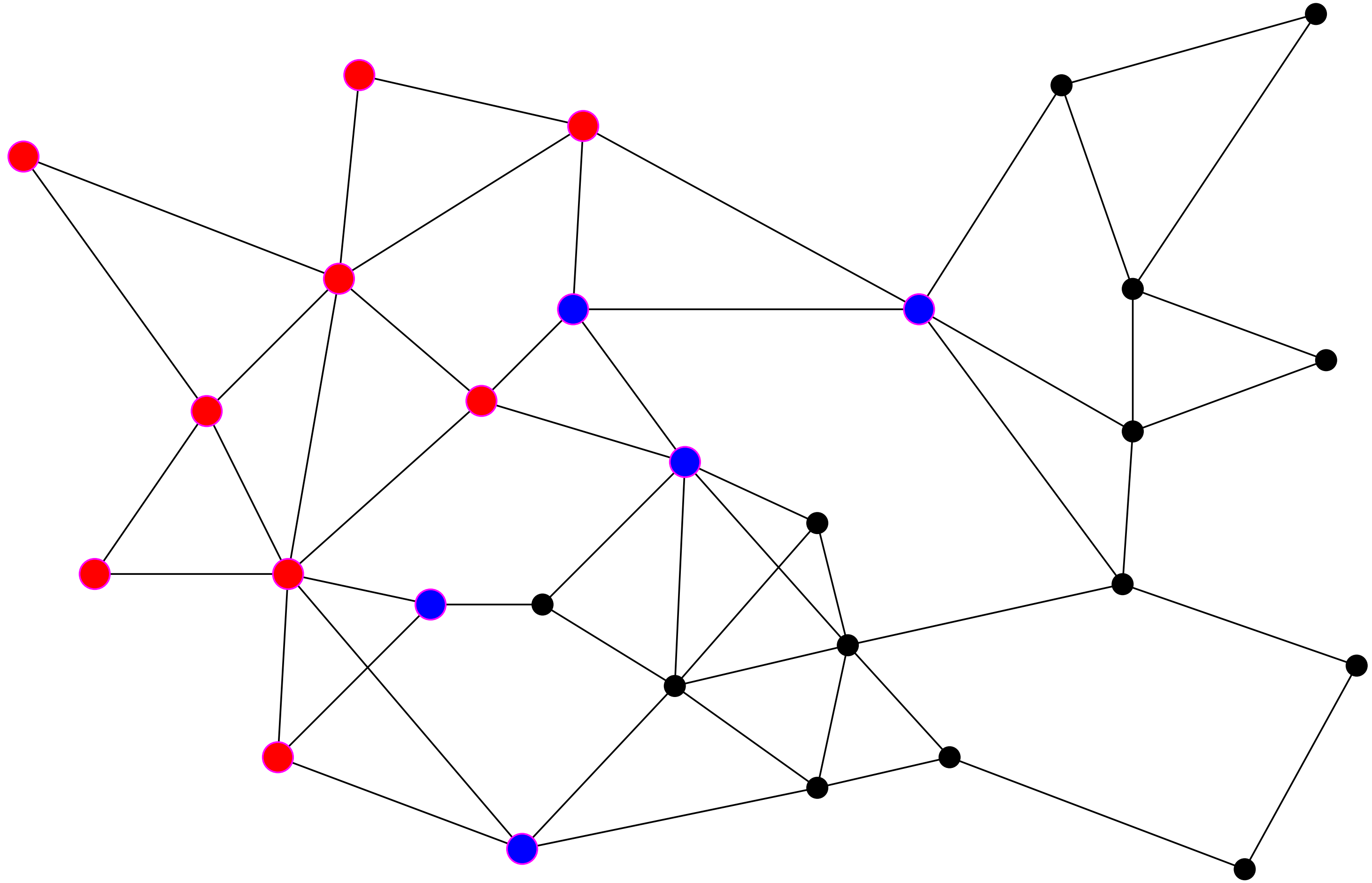
The fire spreads . . .



One more move from a firefighter



The danger is over ...



The Firefighter Problem

- given some input information

  such as : the graph,

  the vertex where the fire starts,

  the number of firefighters per step, etc.

- possible aims :
  - maximise the number of saved vertices
  - minimise the time until the fire is under control
  - minimise the number of firefighters needed

    to protect a given number of vertices

A typical problem : the solo firefighter

- **Input** : graph G, vertex v, integer K

  Question : if a fire starts at v,

  can a single firefighter save at least K vertices ?

- this problem is NP-complete (MacGillivray & Wang, 2003)
  - even restricted to trees with maximum degree 3

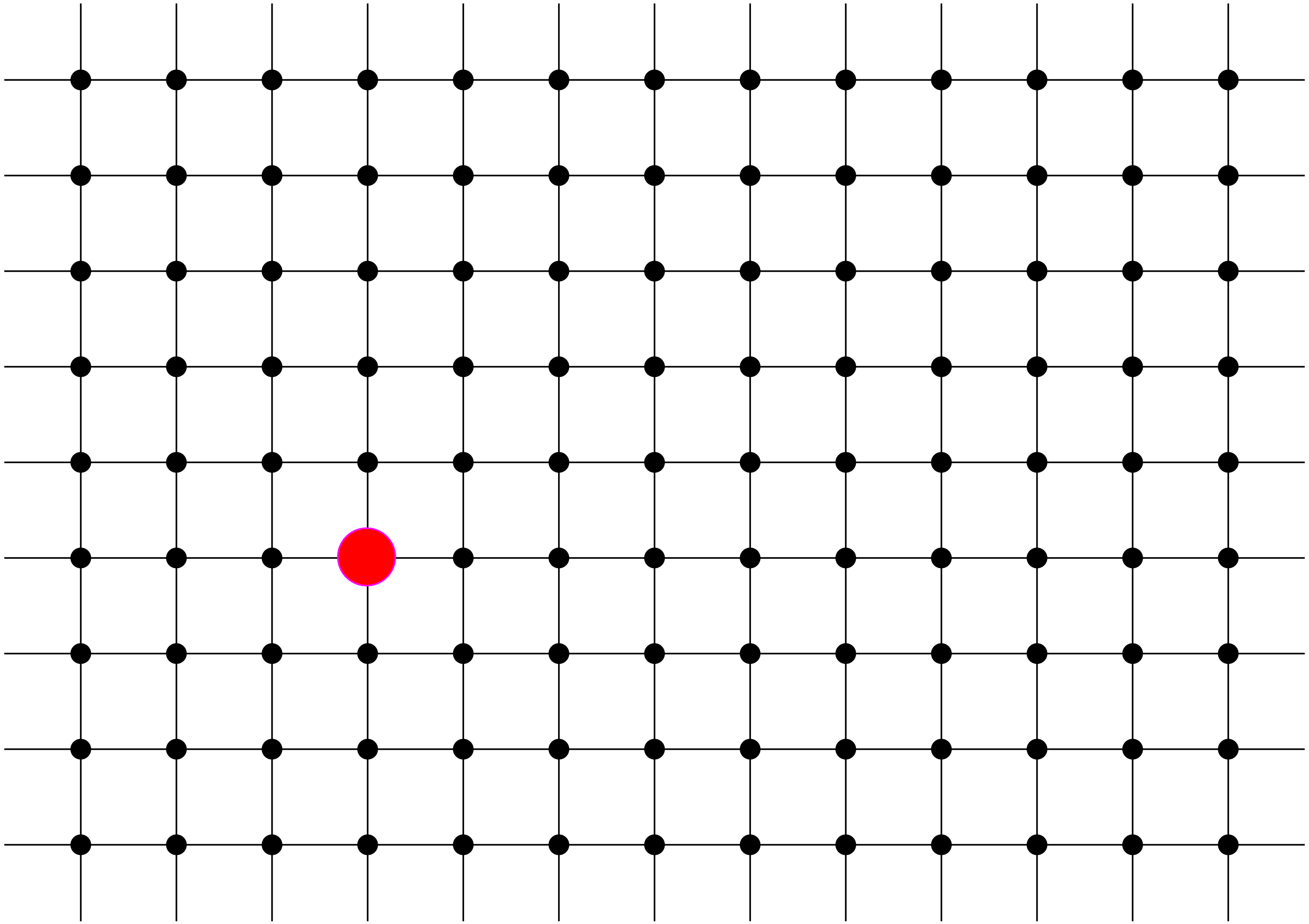
A small side-step : infinite grids

Theorem

(Wang & Moeller, 2002; Fogarty, 2003; Develin & Hartke, 2007)

- for a d-dimensional grid $\mathbb{Z}^d$ we have
  - $d = 1$ or $d \geq 3$ $\Longrightarrow$ one fire can be contained by $2d - 1$ firefighters (in 2 steps)
  - $d = 2$ $\Longrightarrow$ one fire can be contained by 2 firefighters (in 8 steps)
  - all numbers are best possible

The square grid with three firefighters



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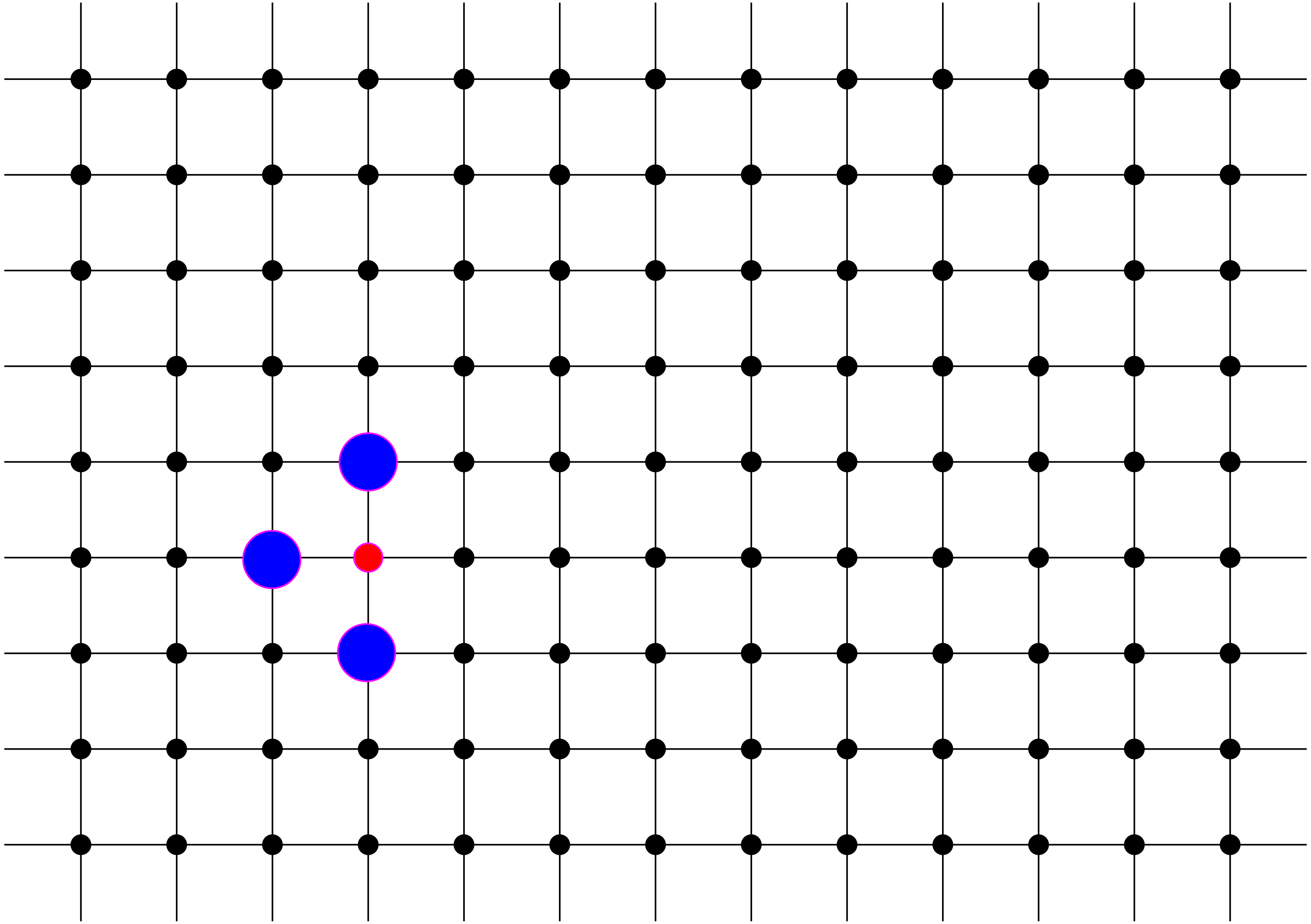

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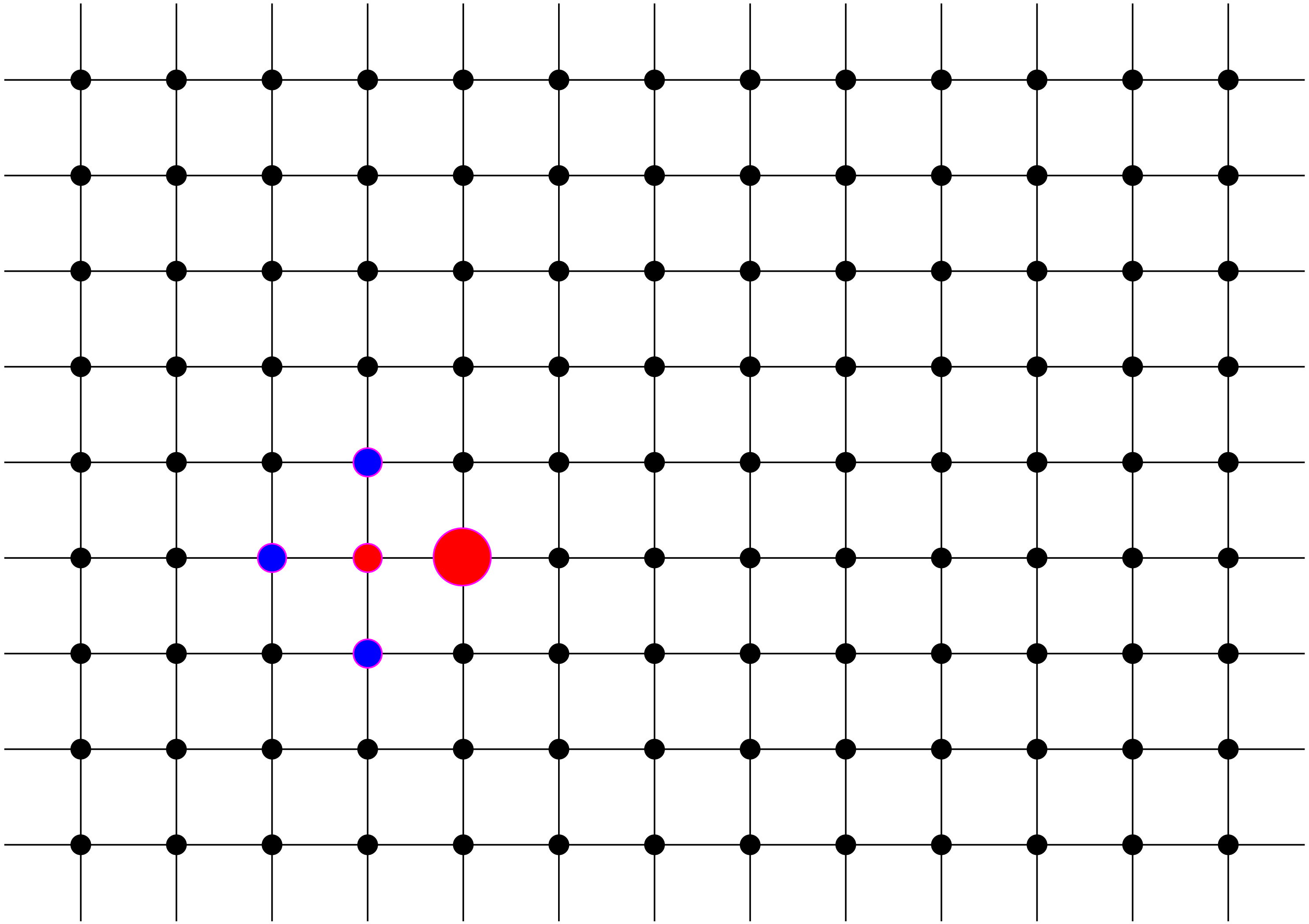
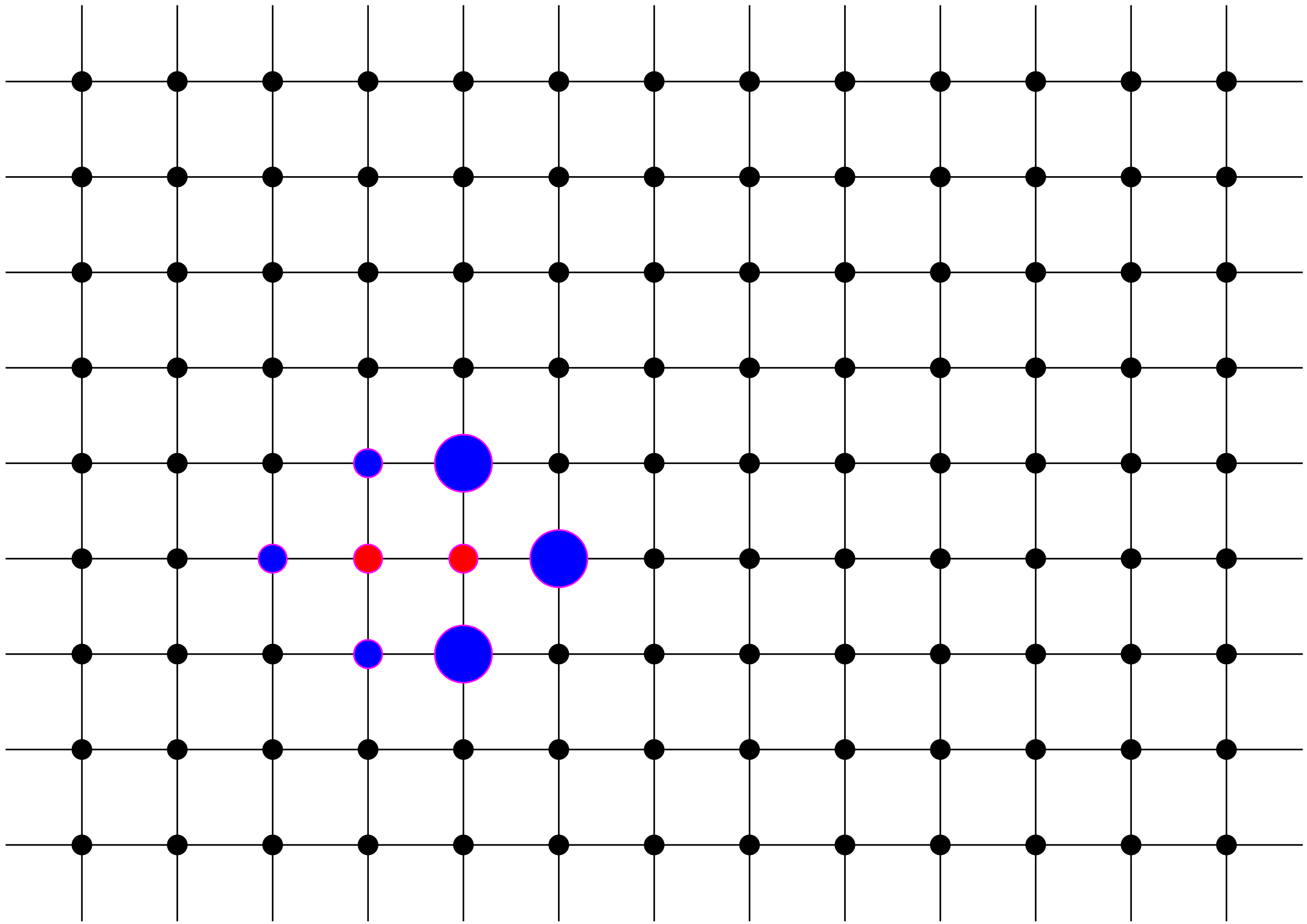
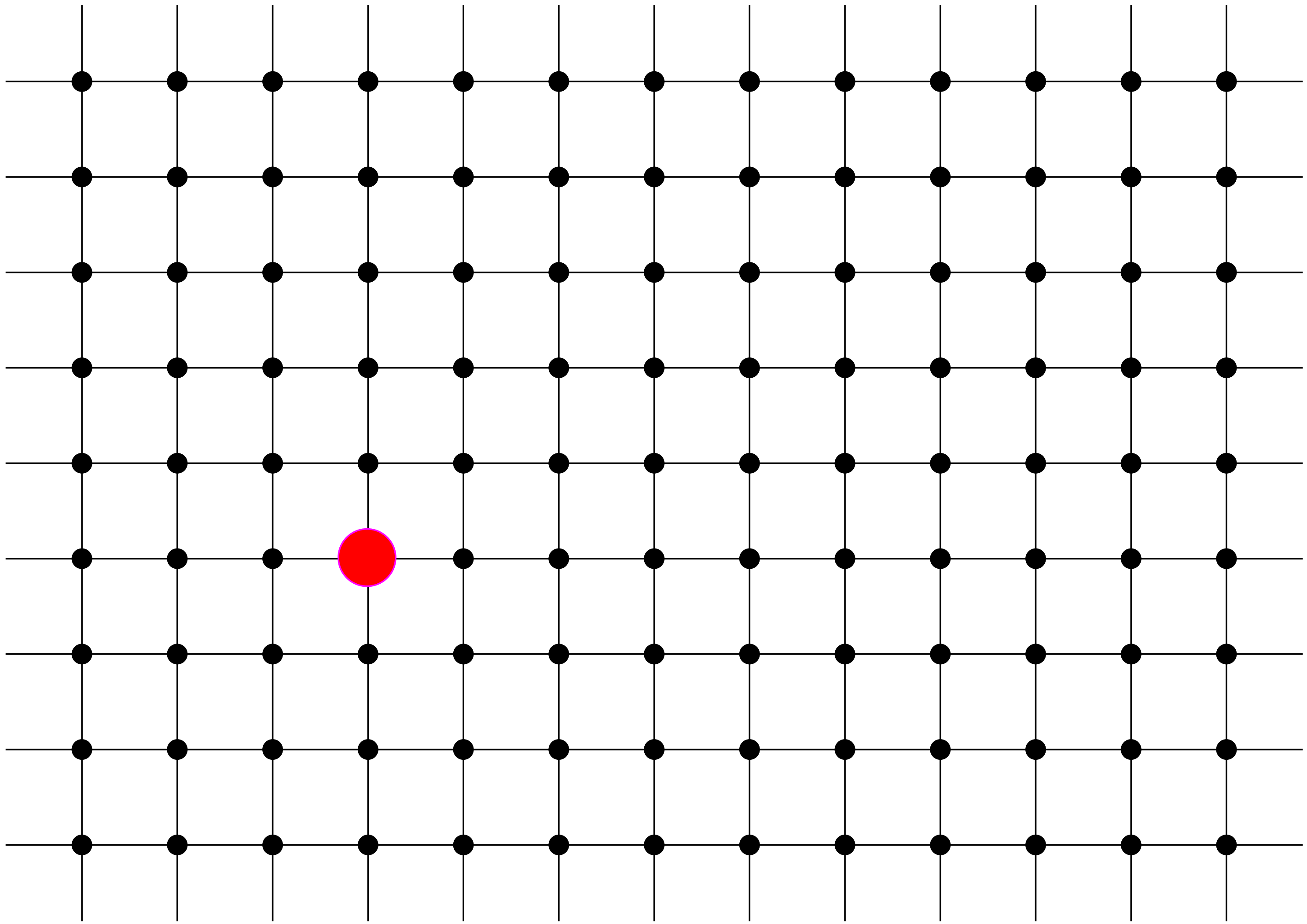


The square grid with ***three*** *firefighters*



The square grid with two firefighters



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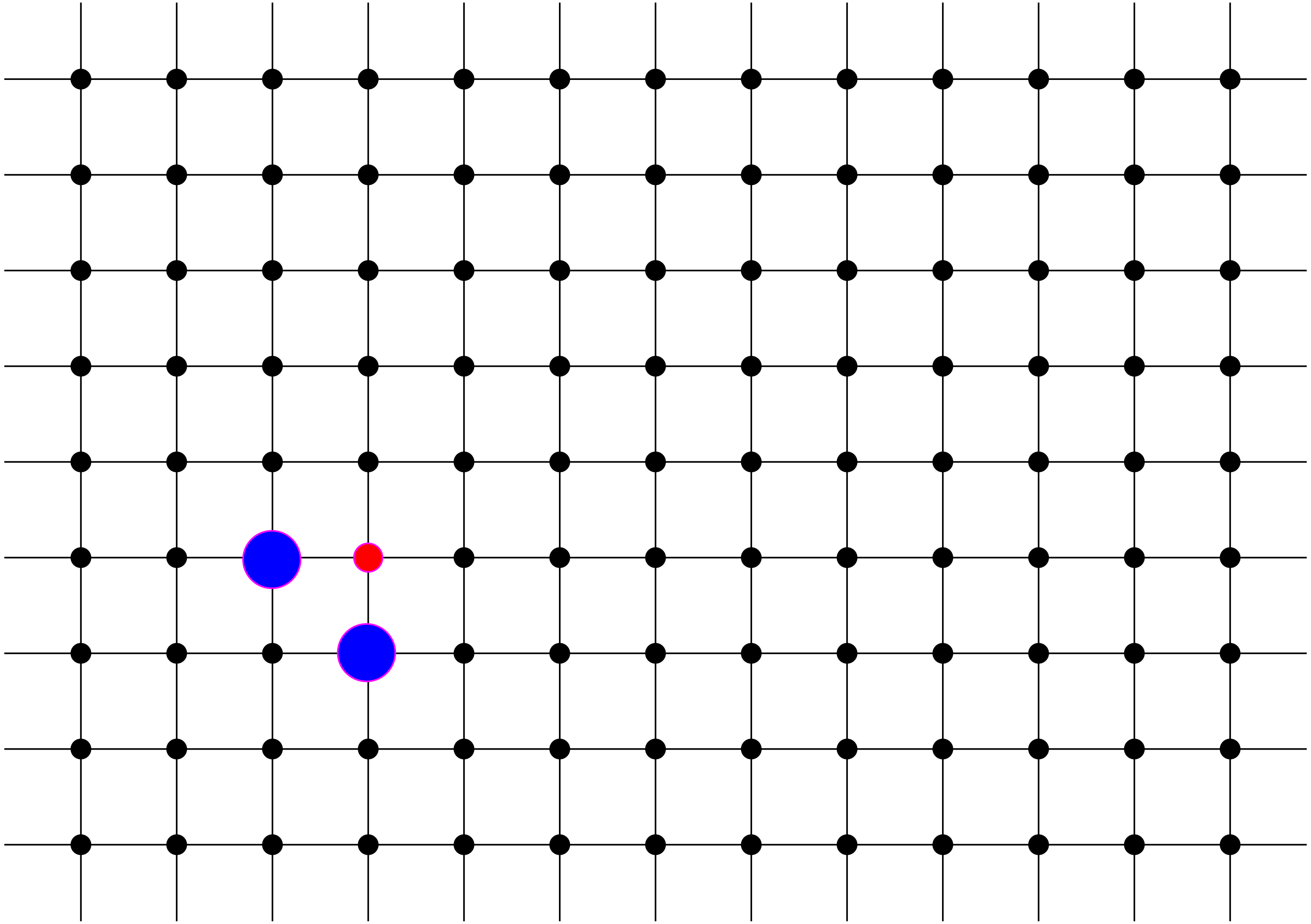

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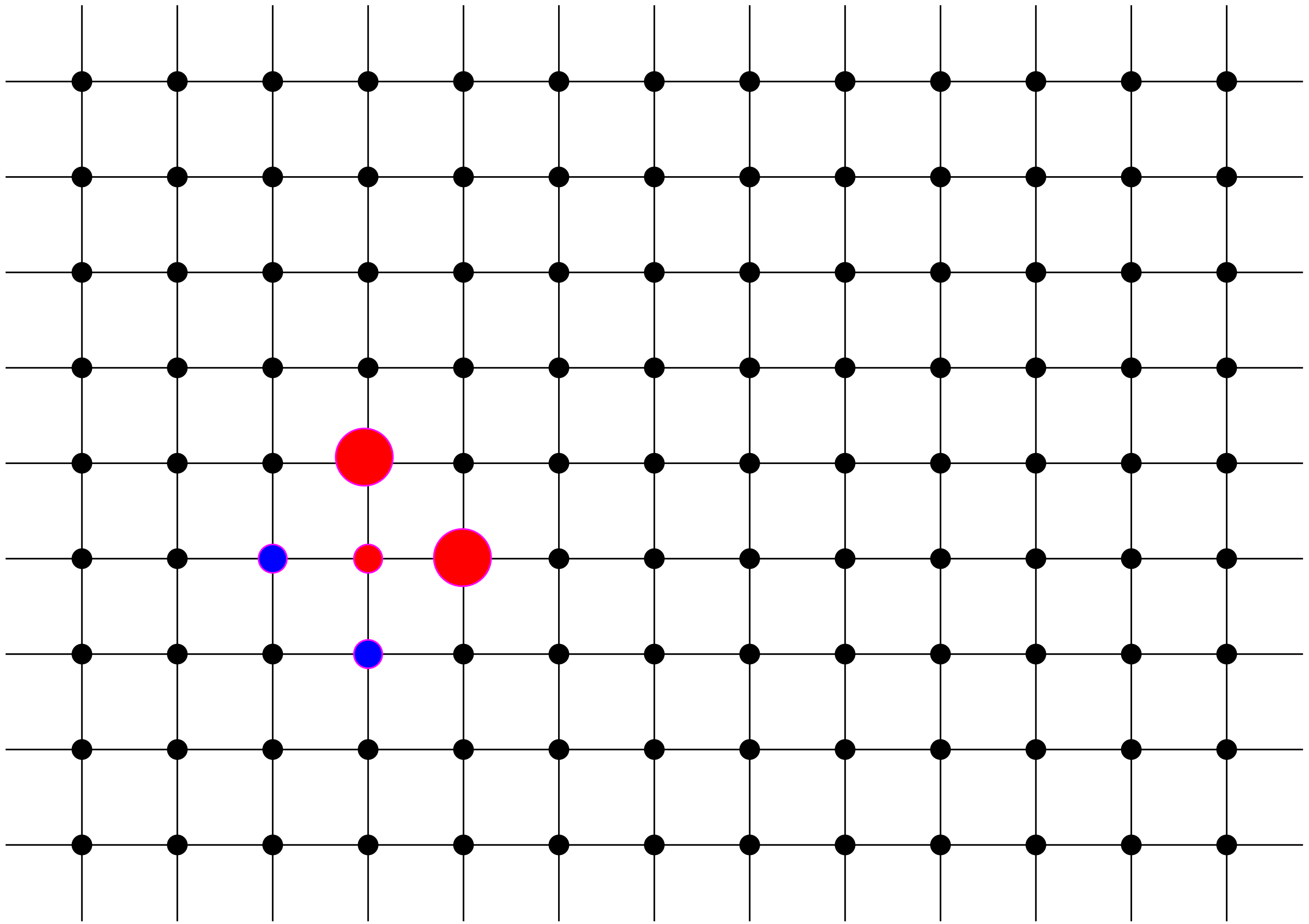

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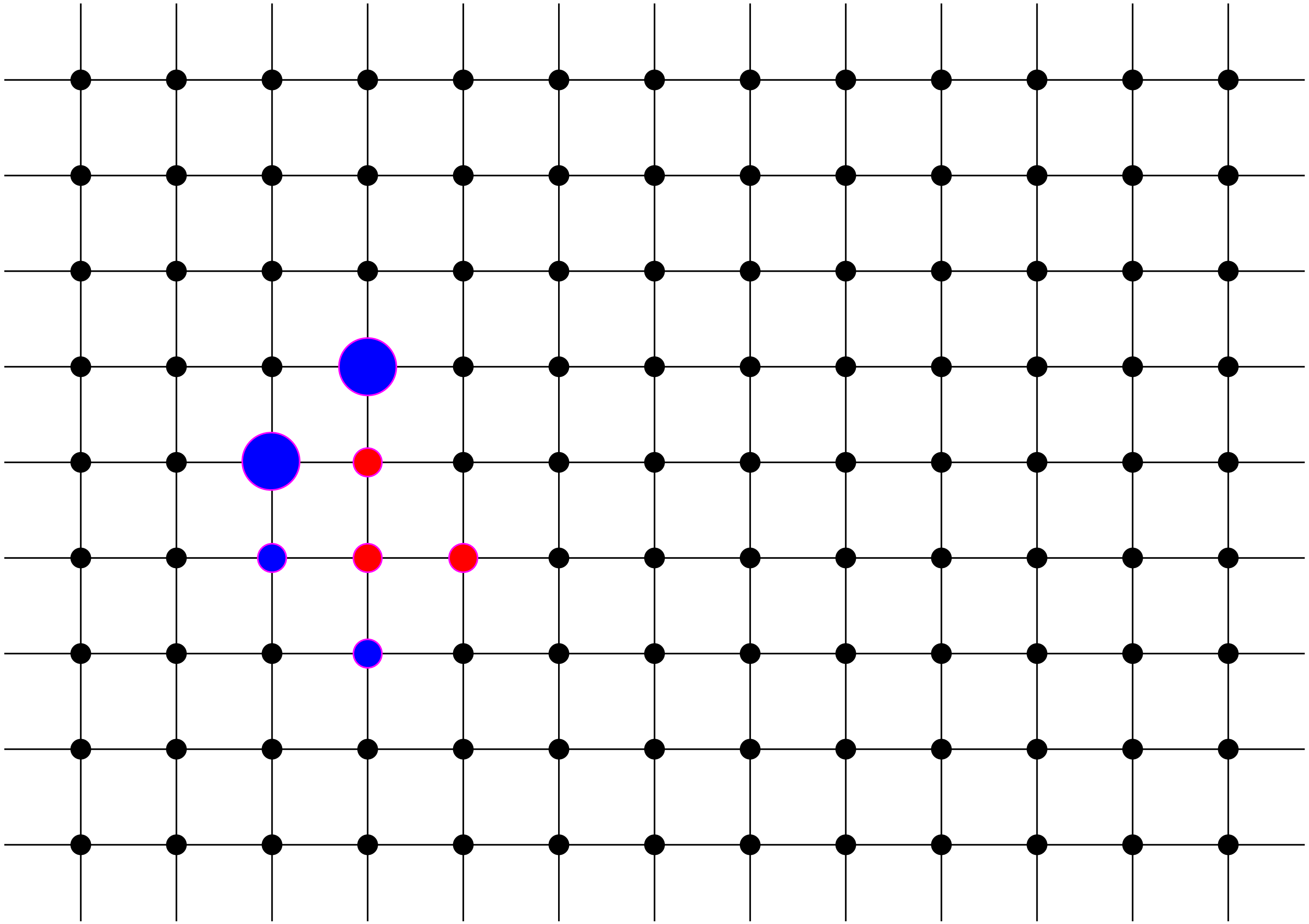

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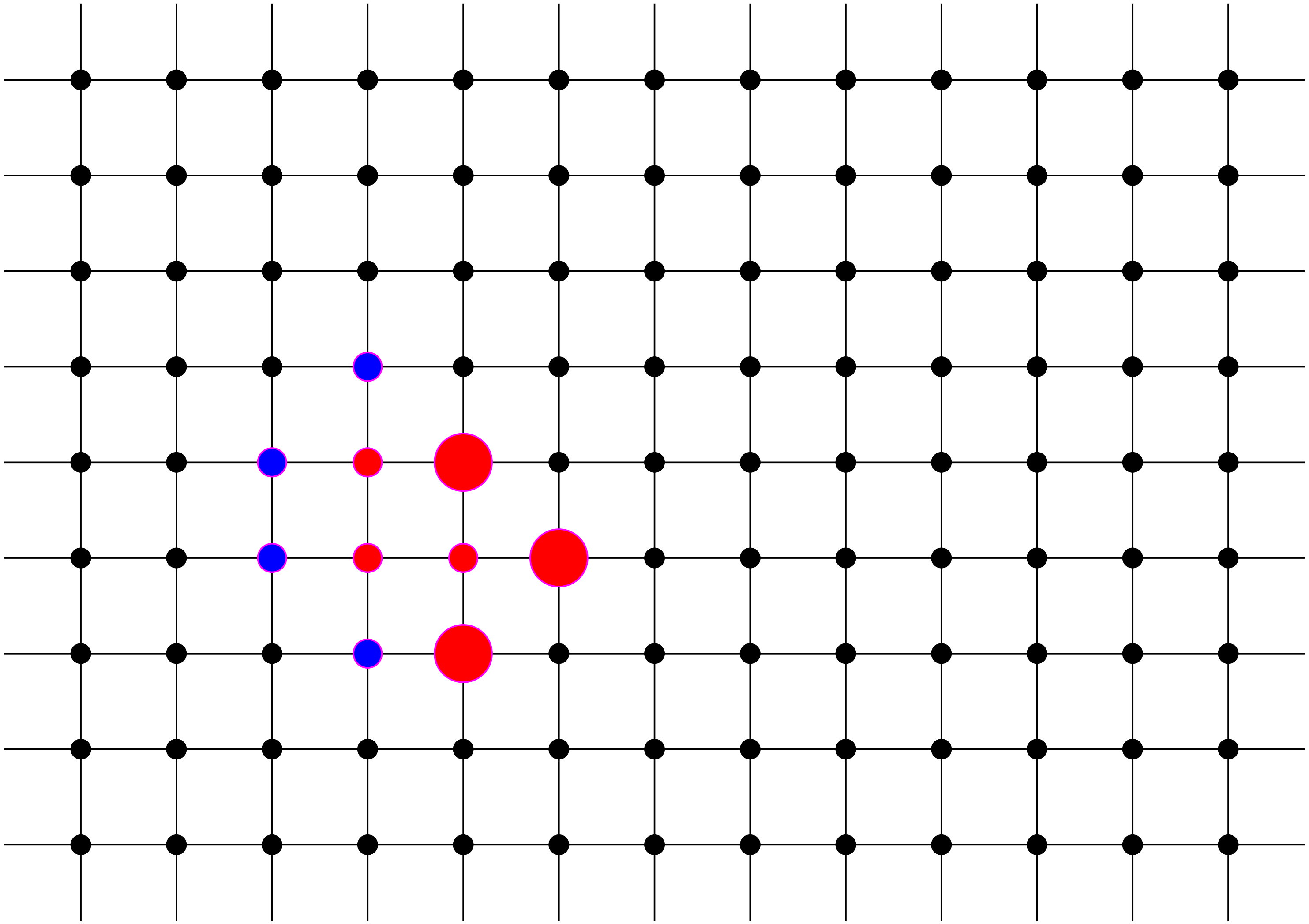

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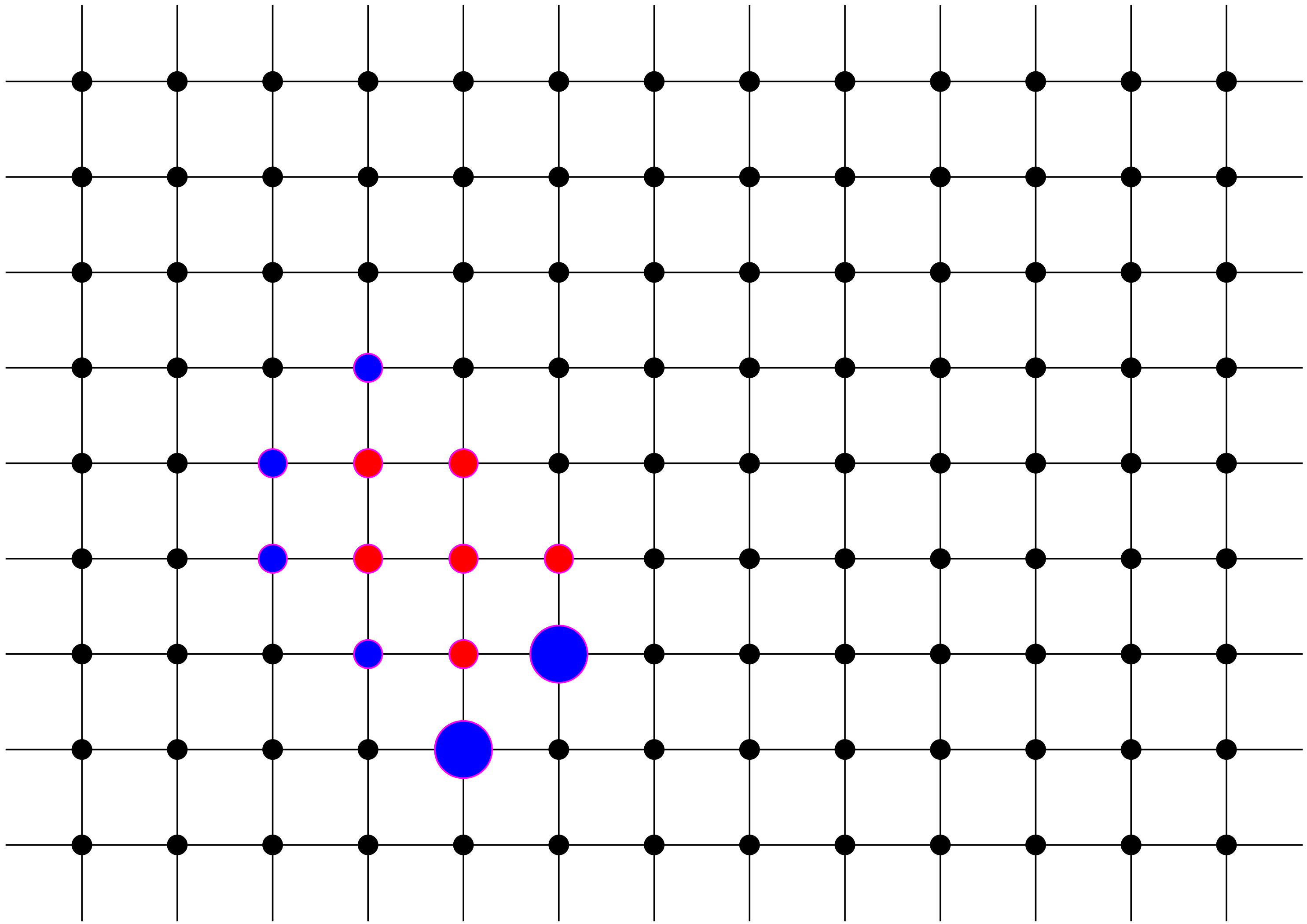

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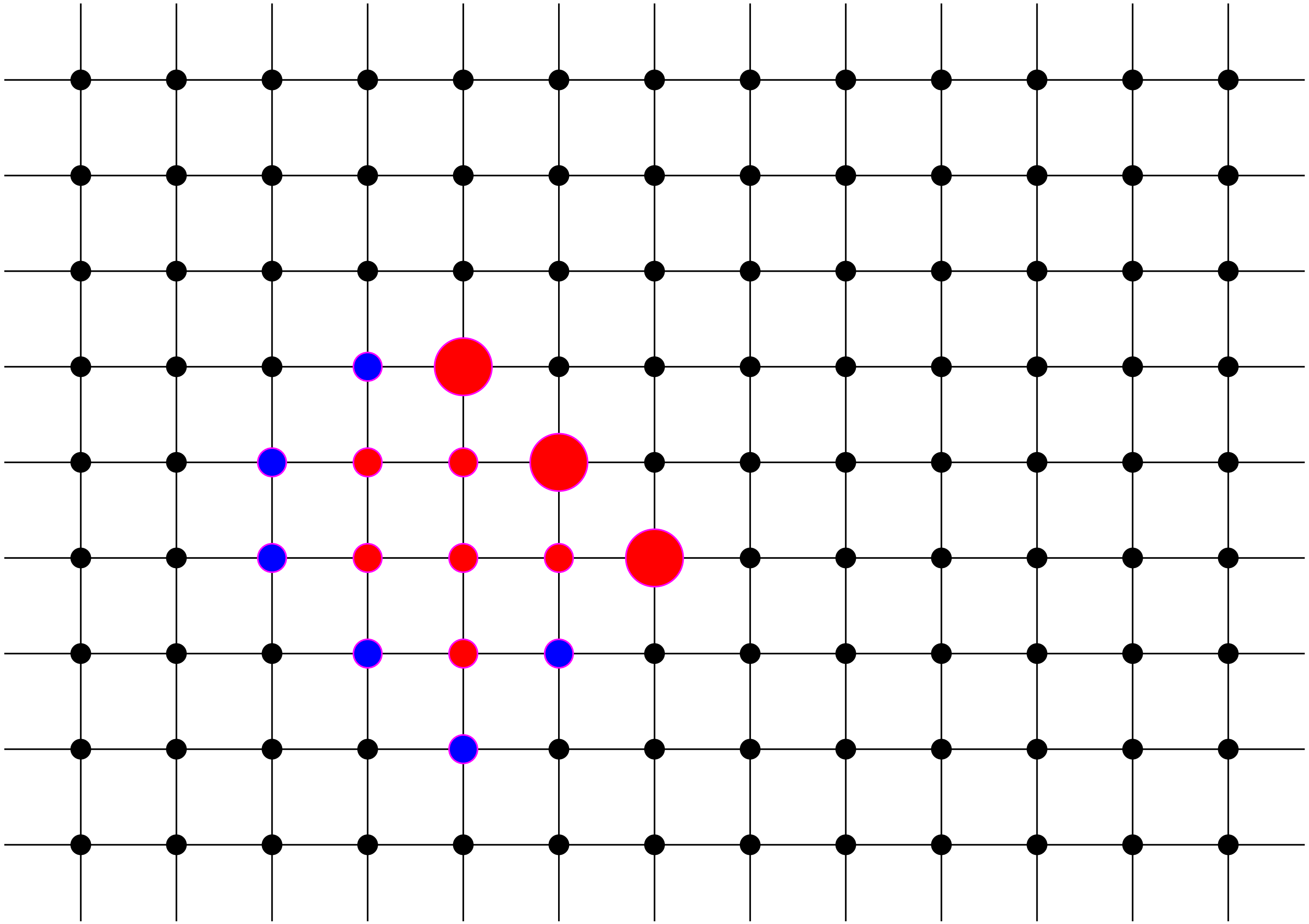

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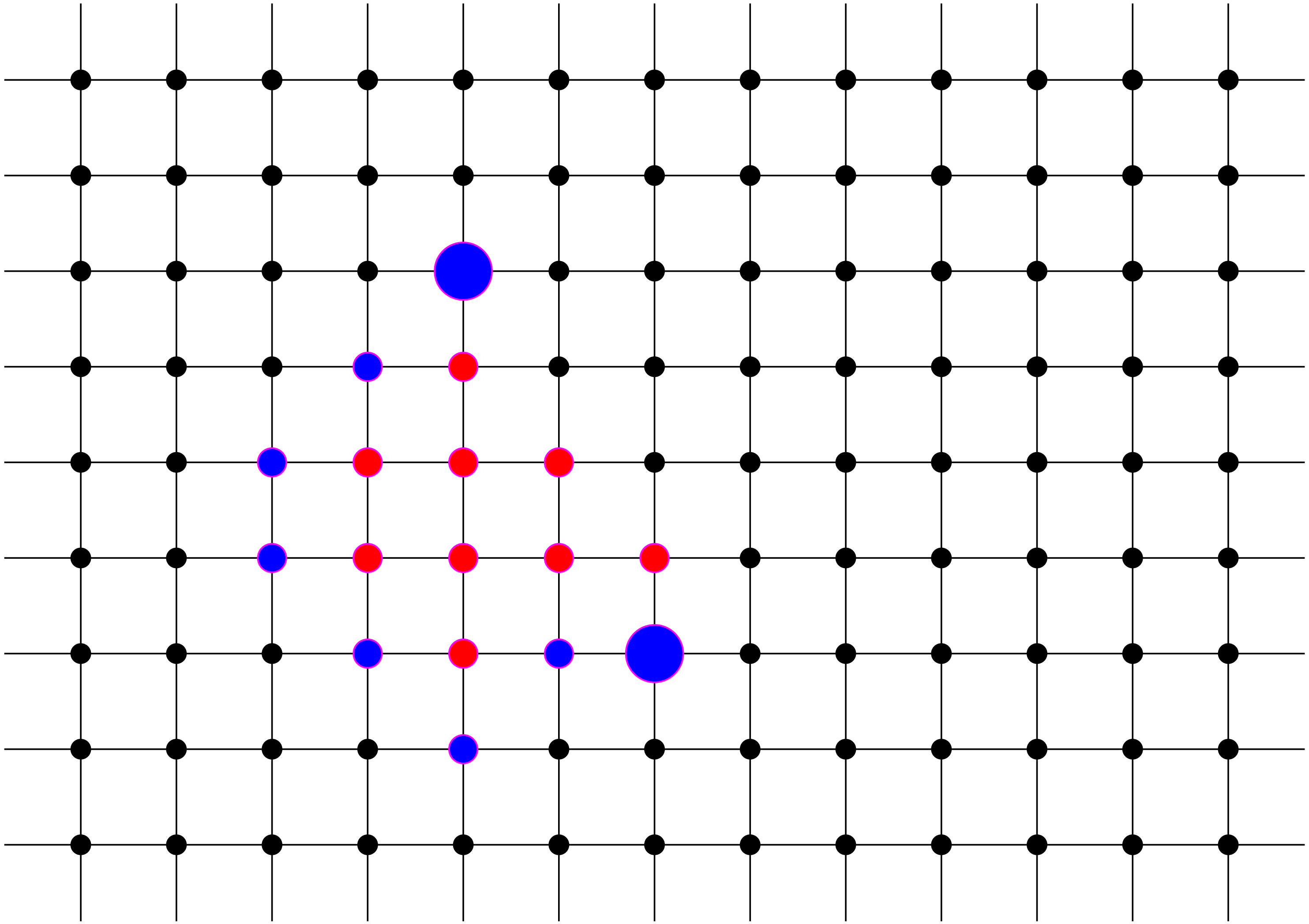

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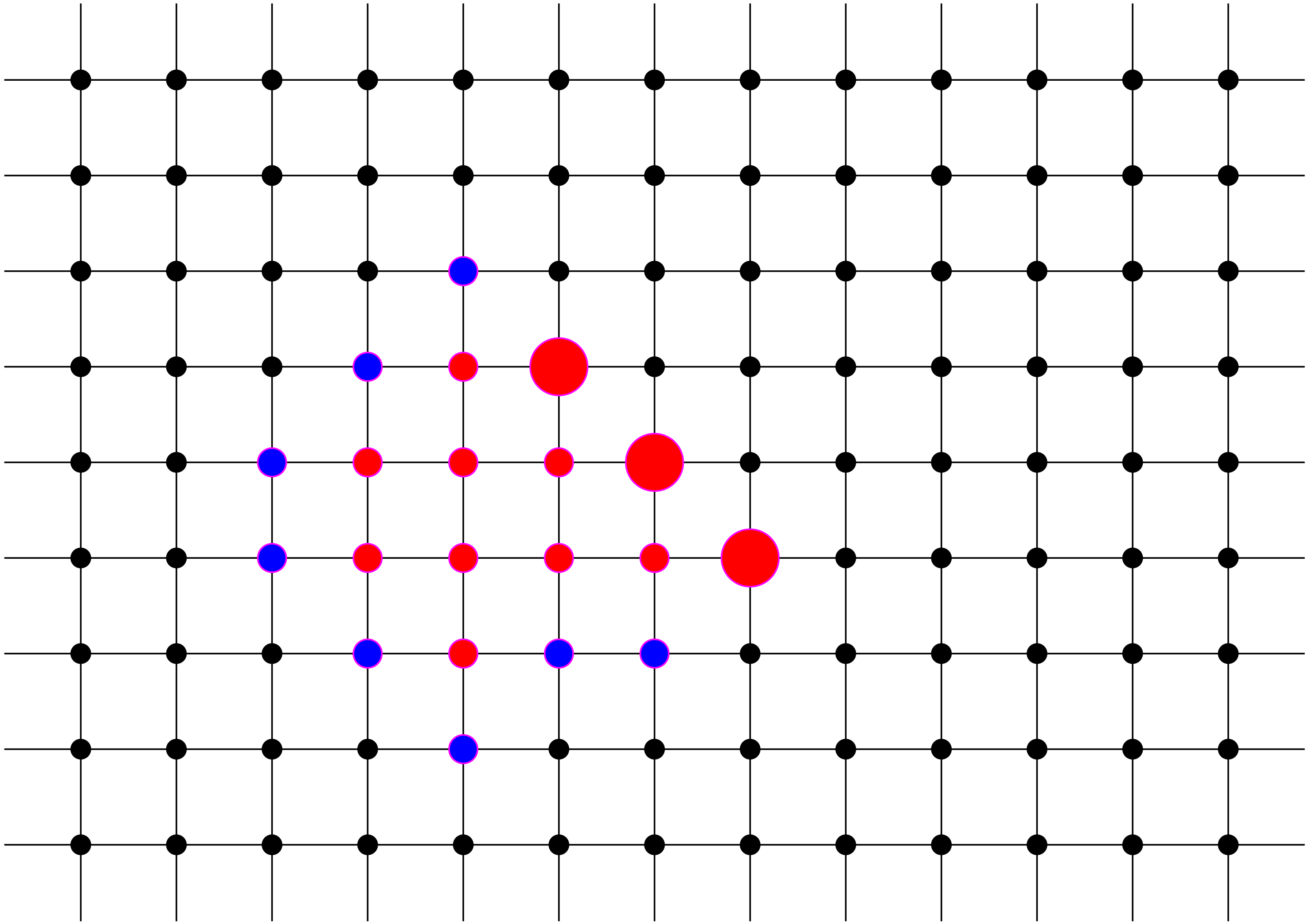

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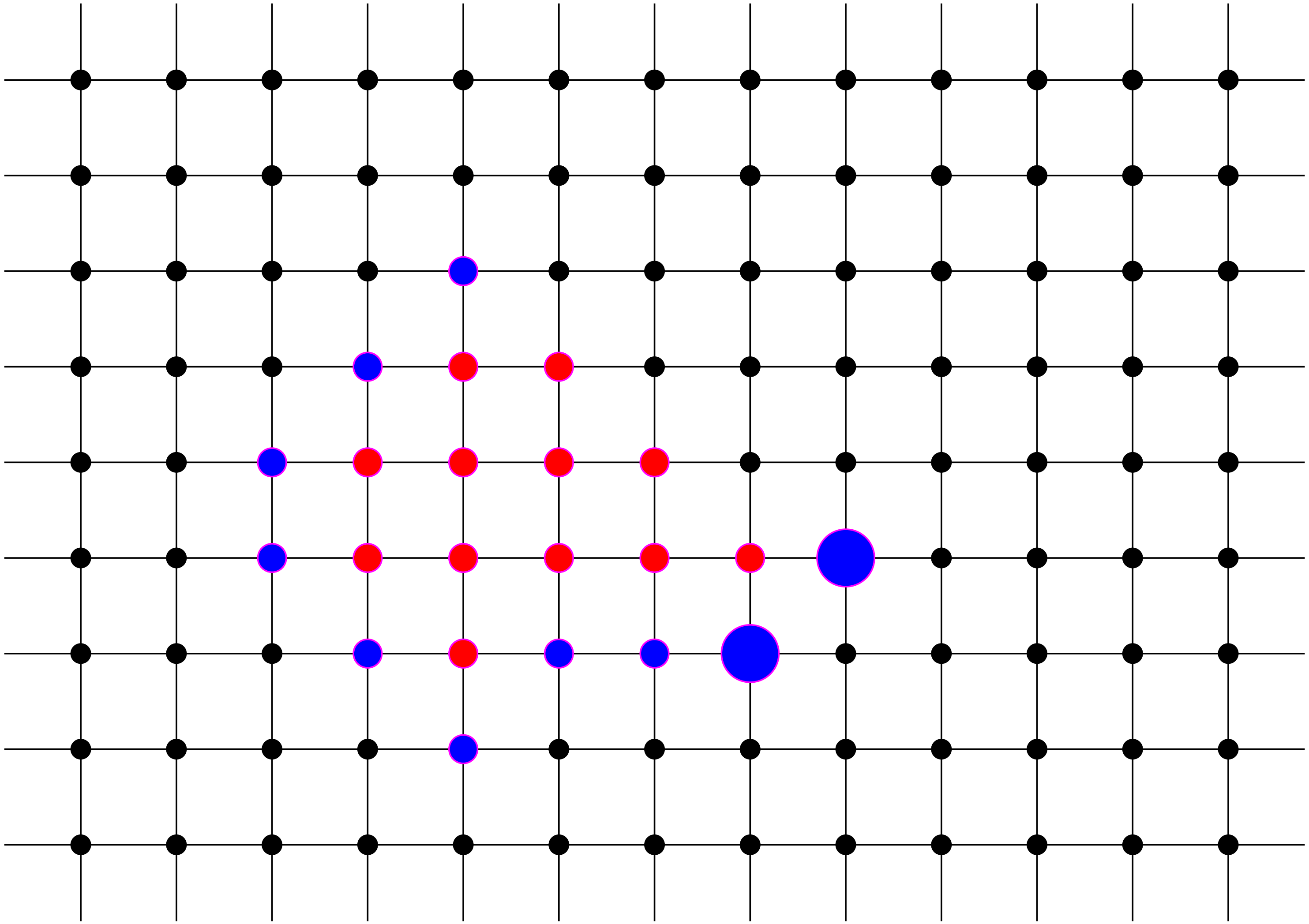

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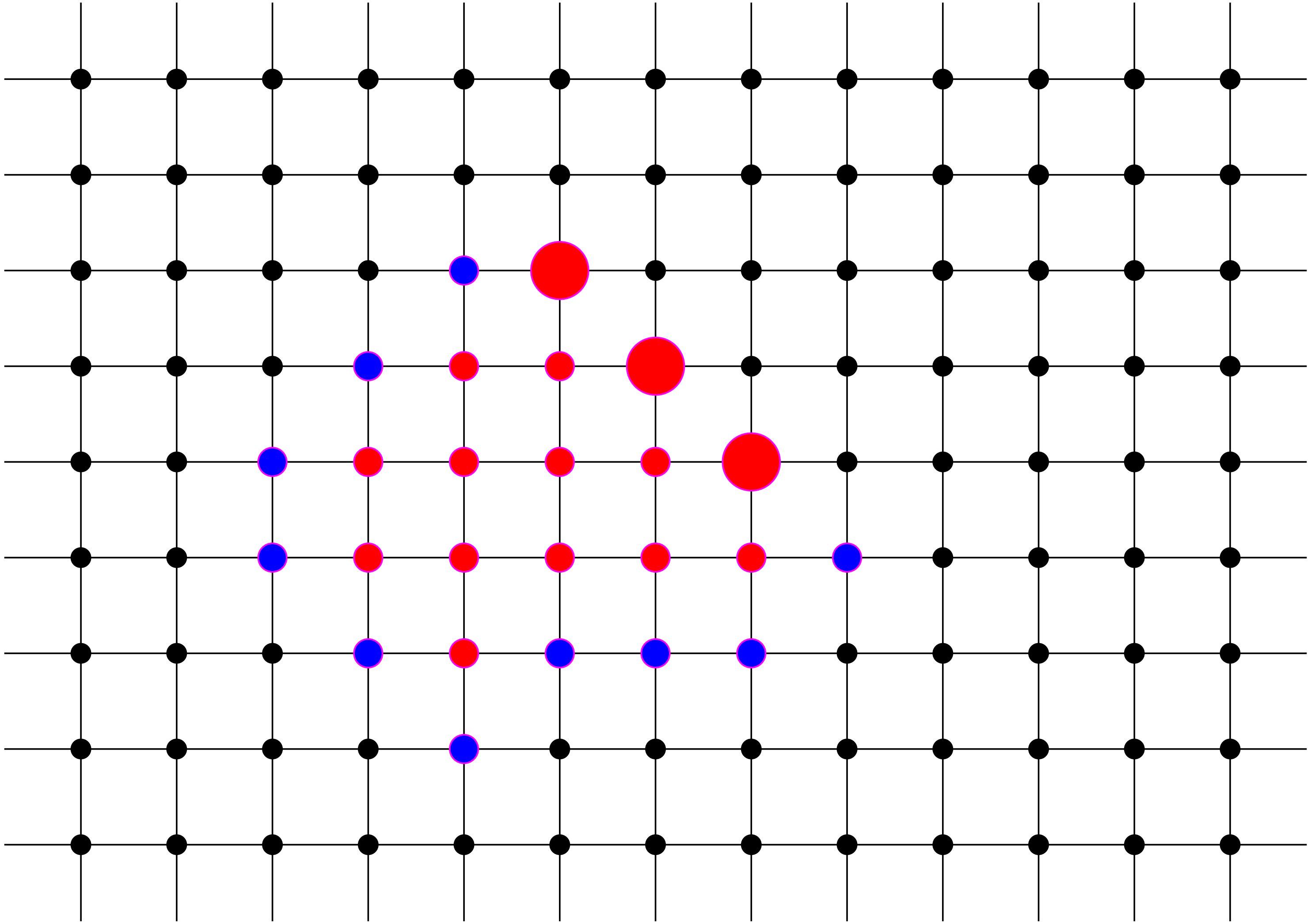

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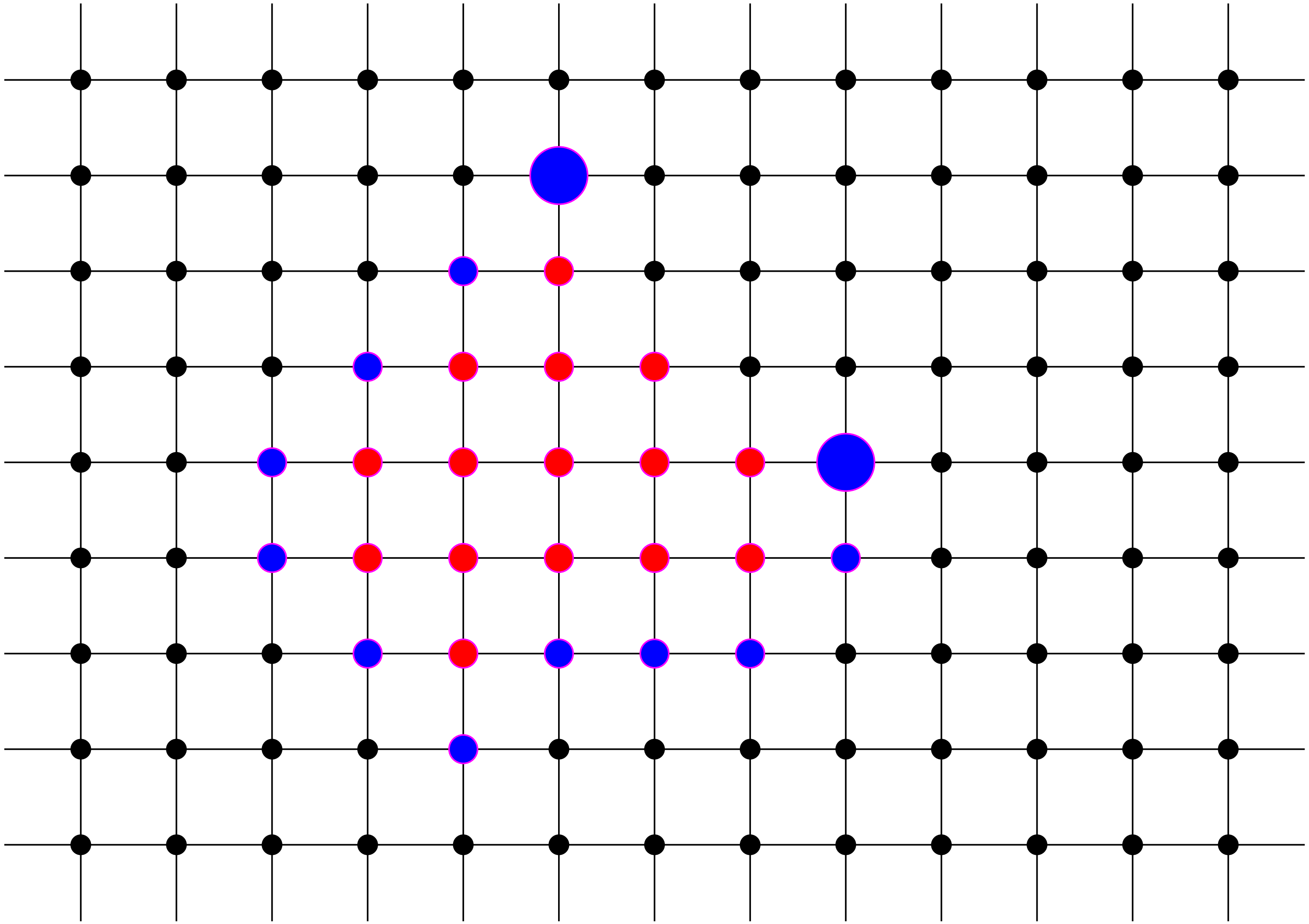

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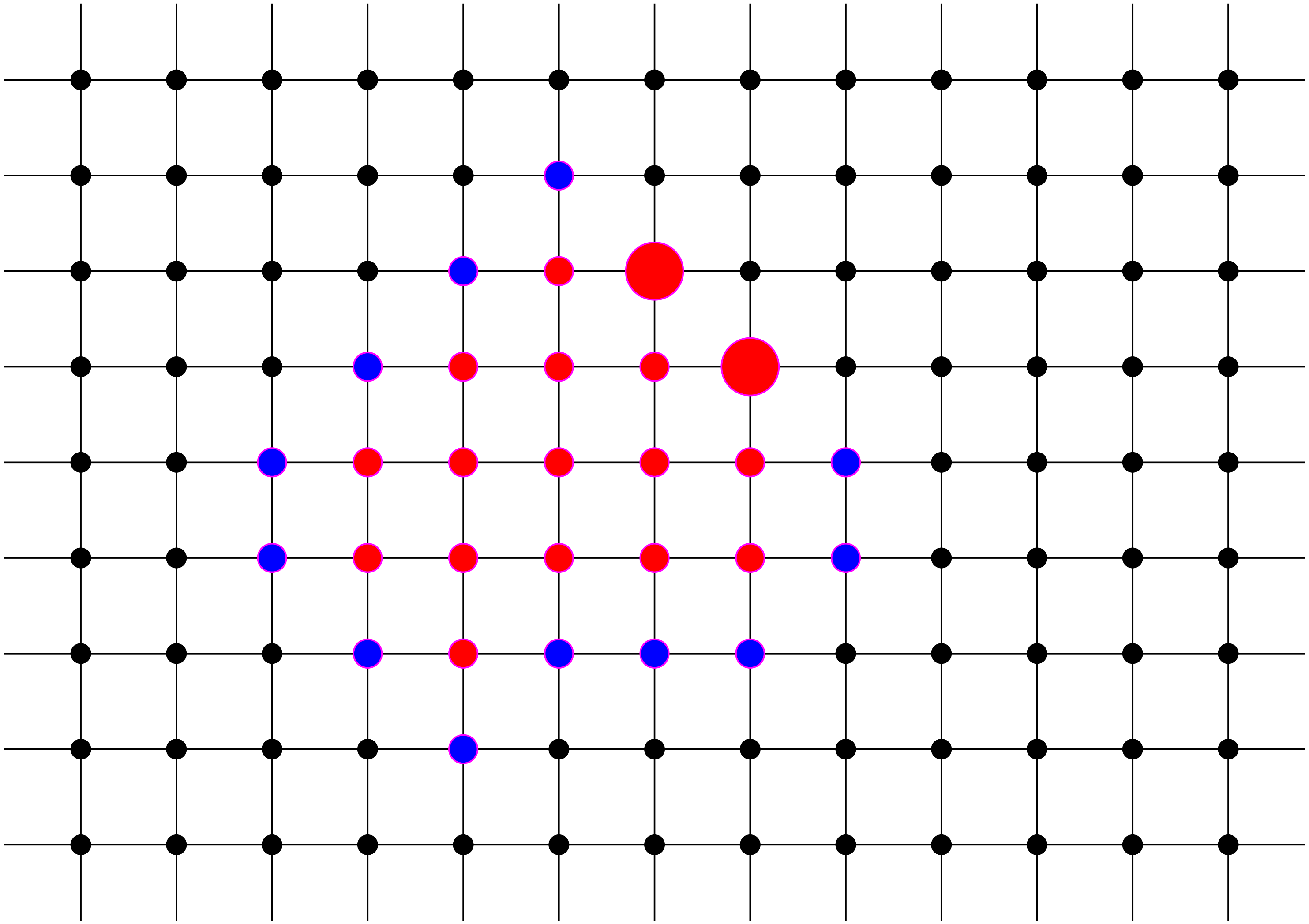

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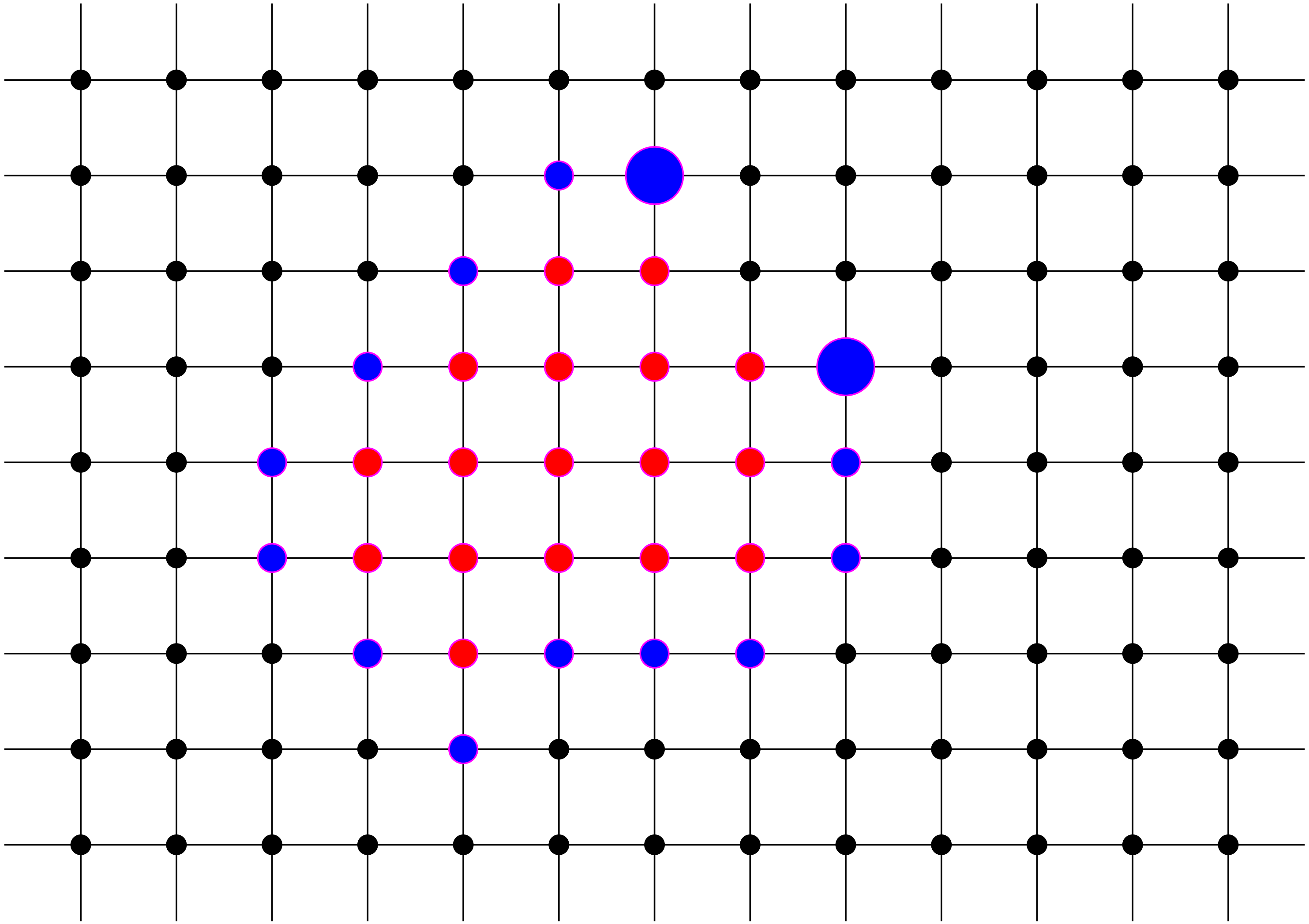
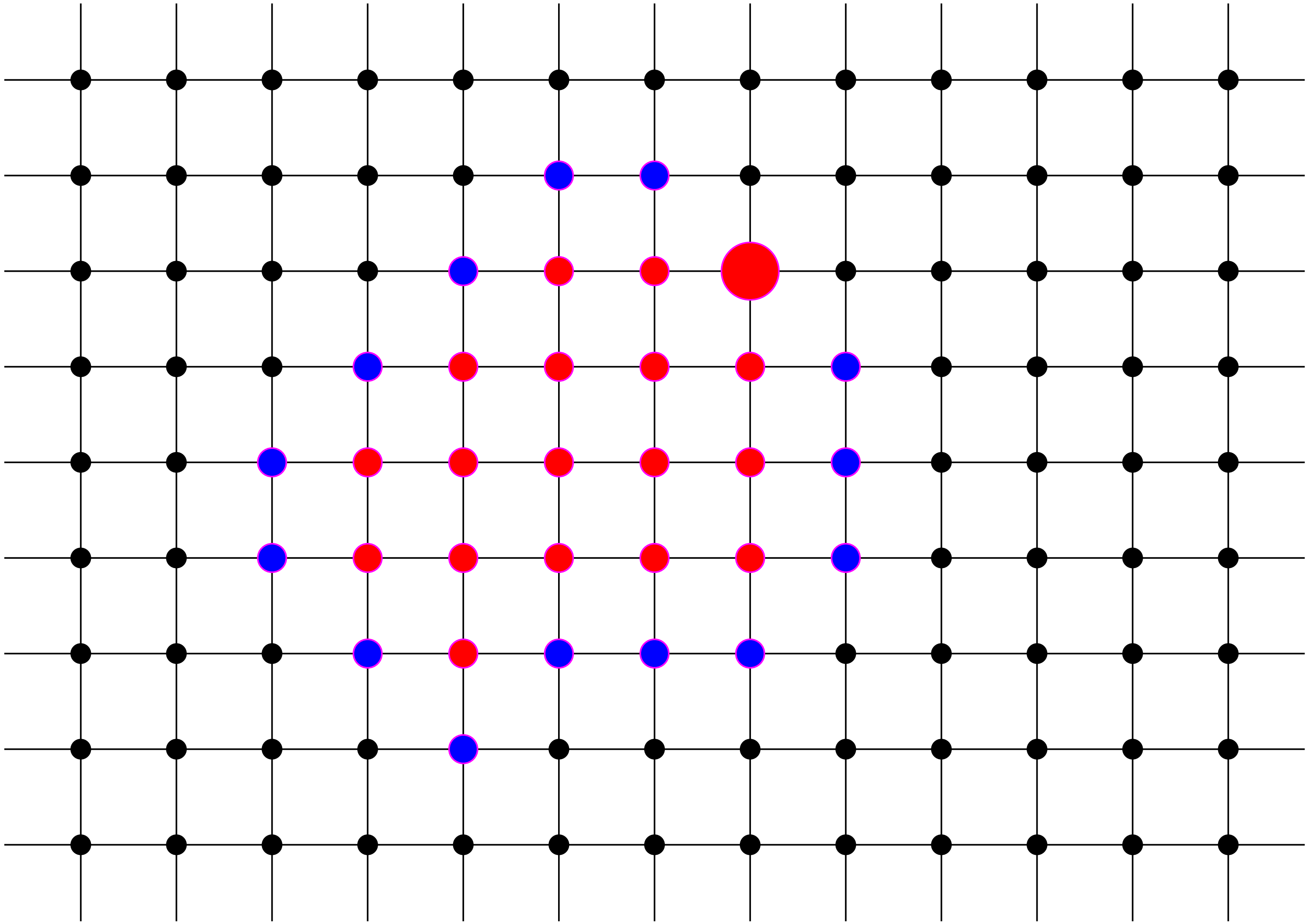
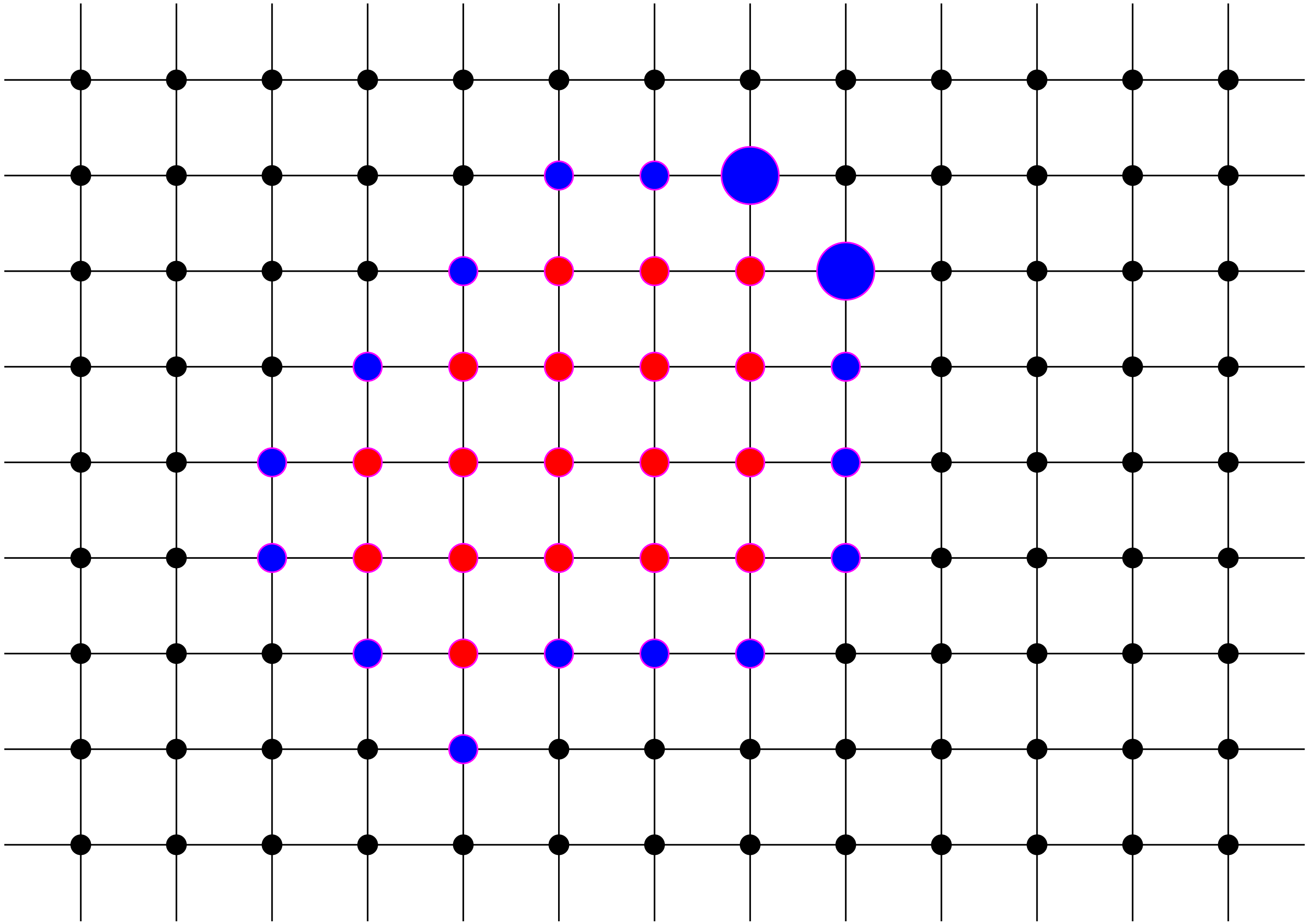


The square grid with two firefighters

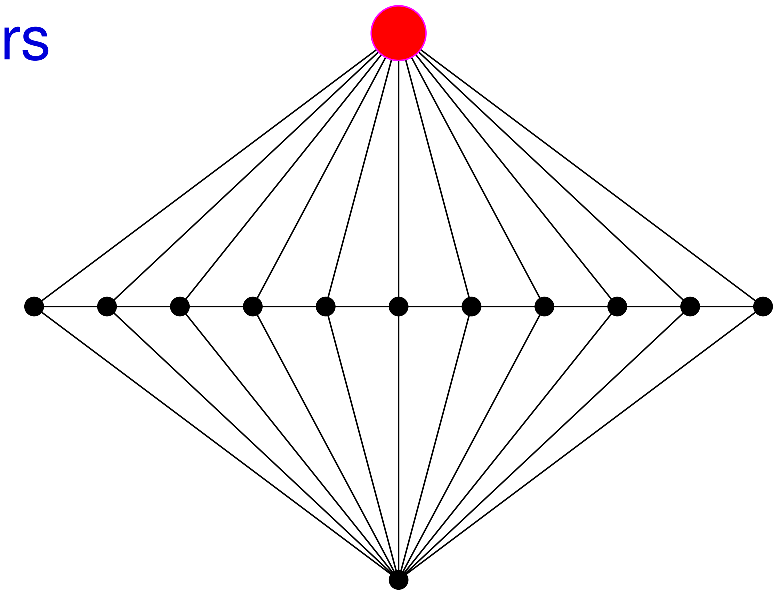


The square grid with two firefighters – done !



Fire containment in graph classes

- suppose we only know the graph is from some graph class
  (say, the graph is planar)
  - and we want to save "most" of the graph
- if we always want to do this,
  we may need many firefighters



Fire containment in graph classes

- what
  if we want to save "most" of the graph, "most of the time" ?

- suppose we have k firefighters
  - $\rho_k(G, v)$: proportion of vertices of G
    that can be saved with k firefighters
    if the fire starts in vertex v
  - $\rho_k(G)$: expected value of $\rho_k(G, v)$
    if v is chosen uniformly at random

$$= \frac{1}{|V(G)|} \sum_{v \in V(G)} \rho_k(G, v)$$

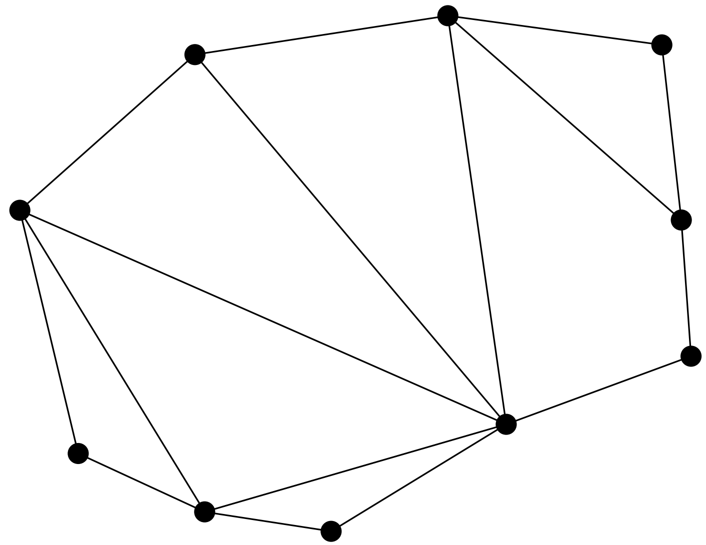
Fire containment in graph classes

- what we really want to answer for some graph class $\mathcal{G}$:
  - what is the minimum k such that $\inf\limits_{G \in \mathcal{G}} \rho_k(G) > 0$?
- in other words :
  - what is the minimum k such that for every $G \in \mathcal{G}$:
    - for a positive fraction of the vertices in G,
      if a fire starts in one of these vertices,
      we can save a positive fraction of G,
      using at most k firefighters at each step
- let's call that the **firefighter number** $\mathit{ff}(\mathcal{G})$ of the class $\mathcal{G}$

The survival rate of some graph classes

Theorem (Cai, Cheng, Verbin & Zhou, 2009+)

- G outerplanar, n vertices $\implies$ $\rho_1(G) \geq 1 - O\left({}^{\log n}/_{n}\right)$
  - **outerplanar** : can be drawn in the plane with all vertices on the outside face



The survival rate of some graph classes

Theorem (Cai, Cheng, Verbin & Zhou, 2009+)

- G outerplanar, n vertices $\Longrightarrow$ $\rho_1(G) \geq 1 - O(\log n / n)$
  - **outerplanar** : can be drawn in the plane with all vertices on the outside face
  - hence for outerplanar graphs $\mathcal{OP}$: $\mathit{ff}(\mathcal{OP}) = 1$
- note that trees are outerplanar

The survival rate of some graph classes

Theorem (Wang, Finbow & Wang, 2010)

- $|E(G)| \le \ell\,|V(G)|$, for some integer ℓ

$$\Longrightarrow \quad \rho_{2\ell-1}(G) \ge {}^{2}/_{5\,\ell}$$

Corollary

- G planar $\Longrightarrow$ $|E(G)| < 3\,|V(G)|$

$$\Longrightarrow \quad \rho_{5}(G) \ge {}^{2}/_{15}$$

  - hence for planar graphs $\mathcal{P}$: $\mathit{ff}\,(\mathcal{P}) \le 5$

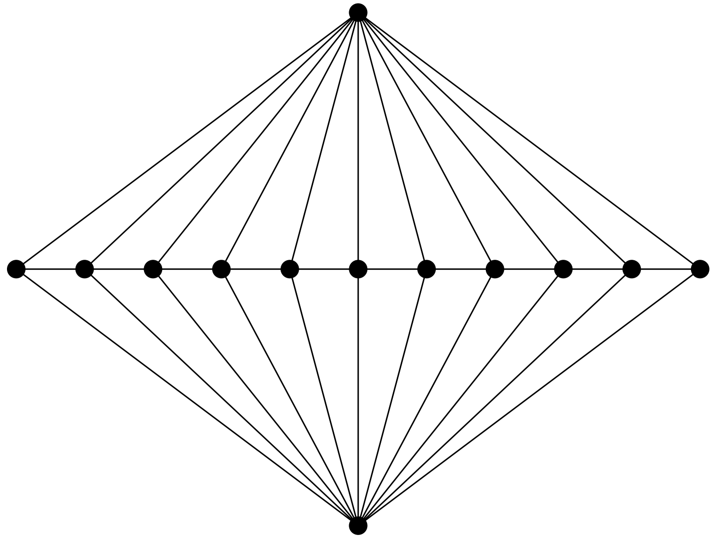
Question

- what is the firefighter number $\mathit{ff}\,(\mathcal{P})$ of planar graphs ?

The survival rate of planar graphs

Observation

- $\mathit{ff}(\mathcal{P}) \geq 2$:



The survival rate of planar graphs

Observation

- $\mathit{ff}\,(\mathcal{P}) \;\geq\; 2$

Theorem

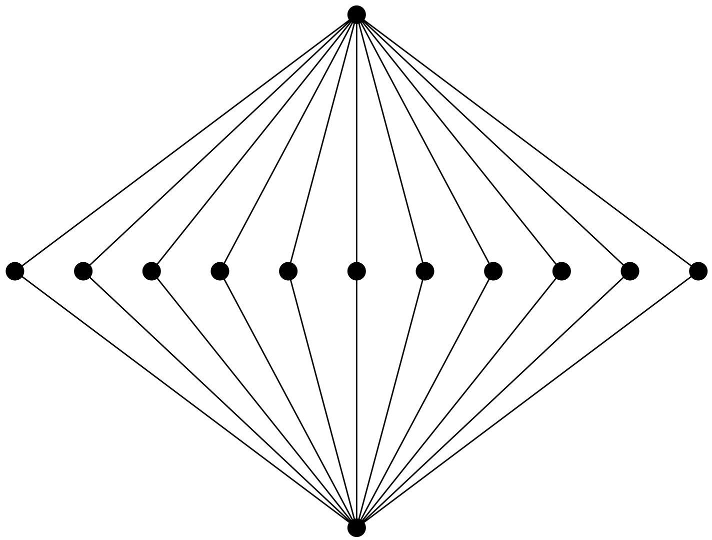
- $\mathit{ff}\,(\mathcal{P}) \;\leq\; 4$

- in fact, we can prove : $\mathit{ff}\,(\mathcal{P}) \;\leq\;$ " $3 + \varepsilon$ "
  - we need 4 firefighters in the first step only,
    for each following step we need only 3 firefighters

The survival rate of triangle-free planar graphs

Theorem

- G triangle-free planar $\Longrightarrow$ $\rho_2(G) \geq {}^1/_{238320}$
  - so for triangle-free planar graphs $\mathcal{P}_4$: $\mathit{ff}(\mathcal{P}_4) \leq 2$
  - this is best possible :

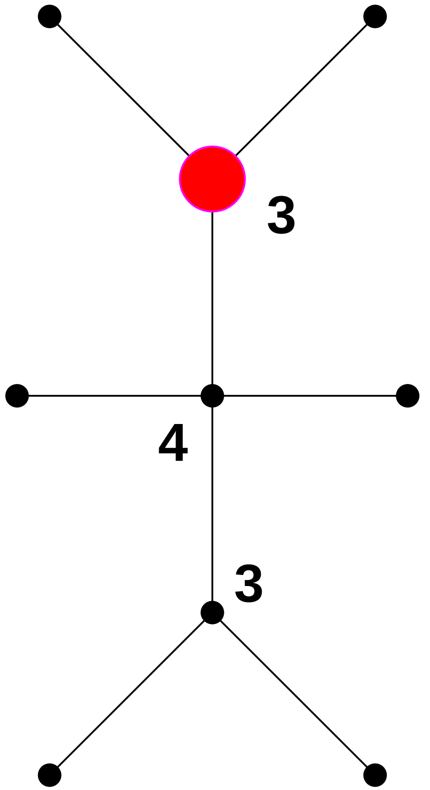


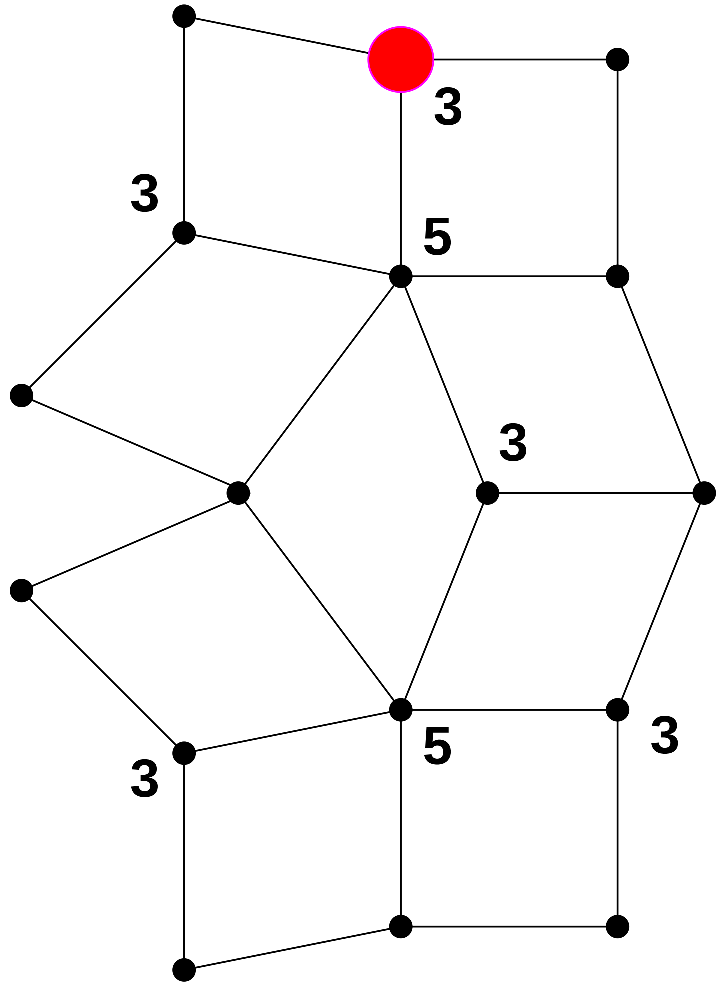
- **so** : $\mathit{ff}(\mathcal{P}_4) = 2$

Some ideas from the proofs

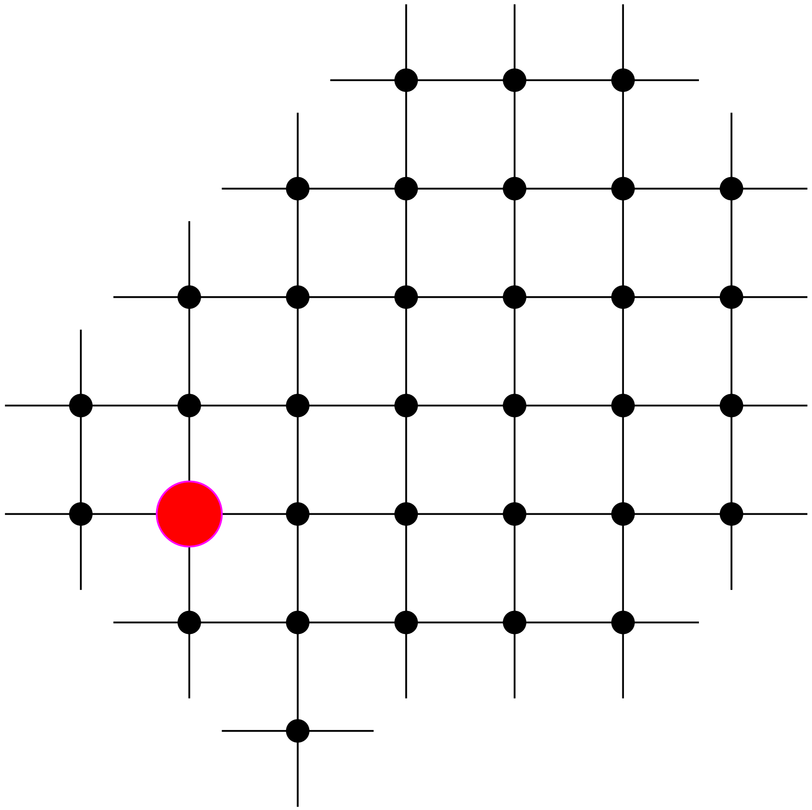
- the proofs consist of two main steps :
  - find a collection of "defendable configurations"
    - a subgraph with a given vertex v,
so that if the fire starts in v,
only a finite number of vertices will be lost
  - show that there is a constant $\alpha > 0$,
so that every graph G in the class
has at least $\alpha \cdot |V(G)|$ defendable configurations
    - uses the discharging method,
but in a non-standard way

Defendable configurations for 2 firefighters





And another defendable configuration for 2 firefighters



The discharging method

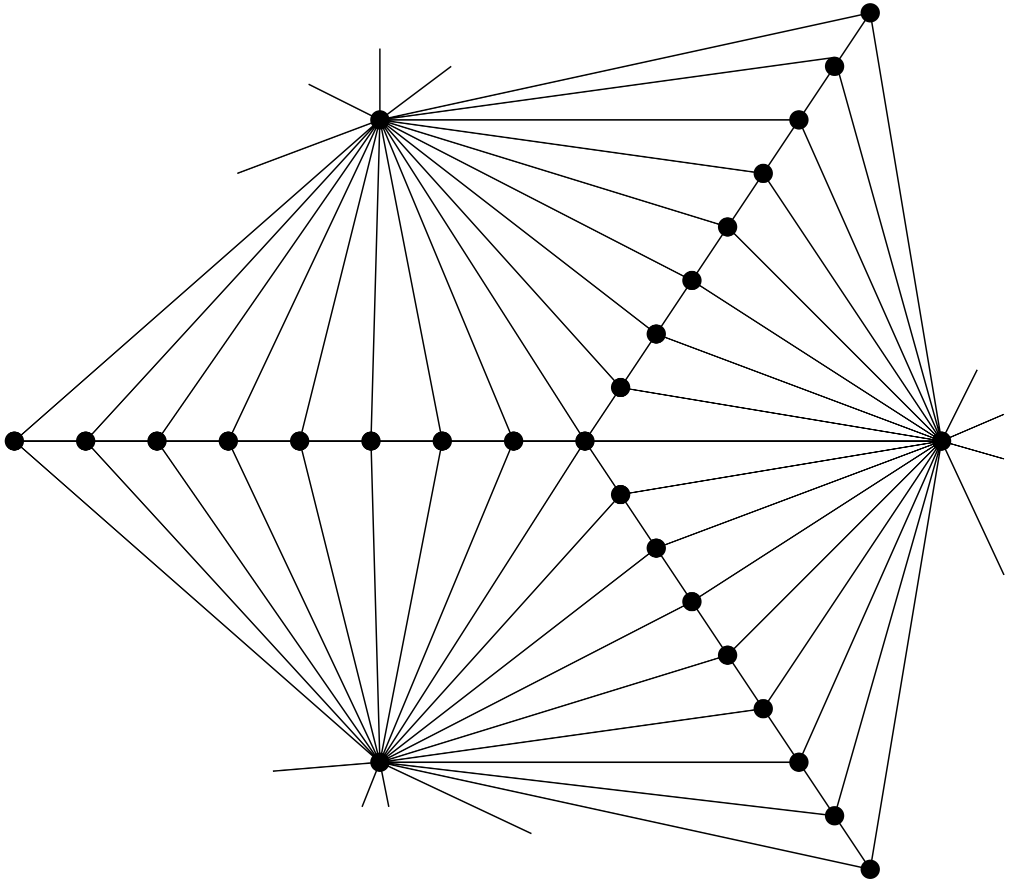
- traditionally used to show that any planar graph :
  - contains at least one subgraph
    from a set of “good” configurations
- we needed to modify it to show that any planar graph :
  - contains many subgraphs (in fact, linearly many)
    from the set of defendable configurations
    - our techniques are not that different from the traditional method
    - but we need to be much more precise

The firefighter number of planar graphs

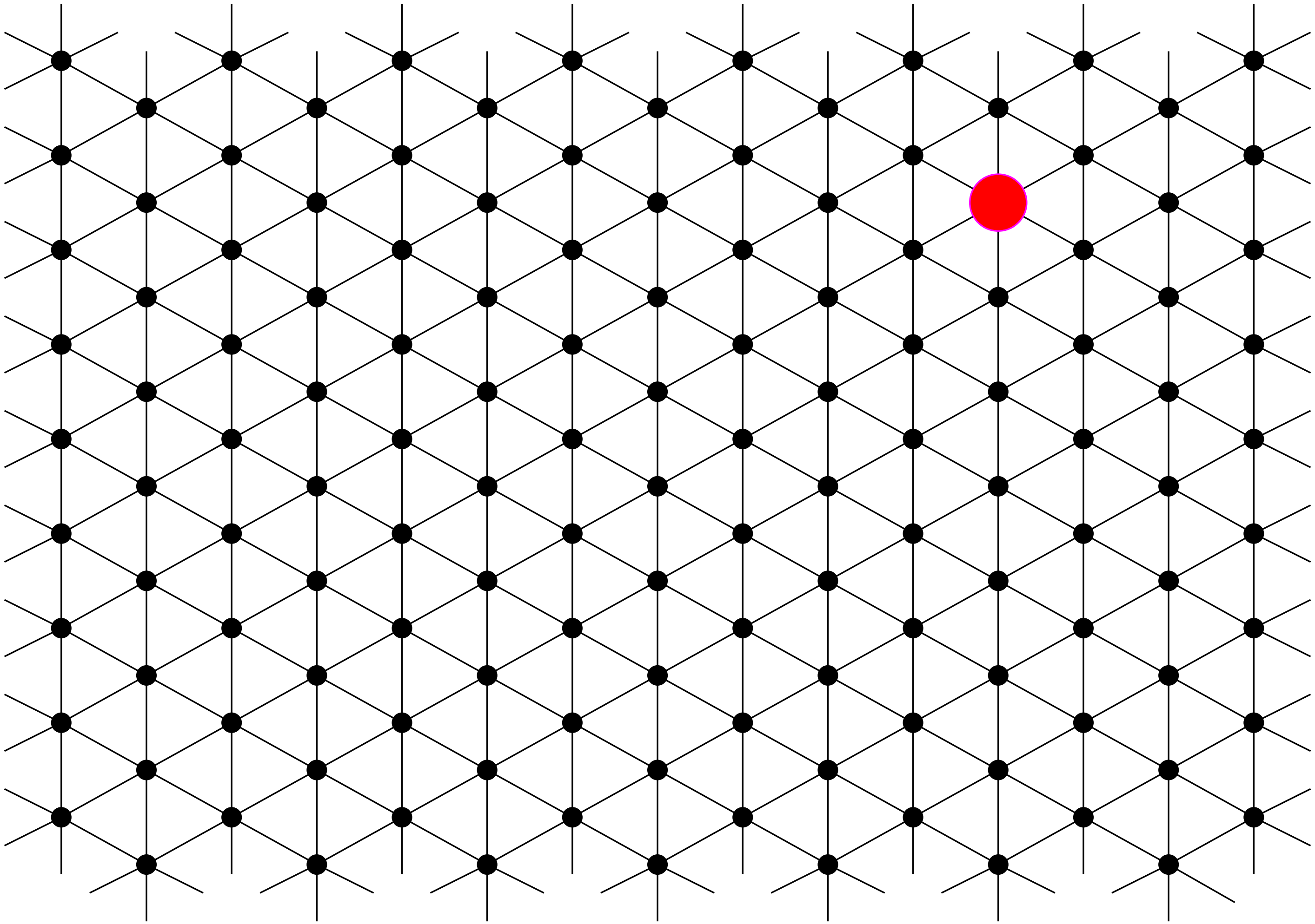
- so now we know : $2 \le \mathit{ff}(\mathcal{P}) \le 4$
- next step : prove that $\mathit{ff}(\mathcal{P}) \le 3$
  - i.e., get rid of the extra firefighter needed in the first step
  - maybe possible to extend our proof,
    using some more careful ("messier") analysis
- and then, can we go to : $\mathit{ff}(\mathcal{P}) = 2$?
  - very likely to need new ideas

The challenge of $\mathit{ff}(\mathcal{P}) \leq 2$

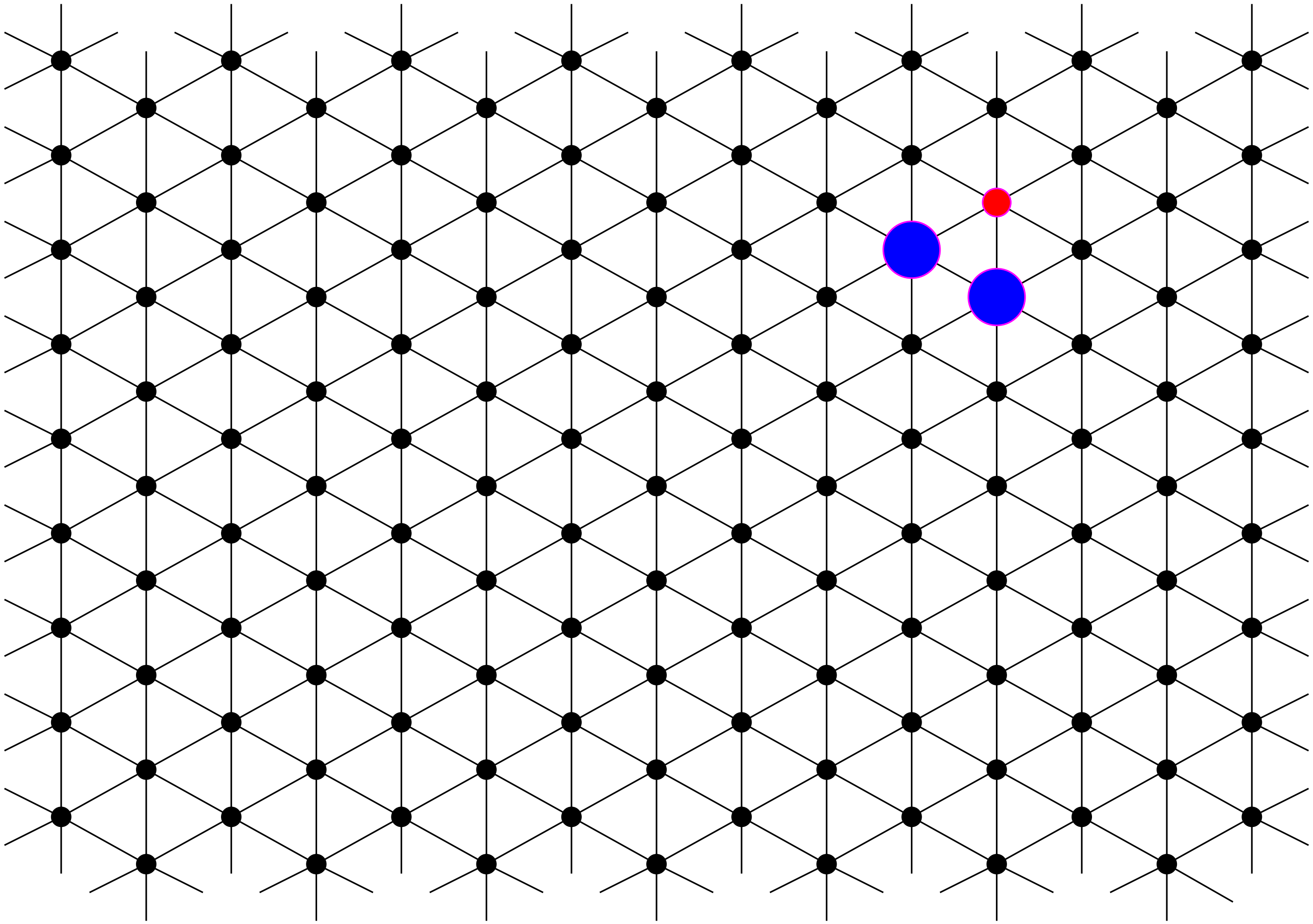
- sometimes the firefighters need to act locally



But sometimes acting locally doesn't work

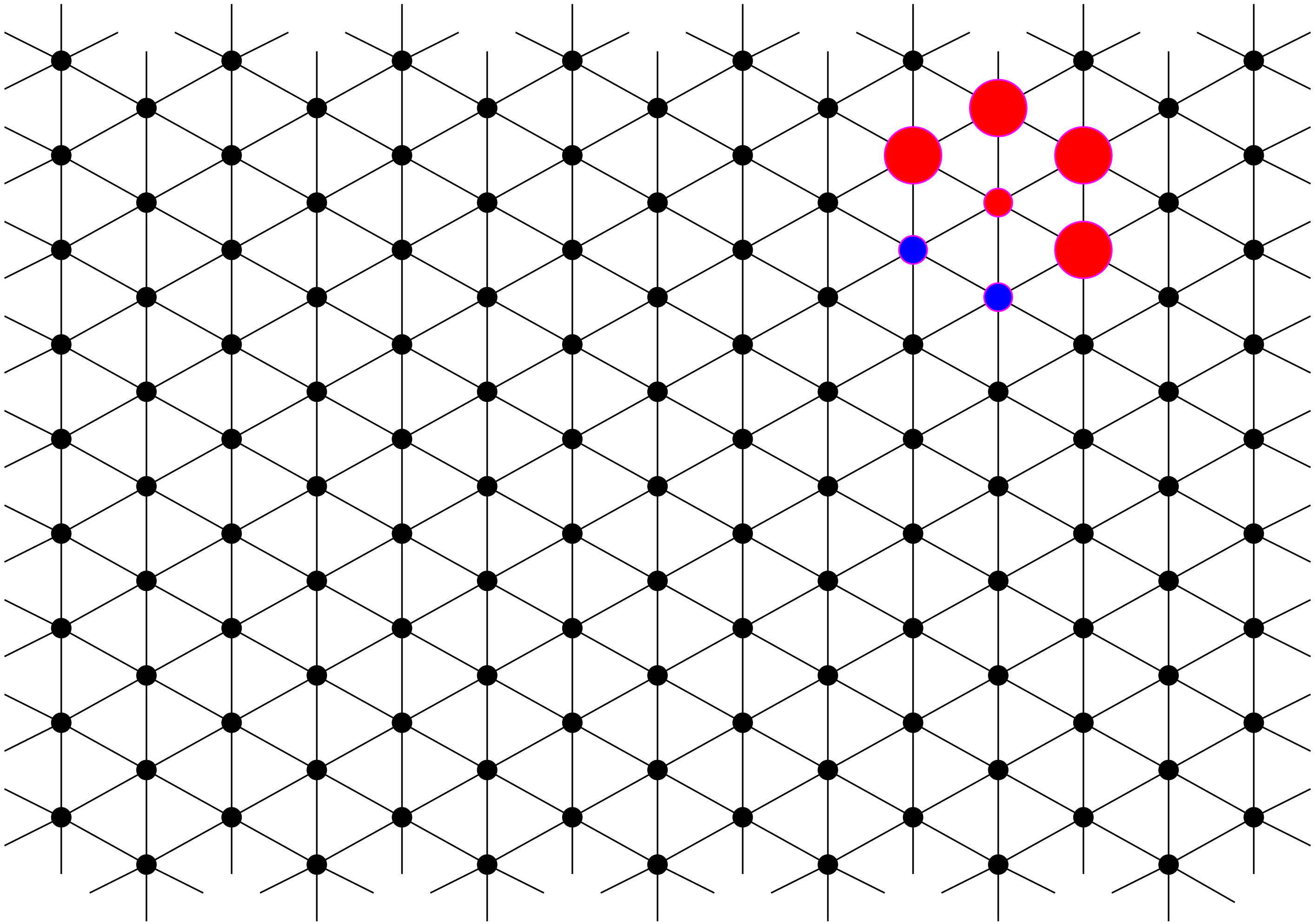


*But sometimes acting **locally** doesn't work*



- looks like a sensible start . . .

But sometimes acting locally doesn't work



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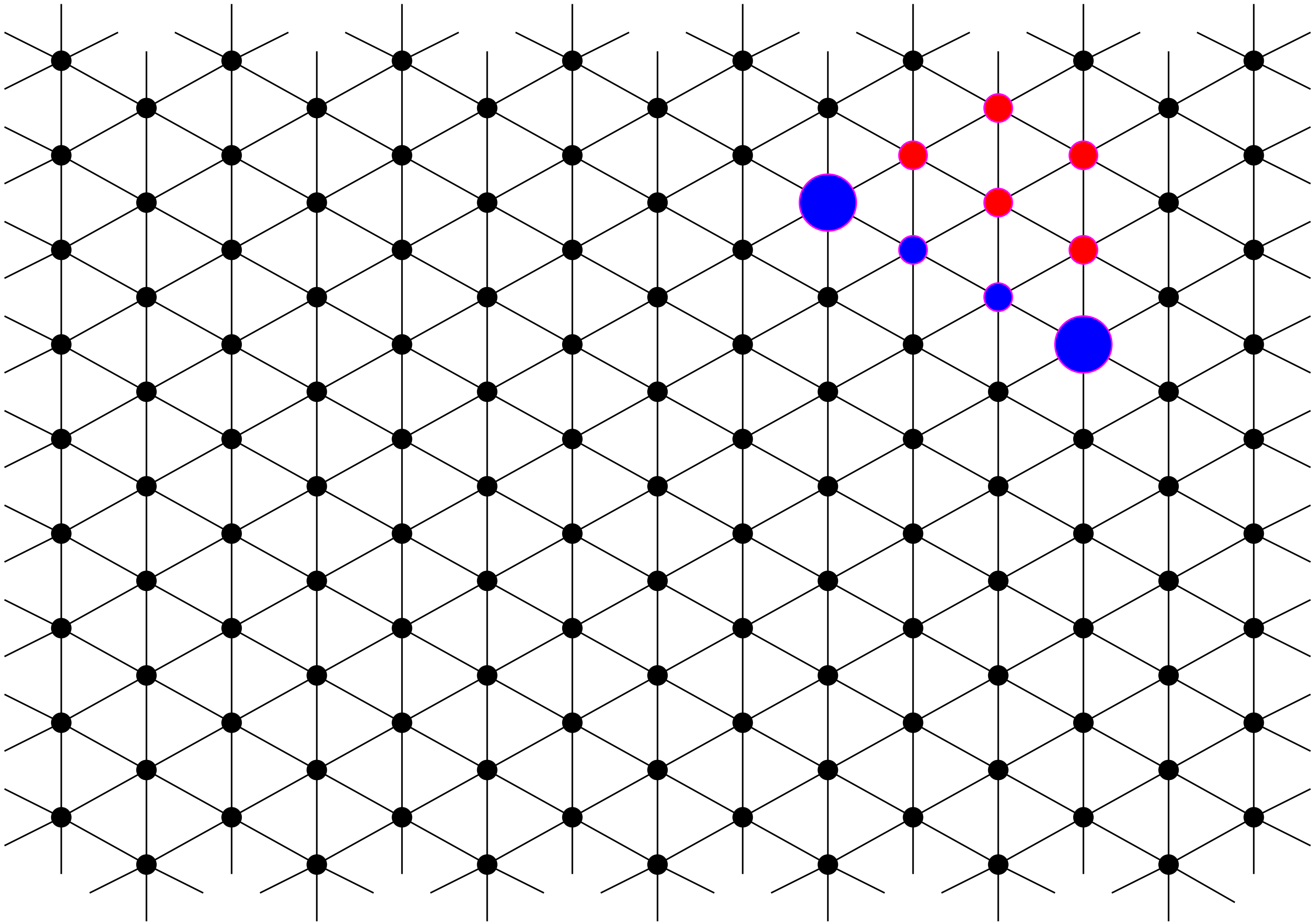
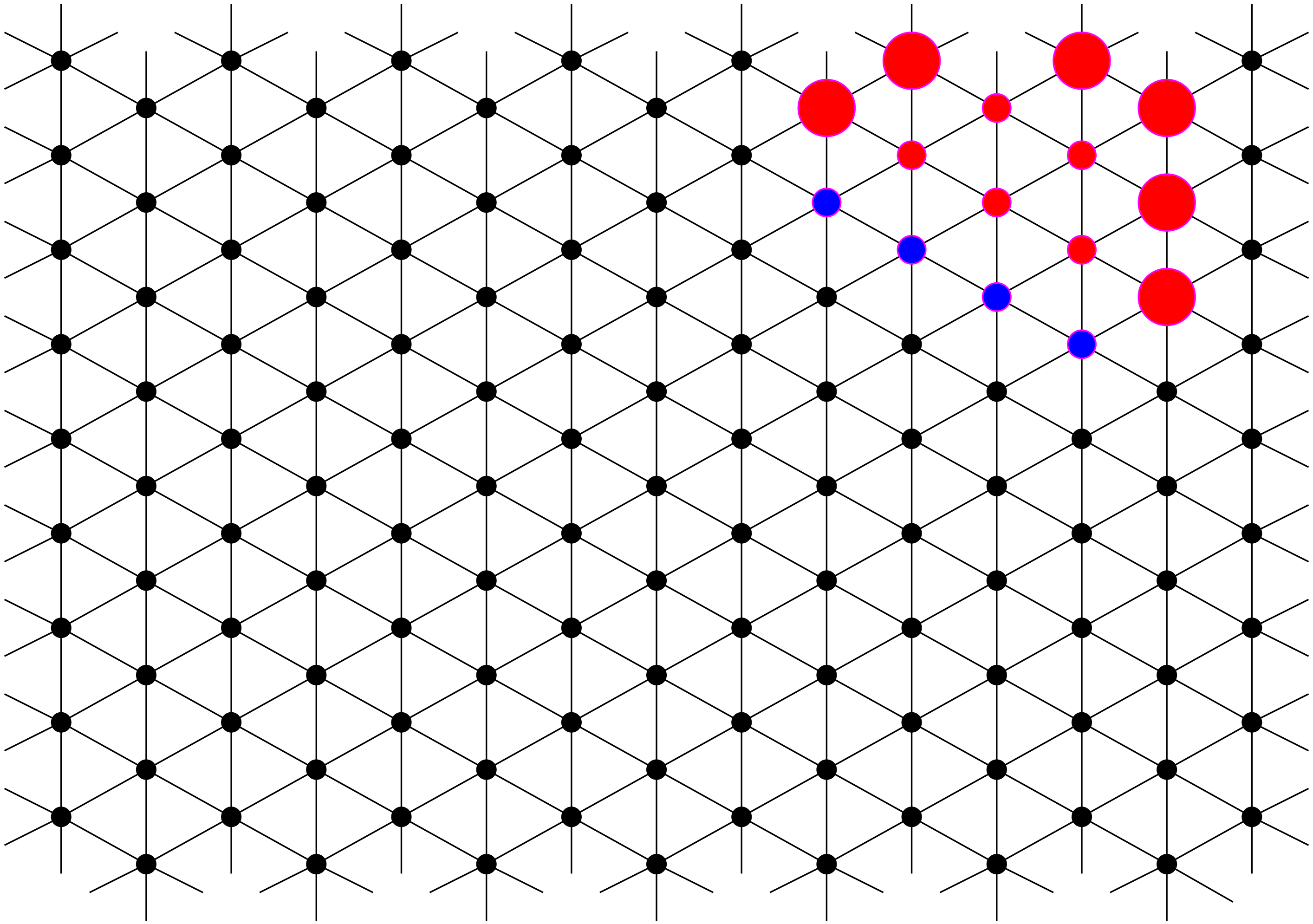


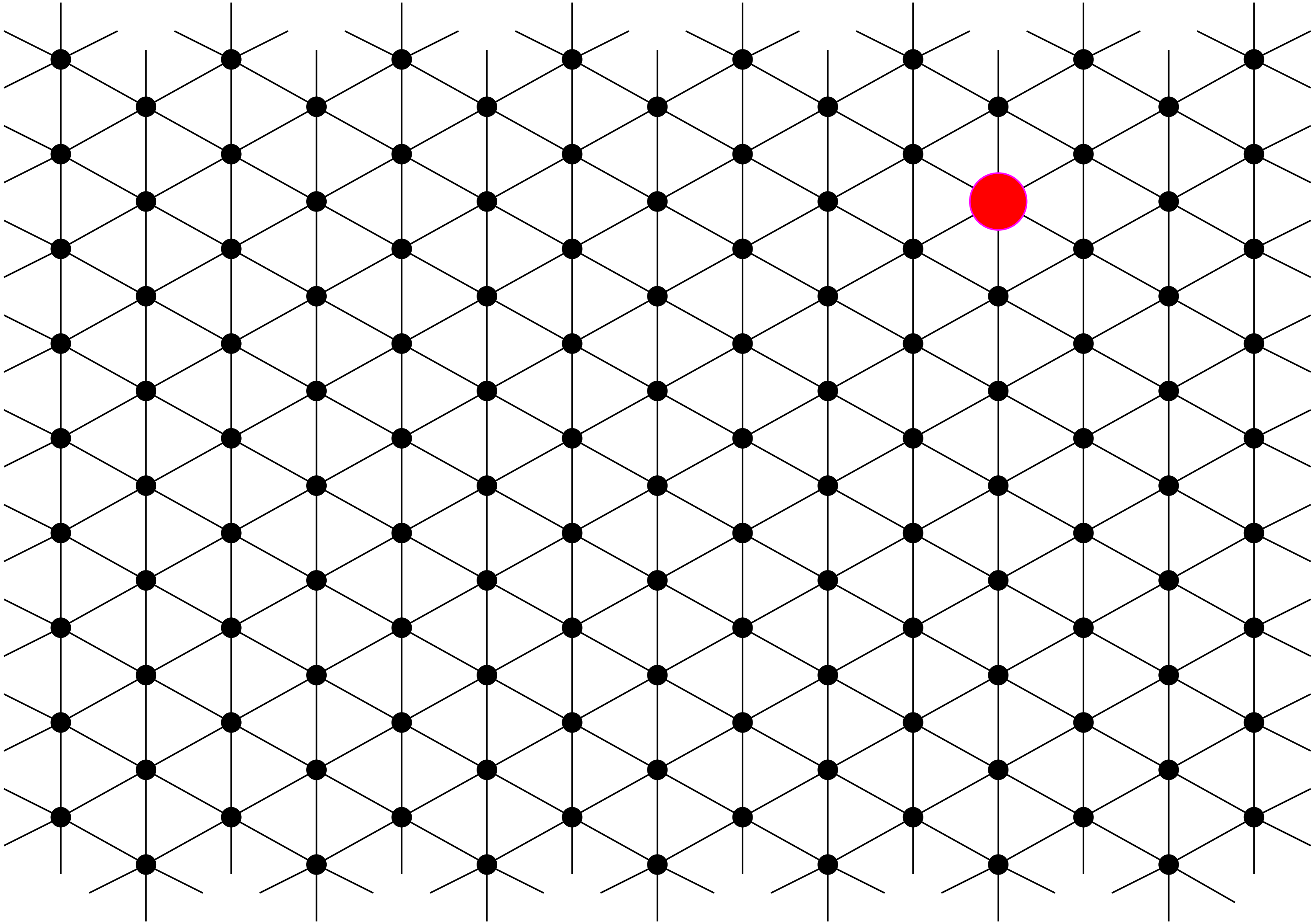
- continue with that strategy . . .

But sometimes acting locally doesn't work



- but what happens on the boundary . . . ?

Sometimes we must think *globally*



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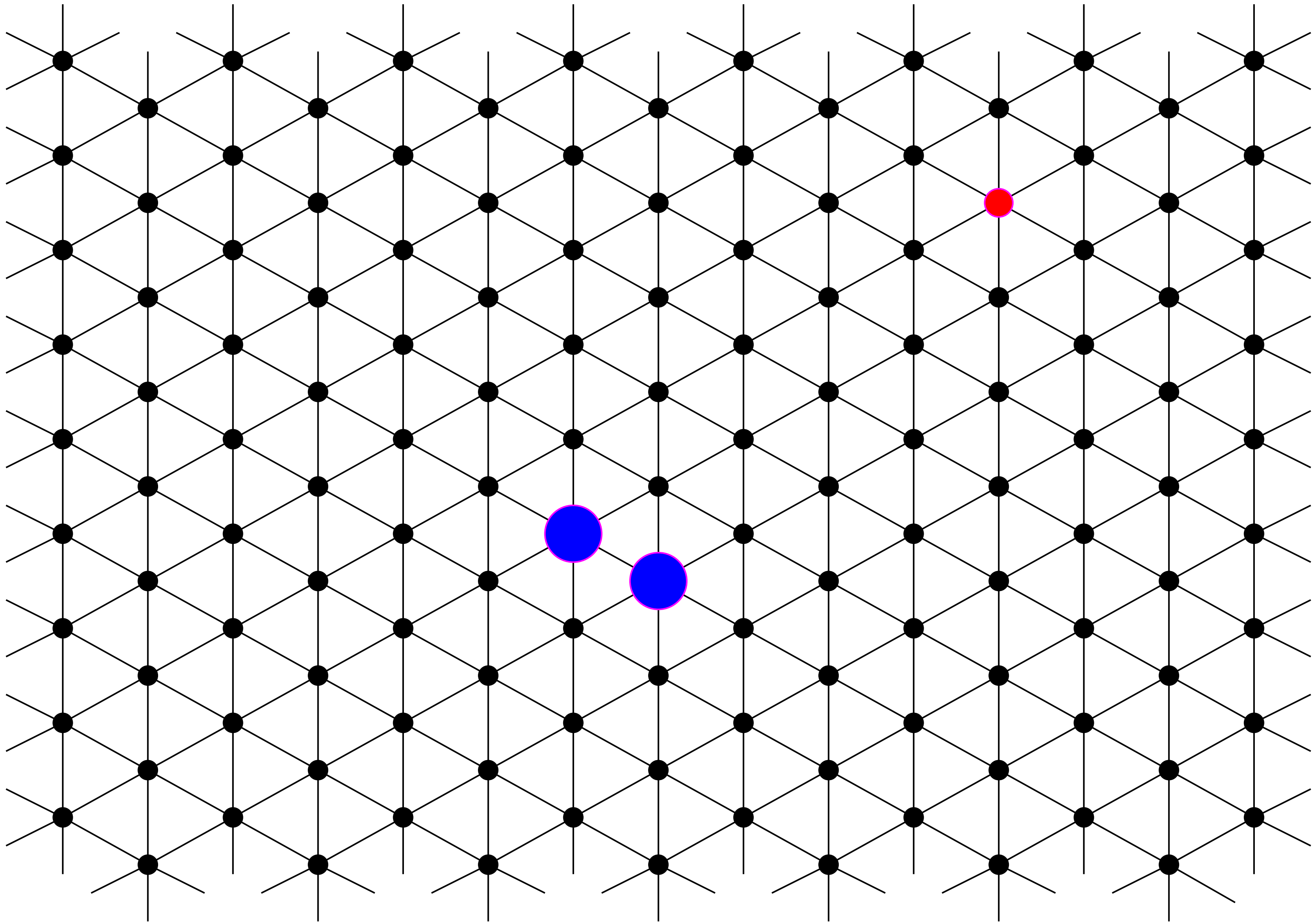

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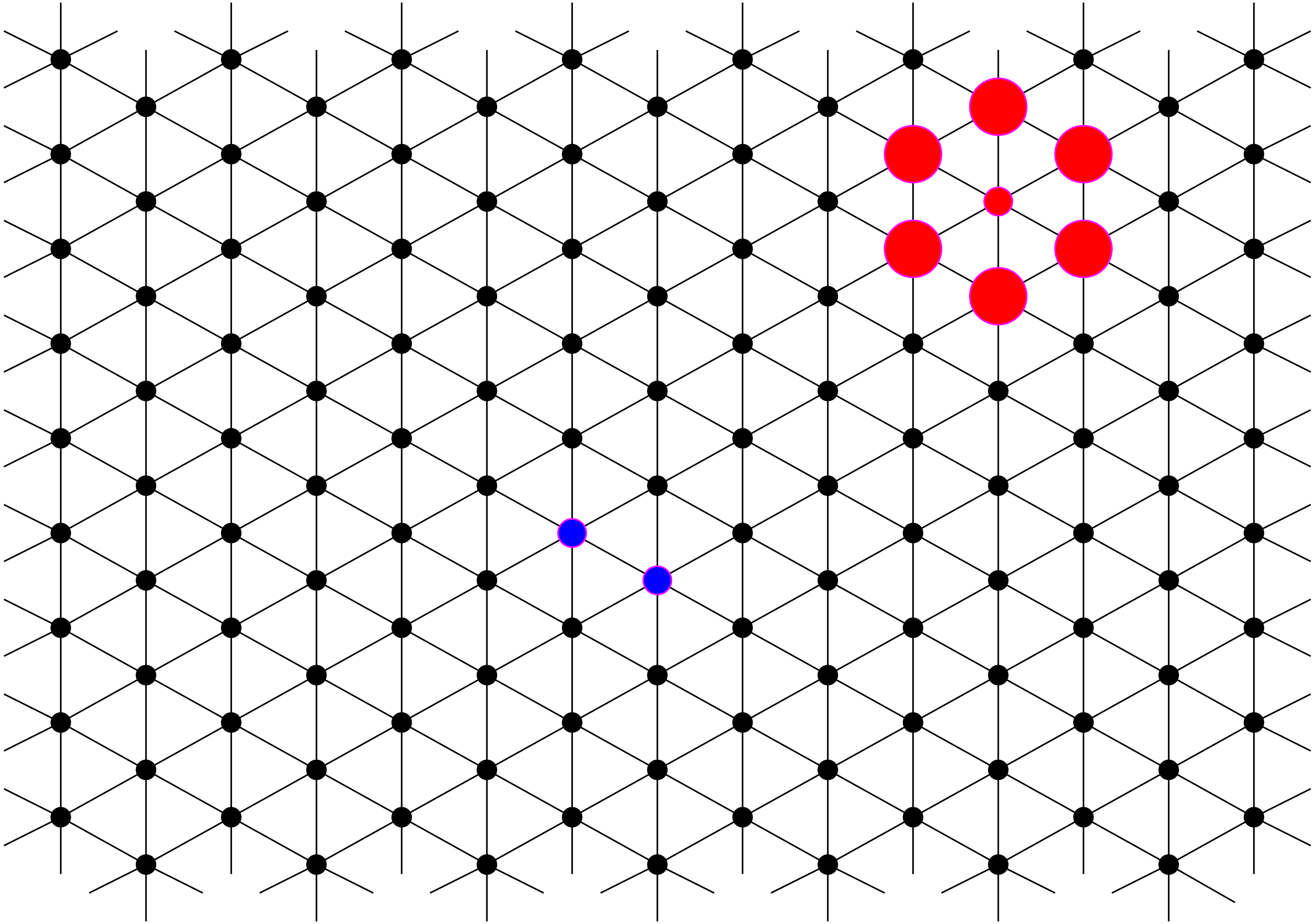

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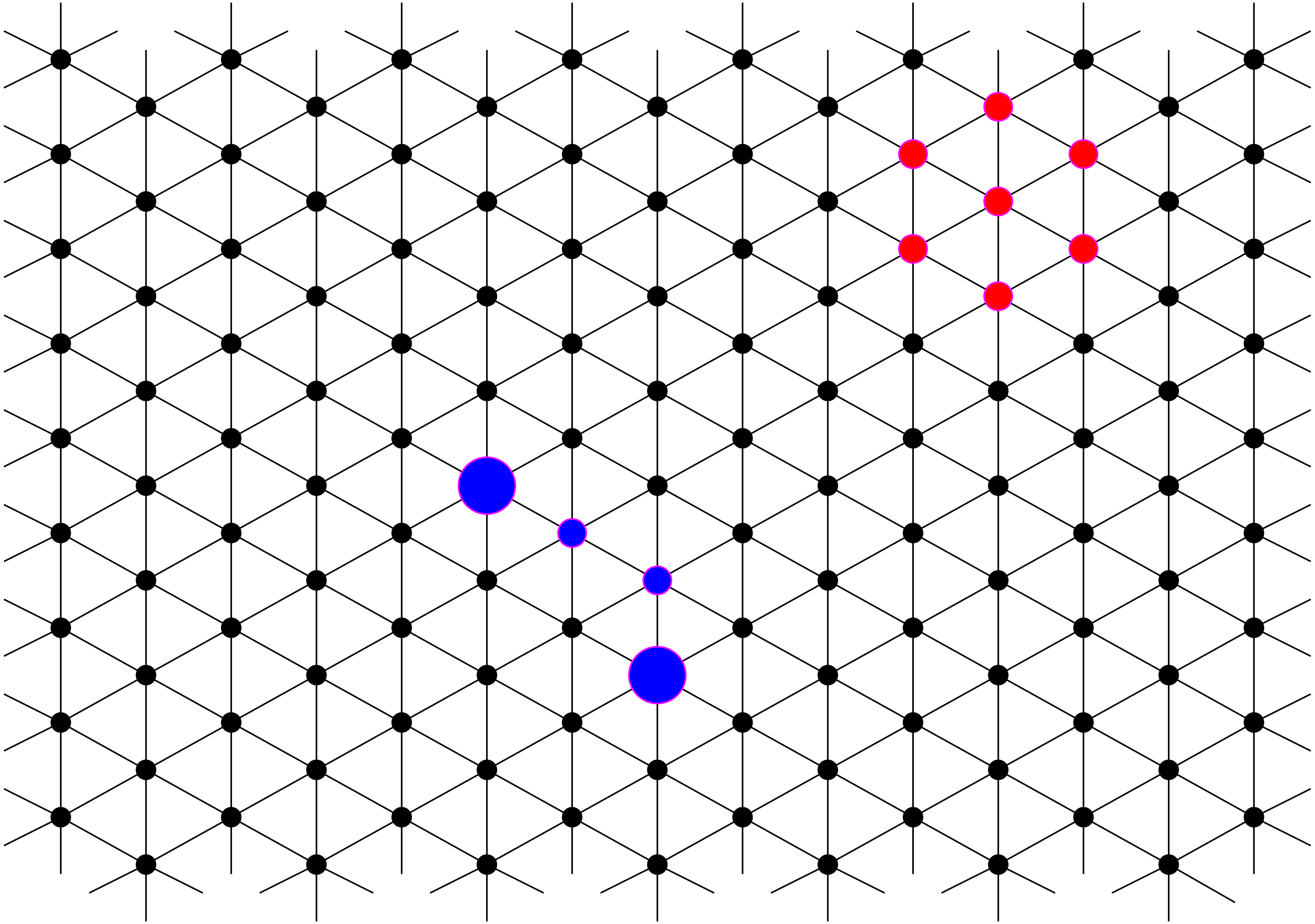

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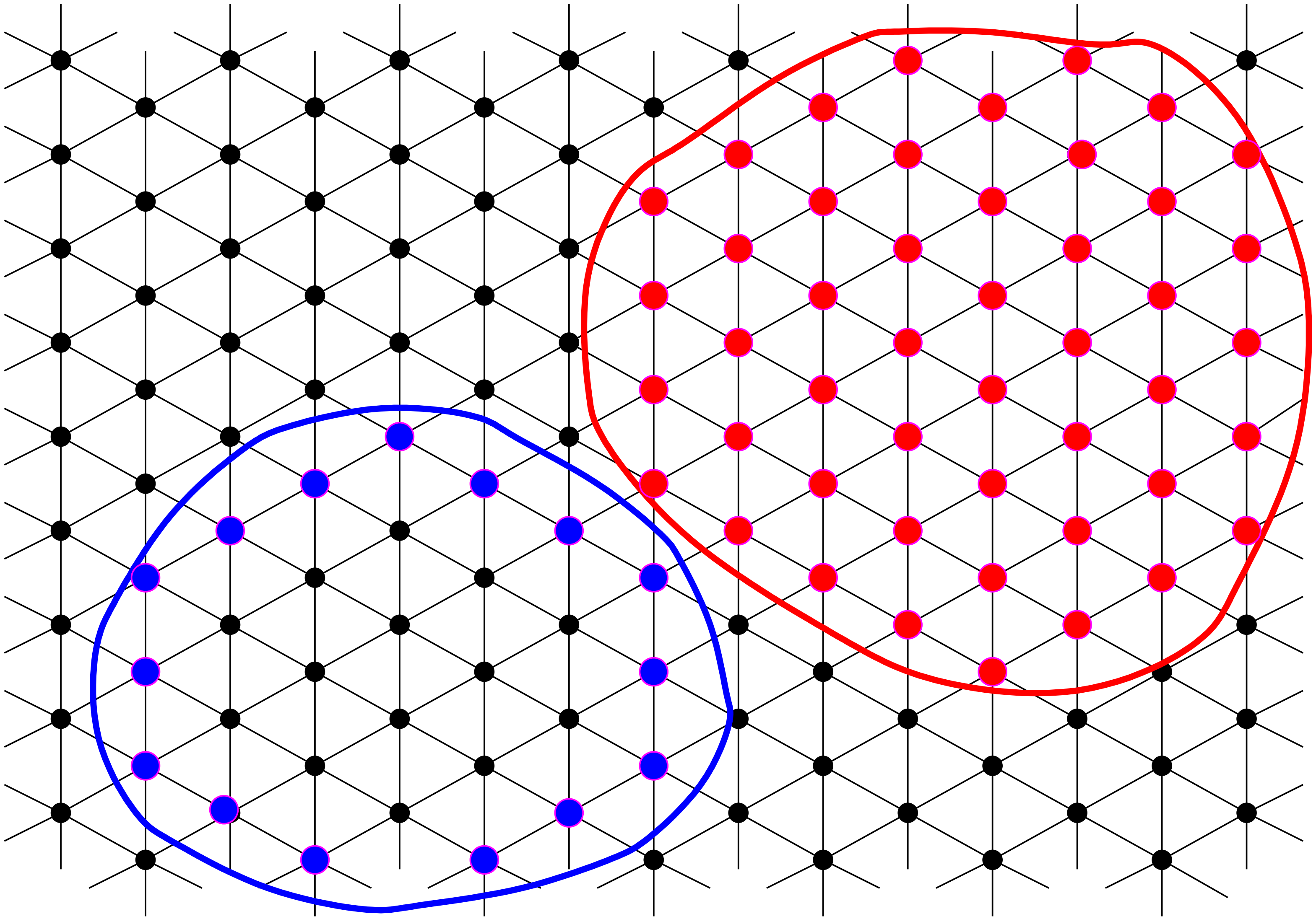
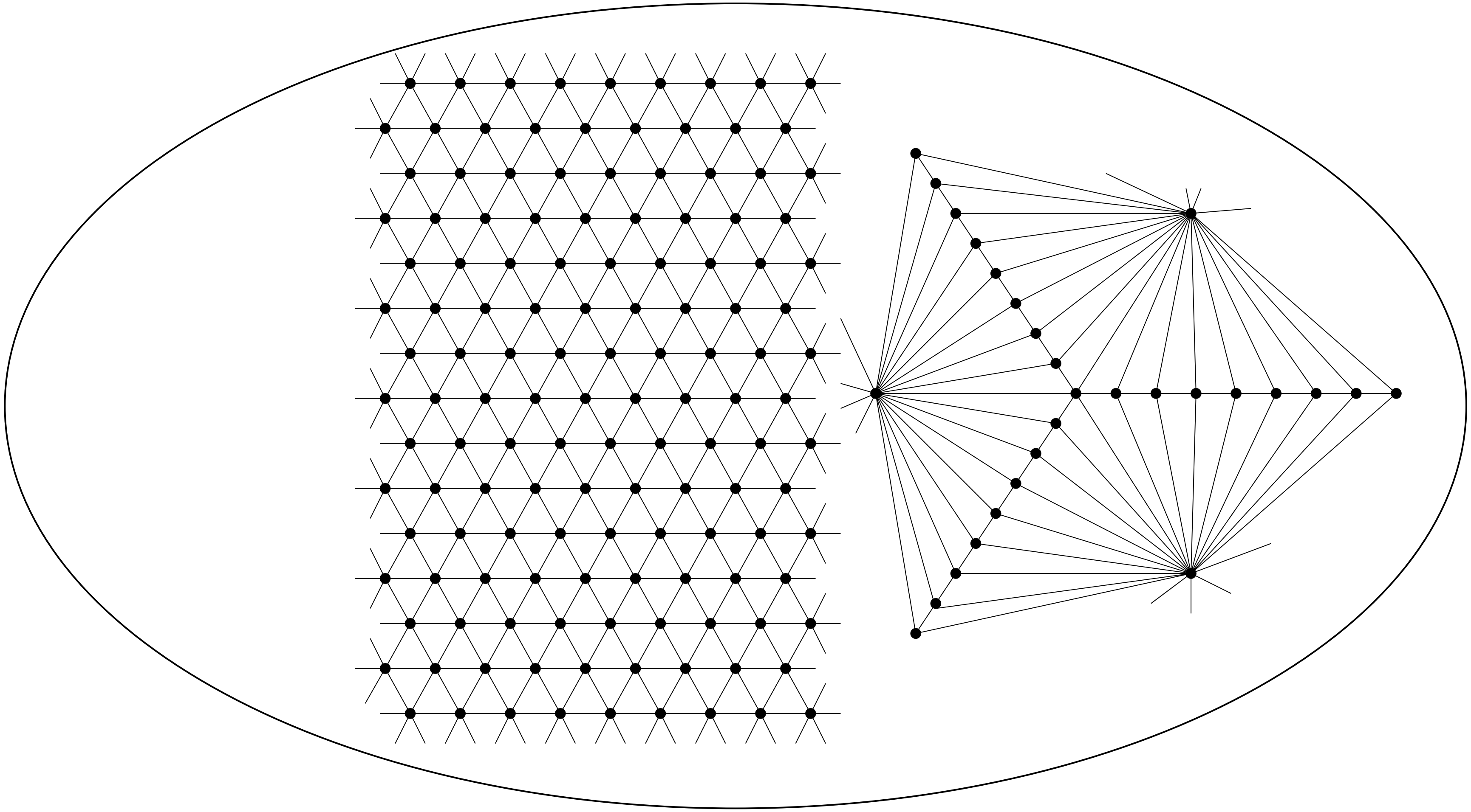


But what to do in general ?



The firefighter number of planar graphs

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  - and is it even true . . . ?