Fractional Colouring and
Pre-colouring Extension of Graphs

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Graph colouring and pre-colouring

- **G** a graph

- **chromatic number** $\chi(G)$:

  minimum k so that a vertex-colouring exists

general question :

- what can we say if some vertices are already pre-coloured ?

- in particular :   can $\chi(G)$ colours still be enough ?

  - in general :   **no**

Pre-colouring questions

next best questions :

- how many extra colours may be needed ?

- and what conditions on the pre-coloured vertices can make life easier ?

Question   (Thomassen, 1997)

- G planar,

  $W \subseteq V(G)$, a set of vertices so that

  distance between any two vertices in W is at least 100

  - can any 5-colouring of W

    be extended to a 5-colouring of G ?

The first answer

■ dist(W) : minimum distance between any two vertices in W

Theorem   (Albertson, 1998)

■ G any graph with chromatic number χ

$W \subseteq V(G)$ with dist$(W) \geq 4$

$\implies$ any $(\chi+1)$-colouring of W

can be extended to a $(\chi+1)$-colouring of G

Some more answers

Theorem (Albertson, 1998)

- G planar graph

  $W \subseteq V(G)$ with dist$(W) \geq 3$

  $\implies$ any 6-colouring of W

  can be extended to a 6-colouring of G

Theorem

- G any graph with chromatic number χ

  $W \subseteq V(G)$ with dist$(W) \geq 3$

  $\implies$ any $(\chi+\chi)$-colouring of W

  can be extended to a $(\chi+\chi)$-colouring of G

Fractional colouring

- **fractional K-colouring** of graph G ($K \in \mathbf{R}$, $K \geq 0$) :

  - every vertex $v \in V$ is assigned a subset $\phi(v) \subseteq [0, K]$ so that :

    - every subset $\phi(v)$ has 'measure' 1

    - and $uv \in E(G) \implies \phi(u) \cap \phi(v) = \varnothing$

- **fractional chromatic number** $\chi_F(G)$:

  $$= \inf \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

  $$= \min \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

Fractional colouring

- **note** :   we always have $\chi_F(G) \leq \chi(G)$

  - but the difference can be arbitrarily large

- - $\chi_F(G) = 1 \iff G$ has no edges

  - $\chi_F(G) = 2 \iff G$ has edges and is bipartite

  - for all rational $K \geq 2$:   there exist G with $\chi_F(G) = K$

Pre-colouring in the fractional world

- so now suppose that for some vertices $W \subseteq V(G)$,
  the vertices in W are already pre-coloured :

  - vertices $w \in W$ have been given some set $\phi(w)$
    of measure 1

- when can this be extended to a fractional colouring of the
  whole graph G ?

- in general we should expect to
  require more than $\chi_F(G)$ colours

The set-up of the problem

- - G a graph with fractional chromatic number $\chi_F \geq 2$

  - $D \geq 3$ an integer

  - $W \subseteq V(G)$ with $\operatorname{dist}(W) \geq D$

- - the vertices $w \in W$

    are pre-coloured with $\phi(w) \subseteq [0, \chi_F + \alpha]$

  - for some real $\alpha \geq 0$

  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$

- how large should α be to make sure this can be done ?

A major part of the answer

Theorem (Král', Krnc, Kupec, Lužar & Volec, 2011)

■ extension is always possible, provided α is at least :

$$
\frac{\sqrt{(\lfloor D/4 \rfloor \, \chi_F - 1)^2 + 4 \lfloor D/4 \rfloor \, (\chi_F - 1)} - \lfloor D/4 \rfloor \, \chi_F + 1}{2 \lfloor D/4 \rfloor},
$$
if $D \equiv 0 \bmod 4$

$$
\frac{\chi_F - 1}{\lfloor D/4 \rfloor \, \chi_F},
$$
if $D \equiv 1 \bmod 4$

$$
\frac{\sqrt{(\lfloor D/4 \rfloor \, \chi_F)^2 + 4 \lfloor D/4 \rfloor \, (\chi_F - 1)} - \lfloor D/4 \rfloor \, \chi_F}{2 \lfloor D/4 \rfloor},
$$
if $D \equiv 2 \bmod 4$

$$
\frac{\chi_F - 1}{\lfloor D/4 \rfloor \, \chi_F + \chi_F - 1},
$$
if $D \equiv 3 \bmod 4$

A major part of the answer

Theorem   (Kráľ, Krnc, Kupec, Lužar & Volec, 2011)

- moreover, these bounds on α are best possible,

    - if $D = 3$ and $\chi_F \geq 2$;

    - if $D \geq 4$ and $\chi_F \in \{2\} \cup [3, \infty)$

A major part of the answer – best possible

- **in other words** :

    - for all integers $D \geq 3$,

        all rational numbers $\chi_F \in \{2\} \cup [3, \infty)$,

        and all $\alpha \geq 0$ failing the bound for that D and χ_F

    - there is a graph G with fractional chromatic number χ_F,

        a subset $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$,

        and a fractional pre-colouring $\phi(w) \subseteq [0, \chi_F + \alpha]$

        for $w \in W$

    - so that ϕ cannot be extended to a fractional colouring
        of the whole G, using colours from $[0, \chi_F + \alpha]$ only

A major part of the answer

Theorem   (Král', Krnc, Kupec, Lužar & Volec, 2011)

- ■ extension is always possible, provided α is at least :

  - ■ $$\frac{\sqrt{(\lfloor D/4 \rfloor \chi_F - 1)^2 + 4 \lfloor D/4 \rfloor (\chi_F - 1)} - \lfloor D/4 \rfloor \chi_F + 1}{2 \lfloor D/4 \rfloor},$$

    if $D \equiv 0 \bmod 4$

  - ■ $$\frac{\chi_F - 1}{\lfloor D/4 \rfloor \chi_F},$$
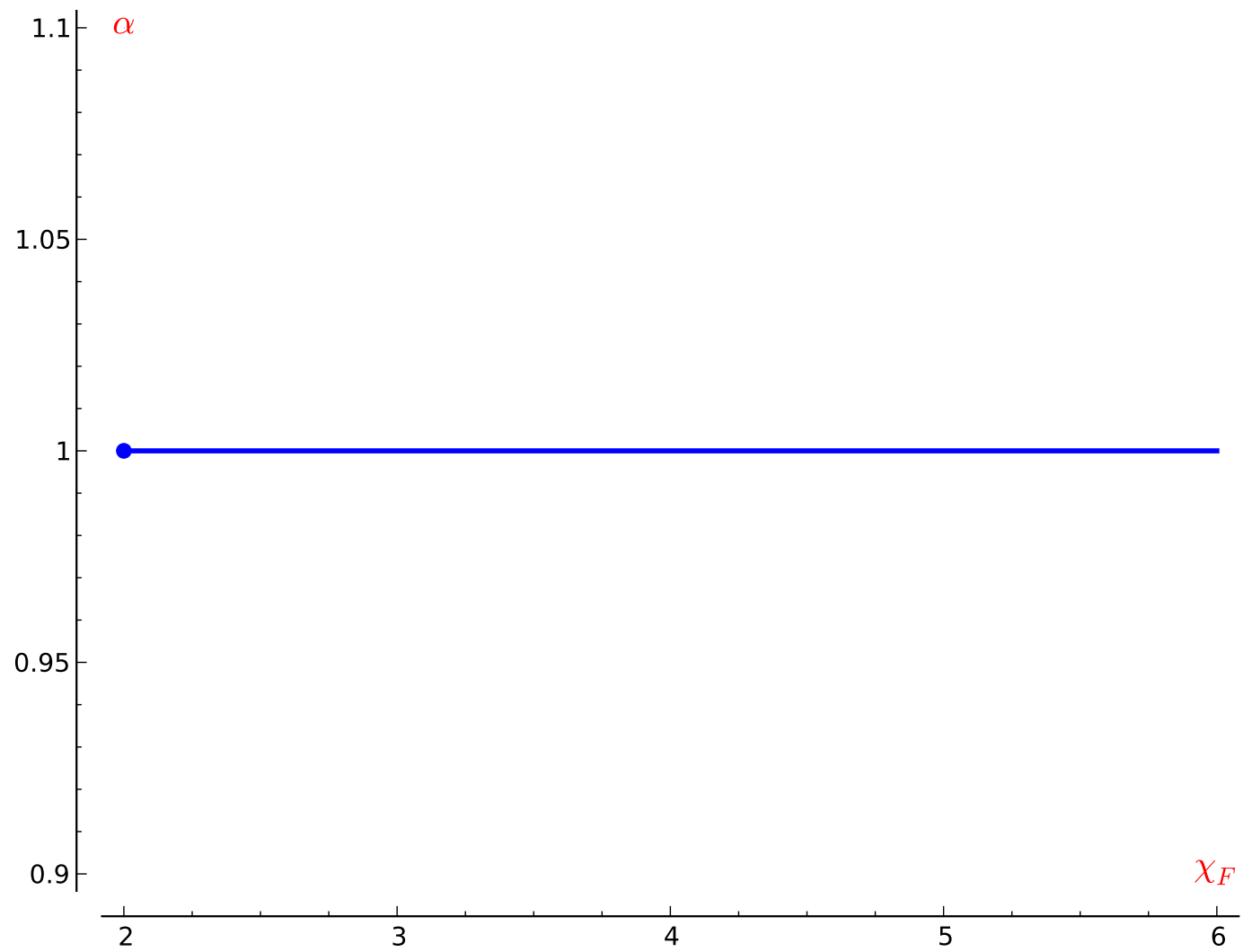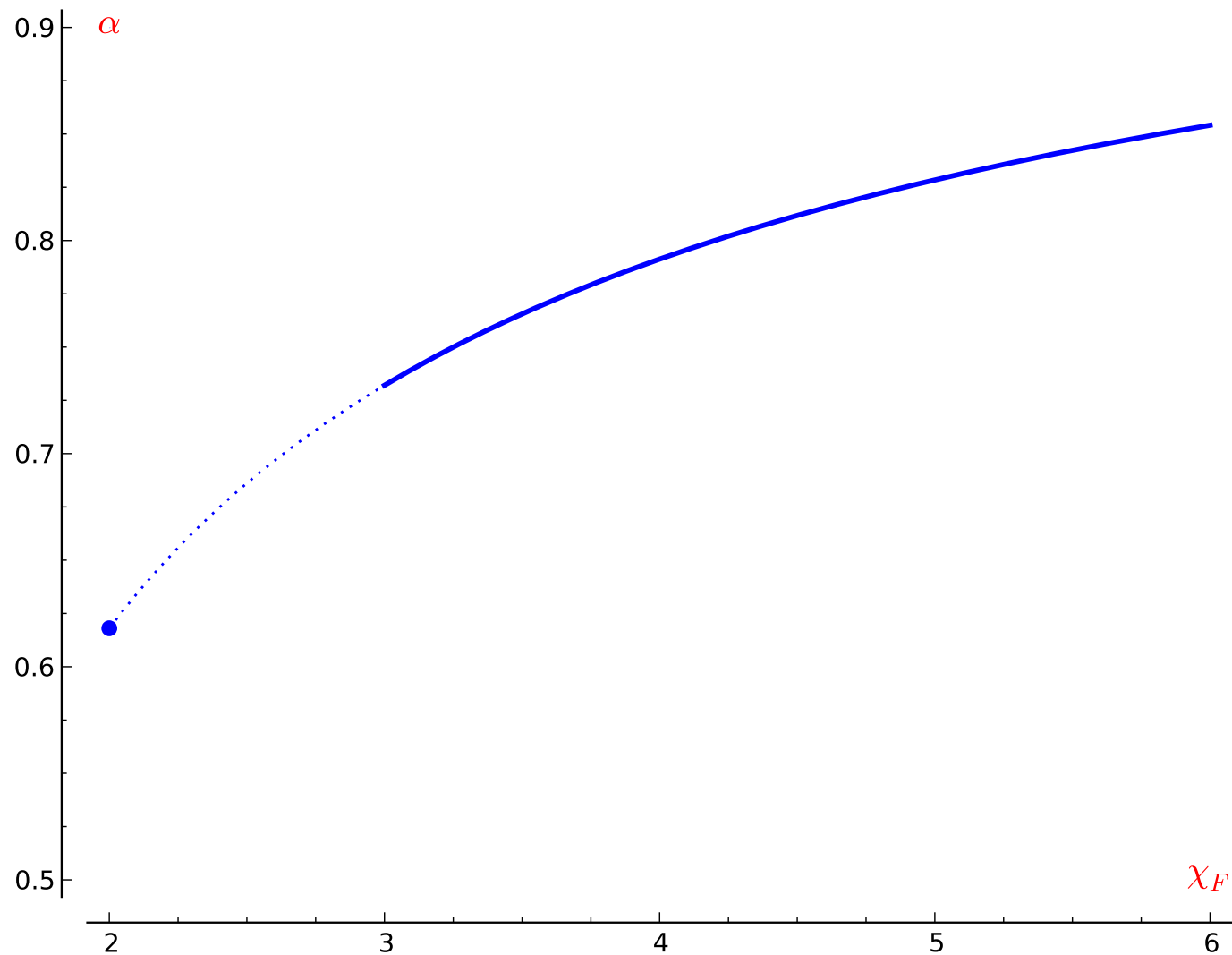
    if $D \equiv 1 \bmod 4$

  - ■ $$\frac{\sqrt{(\lfloor D/4 \rfloor \chi_F)^2 + 4 \lfloor D/4 \rfloor (\chi_F - 1)} - \lfloor D/4 \rfloor \chi_F}{2 \lfloor D/4 \rfloor},$$

    if $D \equiv 2 \bmod 4$

  - ■ $$\frac{\chi_F - 1}{\lfloor D/4 \rfloor \chi_F + \chi_F - 1},$$
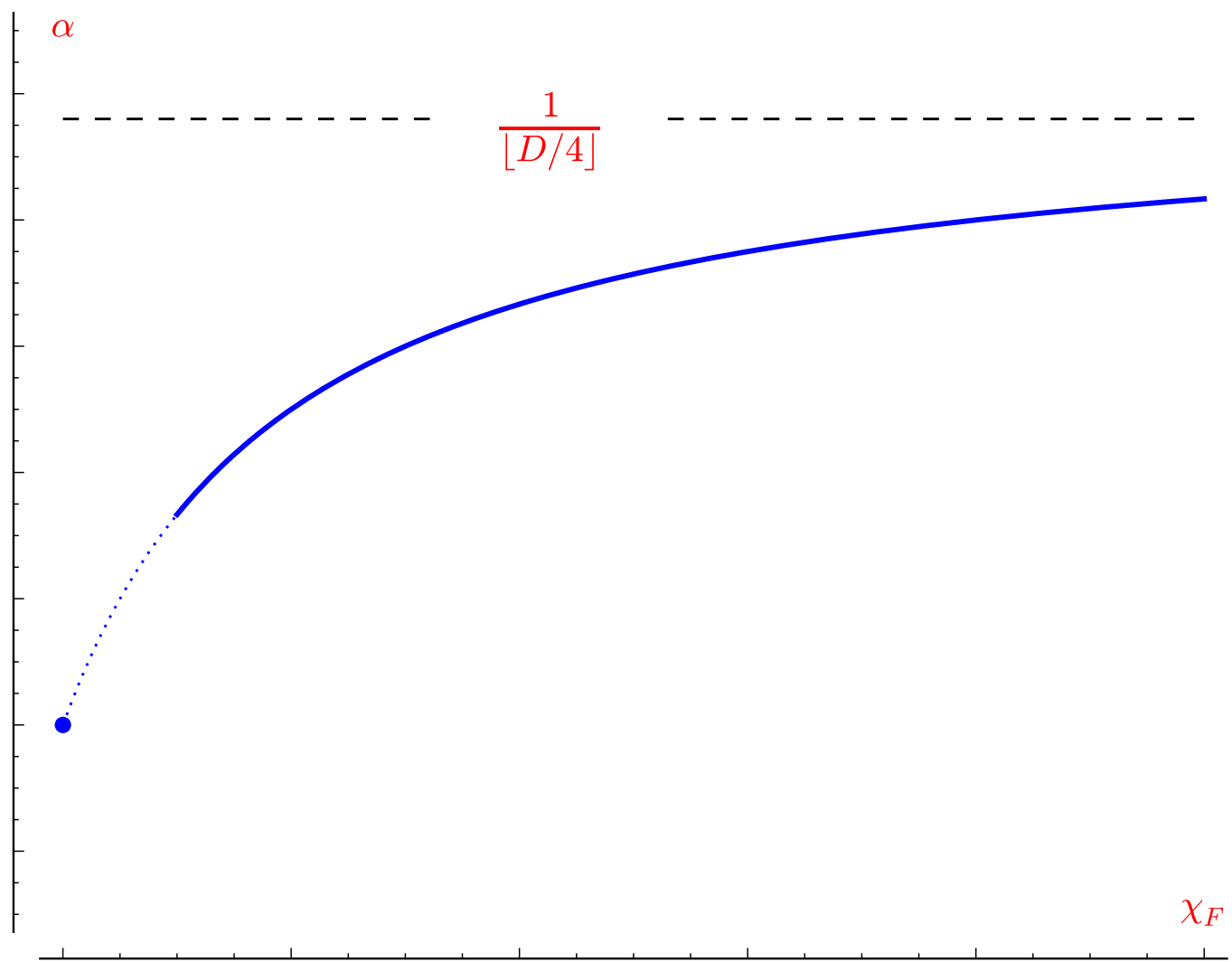
    if $D \equiv 3 \bmod 4$

The picture for $D = 3$

The picture for $D = 4$

The picture for general $D \geq 4$

Almost the complete answer

- so for $D \geq 4$, we know the full answer
  only if $\chi_F = 2$ or $\chi_F \geq 3$

  - so what happens in the gap $2 < \chi_F < 3$?

- the problem again :

  - we have some $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$

  - the vertices $w \in W$ are pre-coloured
    with $\phi(w) \subseteq [0, \chi_F + \alpha]$ of 'measure' 1

  - and we want to extend that to a fractional colouring of
    the whole G, using colours from $[0, \chi_F + \alpha]$

The answer for $D = 4$

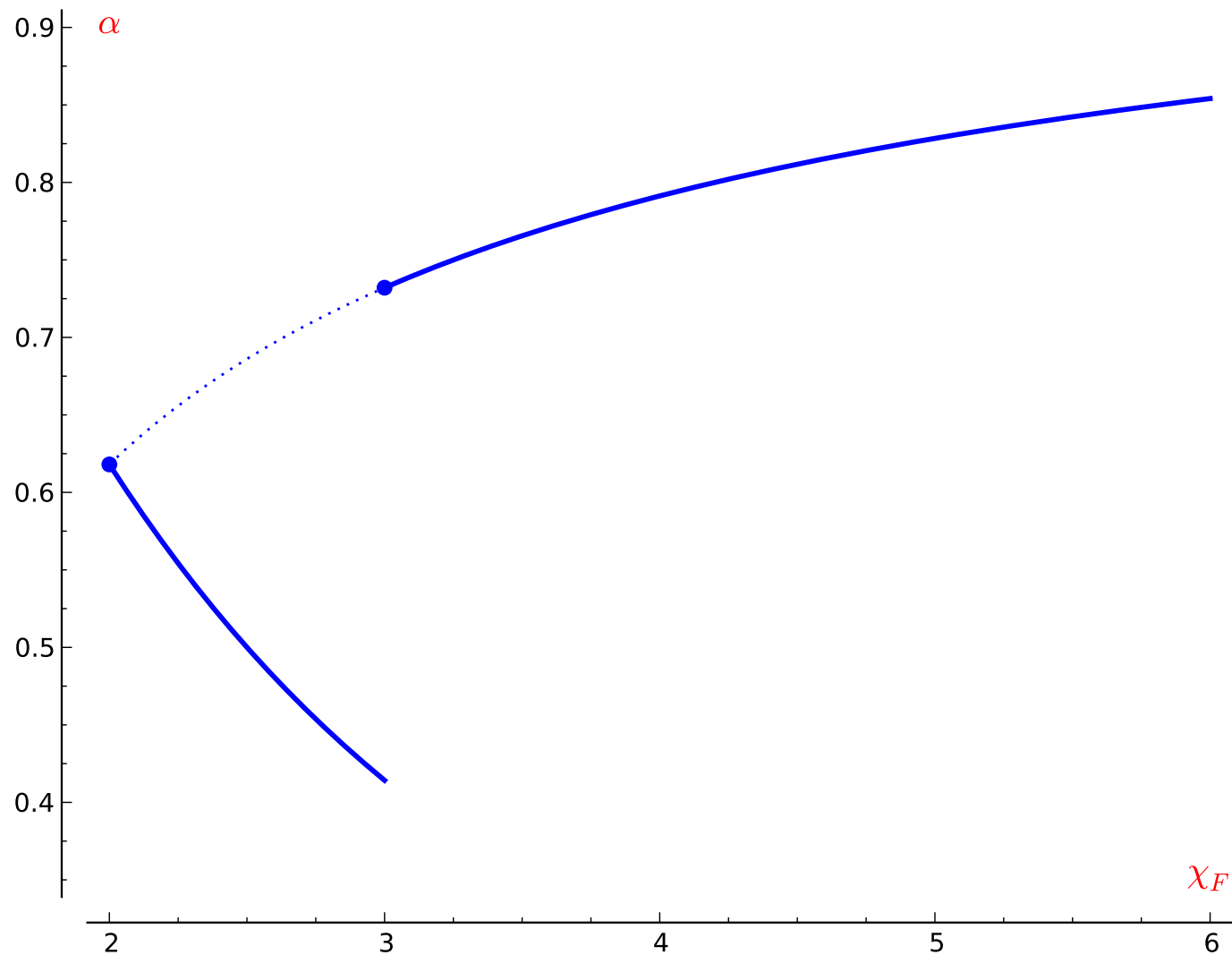**Theorem** (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 4$ we need :

  - $$\alpha \geq \frac{\sqrt{(\chi_F - 1)^2 + 4\,(\chi_F - 1)} - \chi_F + 1}{2}, \quad \text{for } \chi_F \geq 3$$

  - $$\alpha \geq \frac{\sqrt{(\chi_F - 1)^2 + 4} - \chi_F + 1}{2}, \quad \text{for } 2 \leq \chi_F < 3$$

- and these bounds are best possible

The full picture for $D = 4$

Almost the answer for $D = 5$

Theorem   (vdH, Král', Kupec, Sereni & Volec, 2011)
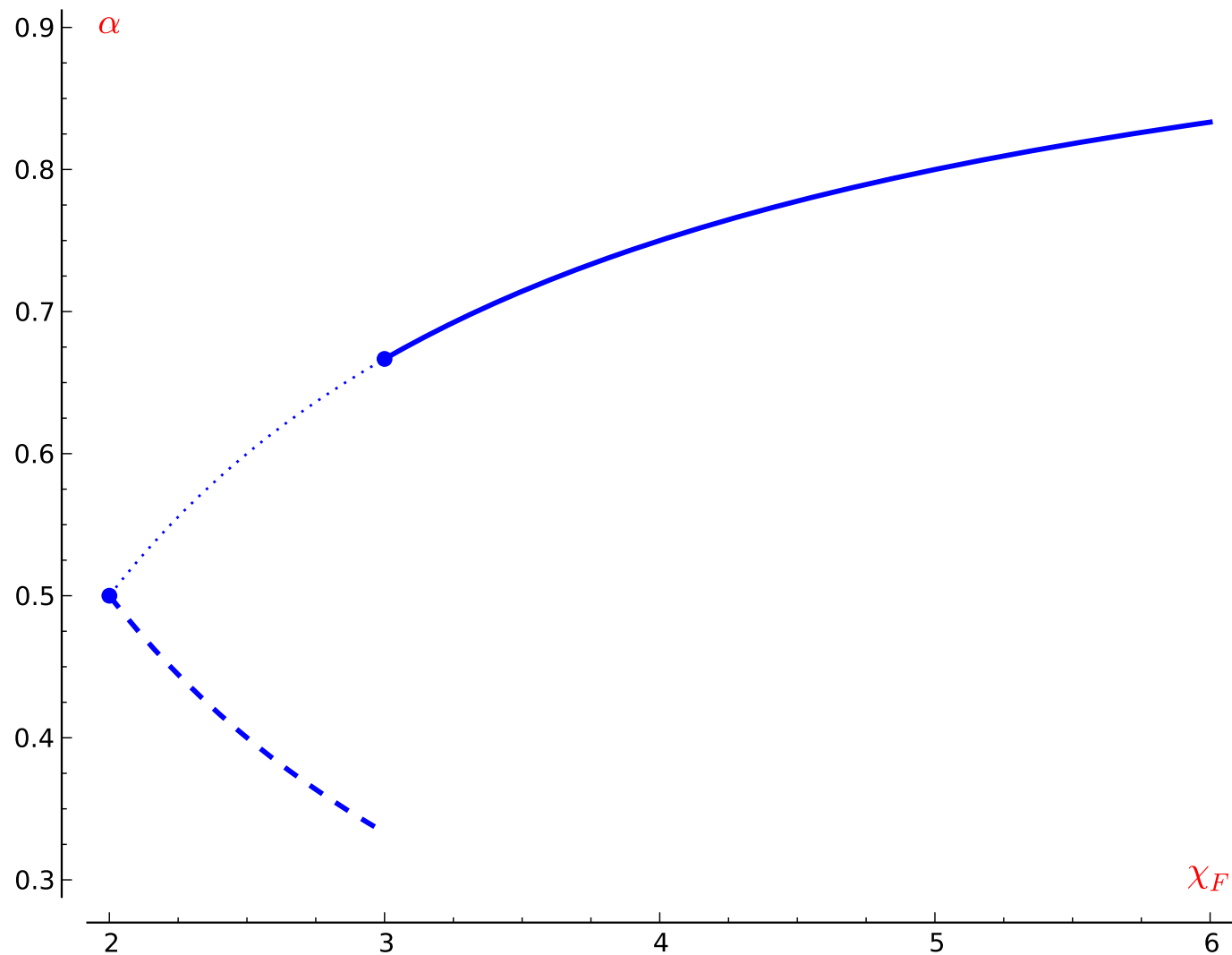
■ for $D = 5$ we need :

  ■ $\alpha \geq \dfrac{\chi_F - 1}{\chi_F}$,   for $\chi_F \geq 3$

  ■ $\alpha \geq \dfrac{1}{\chi_F}$,   for $2 \leq \chi_F < 3$

■ but we don't know if the bound for $2 \leq \chi_F < 3$ is best possible

Almost the full picture for $D = 5$

Almost the answer for $D = 6$

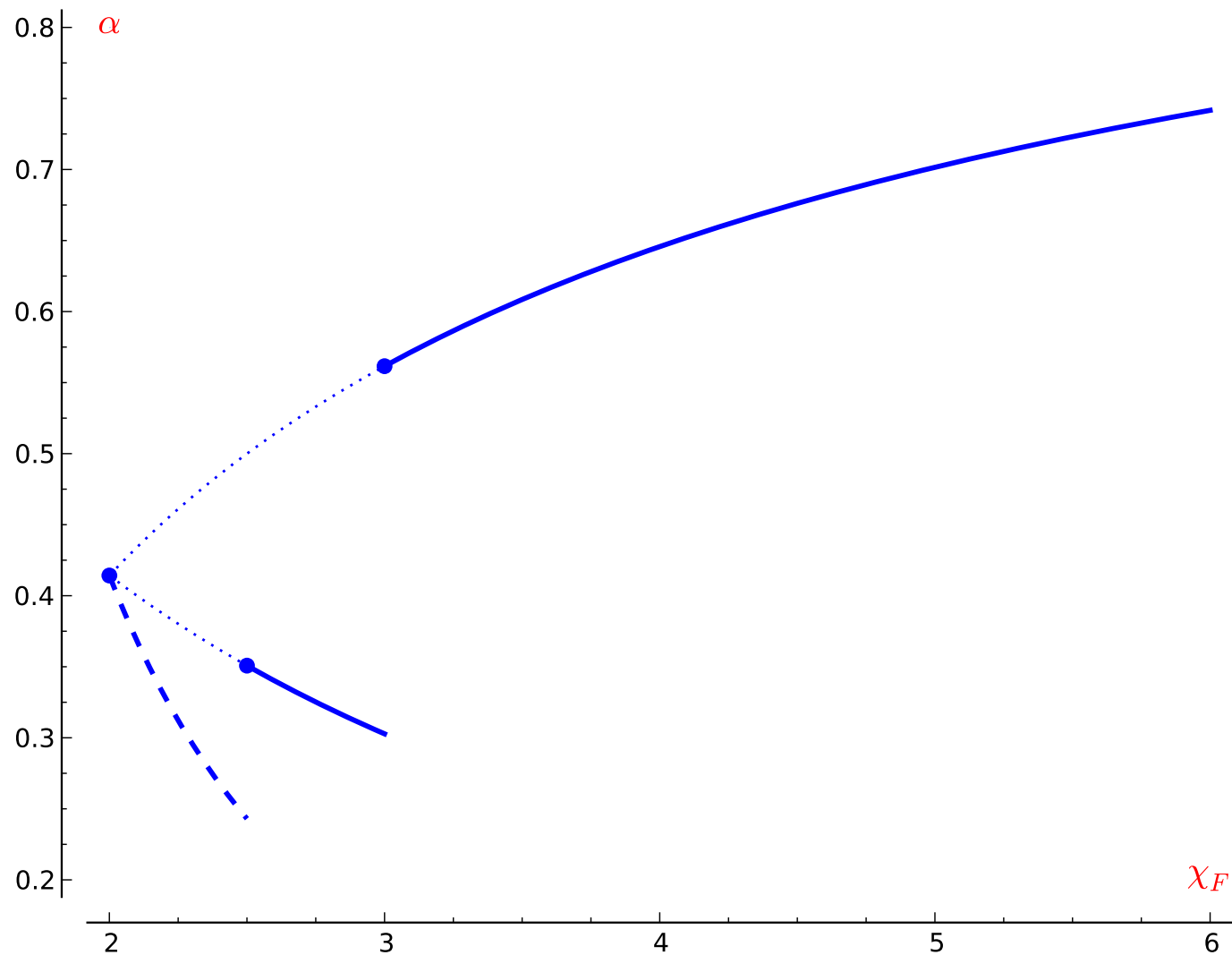**Theorem** (vdH, Král', Kupec, Sereni & Volec, 2011)

■ for $D = 6$ we need :

■ $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4\,(\chi_F - 1)} - \chi_F}{2}$, for $\chi_F \geq 3$

■ $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4} - \chi_F}{2}$, for $2\frac{1}{2} \leq \chi_F < 3$

■ $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4/(\chi_F - 1)} - \chi_F}{2}$, for $2 \leq \chi_F < 2\frac{1}{2}$

■ and the bounds are best possible for $\chi_F \in \{2\} \cup [2\frac{1}{2}, \infty)$

Almost the full picture for $D = 6$

And for $D \geq 7$

- for $D \geq 7$ we have no further precise results

  for $2 < \chi_F < 3$

  - but all indications are that it gets more and more complicated when D gets larger

- to understand the strange behaviour for $2 < \chi_F < 3$ we need to have a look at some aspects of the proof

  - and for that we need to have a further look at fractional colouring first

Fractional colouring again

- fractional K-colouring of graph G :

  - assignment of subsets $\phi(v) \subseteq [0, K]$ to $v \in V$ so that :

    - every subset $\phi(v)$ has 'measure' 1
    - and $uv \in E(G) \implies \phi(u) \cap \phi(v) = \varnothing$

Alternative definition for fractional colouring

- notation : $[m]$: the set $\{1, \ldots, m\}$

  $\dbinom{[m]}{q}$: the collection of q-subsets of $[m]$

- (m, q)-**colouring** of graph G $(1 \leq q \leq m)$:

  - every $v \in V$ is assigned a subset $\psi(v) \in \dbinom{[m]}{q}$, so that :

    - $uv \in E(G) \implies \psi(u) \cap \psi(v) = \varnothing$

- and then : $\chi_F(G) = \min \left\{ \dfrac{m}{q} \; \middle| \; G \text{ has an } (m, q)\text{-colouring} \right\}$

And another definition

- **Kneser graph $Kn(m, q)$** :

  - vertices : all of $\binom{[m]}{q}$, edge $uv \iff u \cap v = \varnothing$

- G has an (m, q)-colouring

  $\iff$ there is a homomorphism $G \longrightarrow Kn(m, q)$

- $\chi_F(G) = \chi_F \iff$ there exist m, q with $\chi_F = \dfrac{m}{q}$,

  so that there is a homomorphism $G \longrightarrow Kn(m, q)$

  - we can interpret this as just a

   labelling of the vertices of G, using labels from $\binom{[m]}{q}$

Fractional colouring and Kneser graphs

■ so to understand fractional colouring,

we can use Kneser graphs

■ but we want to deal with pre-colouring

of vertex sets with a minimum distance D

■ for that we need to build more complicated graphs

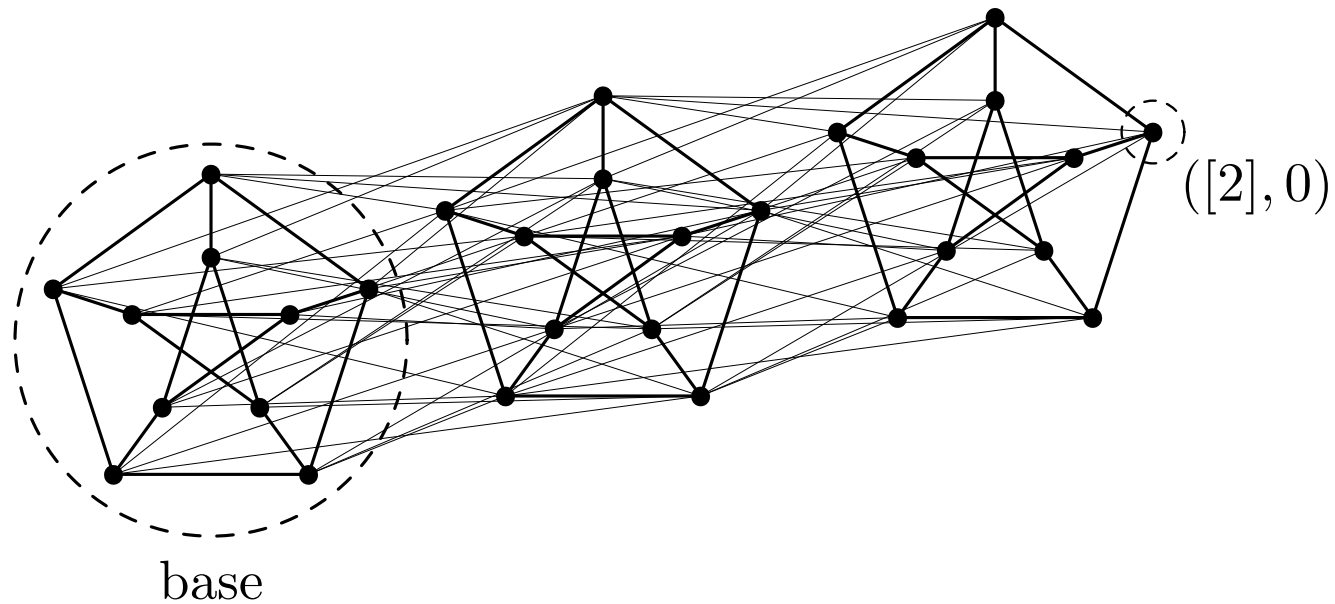■ in the rest of this talk we only look at the case D is even

Armed Kneser graphs

- **given some m, q, D even, and an integer L**

  - **we start with a single $Kn(m, q)$ as a base**

  - **out of the base we grow L disjoint arms,**

    each consisting of $D/2$ disjoint copies of $Kn(m, q)$

  - **we link two consecutive copies of $Kn(m, q)$ in each arm as follows :**

    - u_1 in copy 1 and v_2 in copy 2 :

    $$u_1 \sim v_2 \iff uv \text{ is an edge in } Kn(m, q)$$

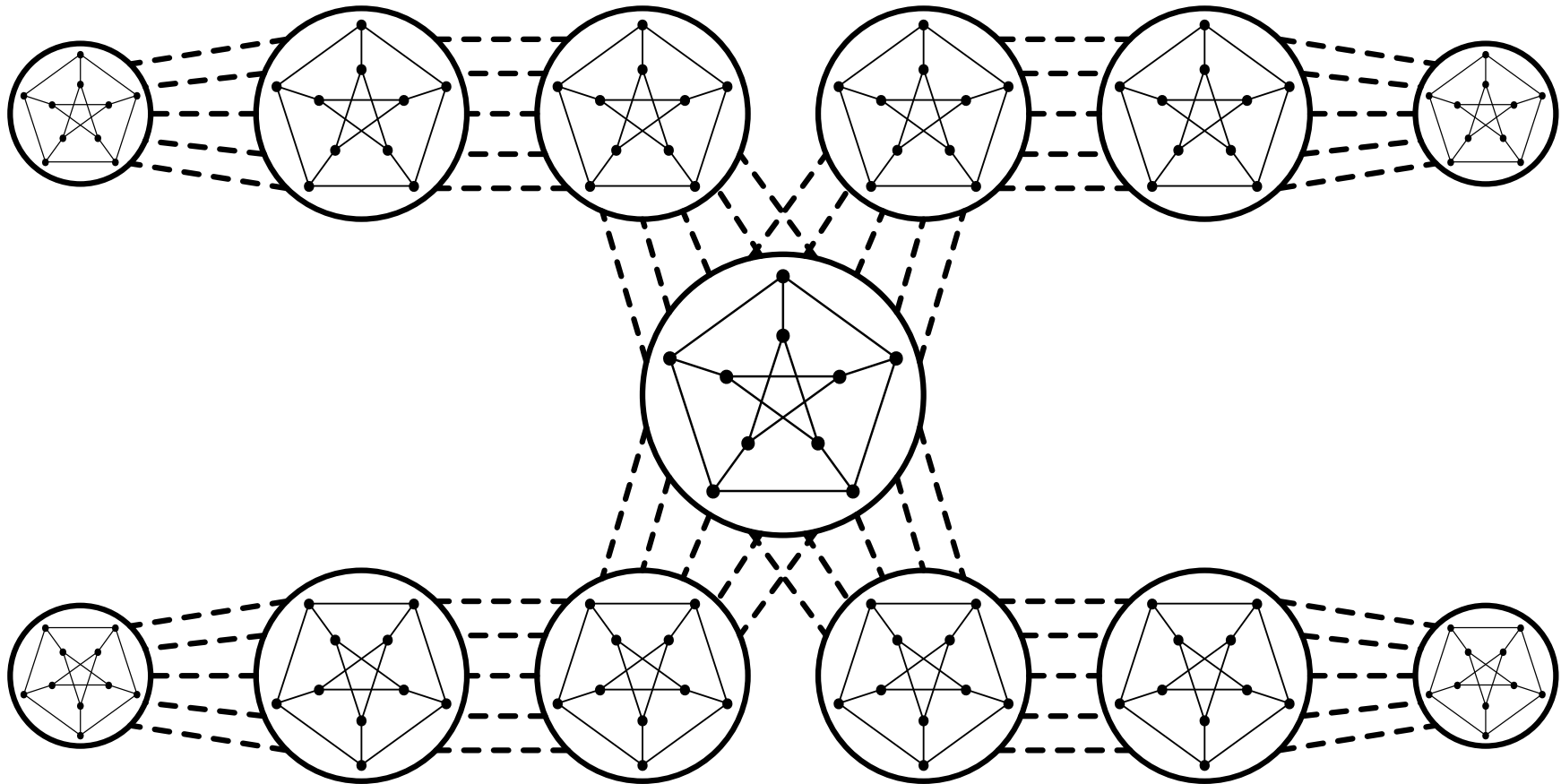# *Armed Kneser graphs*

$([2], 0)$
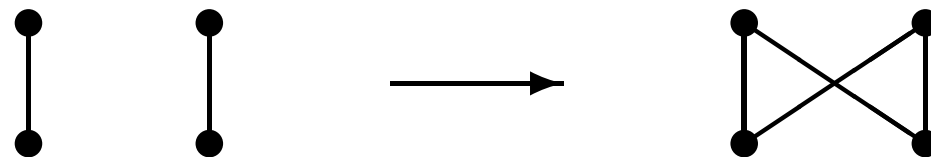
base

one arm of an armed Kneser graph
with $m = 5$, $q = 2$; $D = 4$

Armed Kneser graphs

- given some m, q, D even, and an integer L

  - we start with a single $Kn(m, q)$ as a **base**

  - out of the base we grow L disjoint **arms**,

    each consisting of $D/2$ disjoint copies of $Kn(m, q)$

  - we link two consecutive copies of $Kn(m, q)$ in each arm as follows :

    - u_1 in copy 1 and v_2 in copy 2 :

      $$u_1 \sim v_2 \quad \Longleftrightarrow \quad uv \text{ is an edge in } Kn(m, q)$$

- call the result the **armed Kneser graph** $a\text{-}Kn(m, q; D, L)$

Armed Kneser graphs

the armed Kneser graph $a\text{-}Kn(5, 2; 6, 4)$

Using armed Kneser graphs

- ■
  - ■ now suppose we have a graph G with $\chi_F(G) = \chi_F$

    so : $\quad G \longrightarrow Kn(m, q)$, for some m, q with $\dfrac{m}{q} = \chi_F$

  - ■ and a set $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$

- ■ take the armed Kneser graph $a\text{-}Kn(m, q; D, |W|)$

  with $|W|$ arms

- ■ we will map G to this armed Kneser graph

  - ■ using the labels given by the homomorphism

    $$G \longrightarrow Kn(m, q)$$

Using armed Kneser graphs

- each $w \in W$ gets its own arm

  - map w in the copy of $Kn(m, q)$ at the end of its arm

  - map the neighbours of w in G

    in the copy of $Kn(m, q)$ one step closer to the base

  - map the neighbours of the neighbours of w in G

    in the next copy (closer to the base) of $Kn(m, q)$

  - etc.

- map all vertices at distance at least $D/2$ from w in G

  in the base of the armed Kneser graph
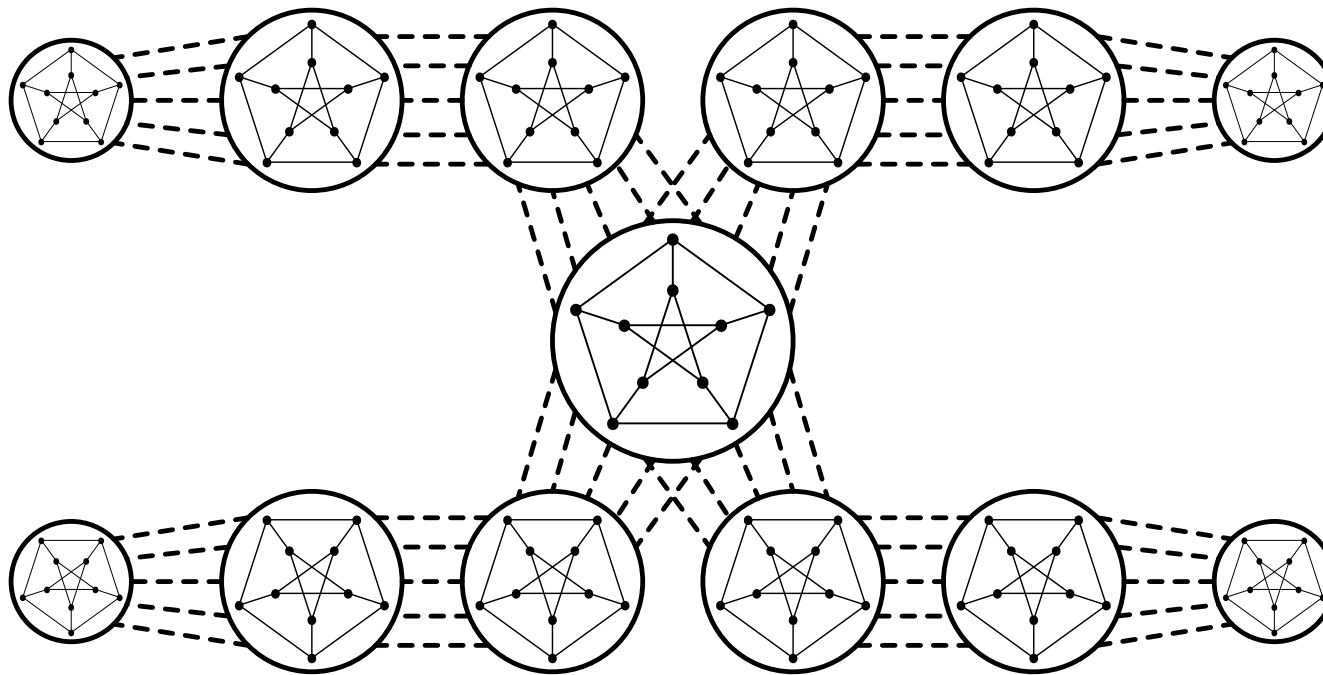
Using armed Kneser graphs

- this mapping of G in the armed Kneser graph satisfies :

    - images of elements of W have distance D

    - a pre-colouring of W

        gives a pre-colouring of the images of W

    - a fractional colouring of the armed Kneser graph

        can be mapped back to a fractional colouring of G

in other words :

- all aspects of fractional pre-colouring extensions of graphs are determined by fractional pre-colouring extensions of armed Kneser graphs !

Pre-colouring extensions of armed Kneser graphs

- ■ suppose we have an armed Kneser graph

  $$a\text{-}Kn(m, q; D, L)$$

  - ■ and one pre-coloured vertex in the end of each arm

Pre-colouring extensions of armed Kneser graphs

- ■ suppose we have an armed Kneser graph

$$a\text{-}Kn(m, q; D, L)$$

- ■ and one pre-coloured vertex in the end of each arm

- ■ in a fractional colouring extending that pre-colouring :

  - ■ the base must 'accommodate' all arms
    - ■ so will have to be given some 'average' colouring

  - ■ so along the arms, the colouring extension must connect
    - ■ the pre-coloured vertex in the end
    - ■ with some 'average' colouring of the base

Colouring along an arm of an armed Kneser graph

- **■** so consider an arm of an armed Kneser graph

  - **■** with one pre-coloured vertex w' in its end

$([2], 0)$

base
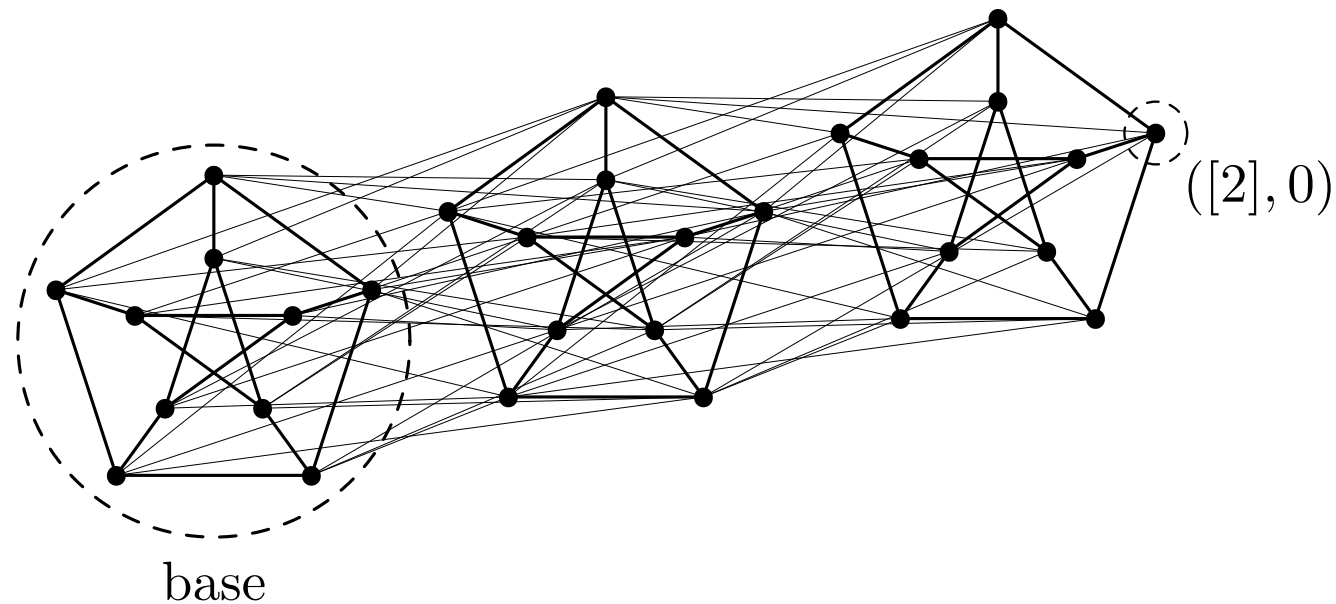
Colouring along an arm of an armed Kneser graph

- so consider an arm of an armed Kneser graph

    - with one pre-coloured vertex w' in its end

- the colouring along the arm is mostly determined by :

    - w' itself, in the end copy of $Kn(m, q)$

    - then by vertices in the 2 nd copy of $Kn(m, q)$

      that are neighbours of w'

    - and then by vertices in the 3 rd copy of $Kn(m, q)$

      that are neighbours of neighbours of w'

    - etc.

Now things get interesting

- w' is a vertex in $Kn(m, q)$, i.e., a q-subset of $[m]$

- its neighbours are the q-subsets of $[m]$ disjoint from w'

  - those neighbours together form a subgraph
    that is isomorphic to the Kneser graph $Kn(m - q, q)$

- for $\chi_F = \dfrac{m}{q} \geq 3$ we have

$$\chi_F\big(Kn(m - q, q)\big) = \frac{m - q}{q} = \chi_F - 1$$

- for $2 \leq \chi_F = \dfrac{m}{q} < 3$,

  $Kn(m - q, q)$ has just isolated vertices

  - hence in those cases : $\chi_F\big(Kn(m - q, q)\big) = 1$

Now things get interesting

- so for $\chi_F \geq 3$ we have
$$\chi_F(\text{set of neighbours of } w') = \chi_F - 1$$

- while for $2 \leq \chi_F < 3$ we have
$$\chi_F(\text{set of neighbours of } w') = 1$$

- this causes the difference between the two cases when $D = 4$

  (then the armed Kneser graph has arms of length 2, with the set of neighbours of w' in the middle)

And things get even more interesting

- next consider the set of neighbours of neighbours of w'

- for $\chi_F = \dfrac{m}{q} \geq 3$, this is the whole Kneser graph $Kn(m, q)$

- for $2 \leq \chi_F = \dfrac{m}{q} < 2\frac{1}{2}$,

  this is again a collection of isolated vertices

- but for $2\frac{1}{2} \leq \chi_F = \dfrac{m}{q} < 3$, it gets complicated

  - the structure is not a Kneser graph

  - its structure can vary even in cases where $\dfrac{m}{q} = \dfrac{m'}{q'}$

To summarise these findings

- when colouring along an arm of an an armed Kneser graph :

  - for $\chi_F = \dfrac{m}{q} \geq 3$, we are always dealing with structures

    that are Kneser graphs itself

  - for $2 \leq \chi_F = \dfrac{m}{q} < 3$, we have to consider structures

    that are not Kneser graphs

- we just seem to lack an understanding of the
  internal structure of Kneser graphs
  to deal with those latter cases in general