Fractional Colouring and Precolouring Extension of Graphs

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Graph colouring and precolouring

- $G$ a graph
- chromatic number $\chi(G)$:
  - minimum $k$ so that a vertex-colouring exists

**general question:**

- what can we say if some vertices are already precoloured?
- in particular: can $\chi(G)$ colours still be enough?

- in general: no
**Precolouring questions**

next best questions:
- how many extra colours may be needed?
- and what conditions on the precoloured vertices can make life easier?

**Question** (Thomassen, 1997)
- $G$ planar,
  
  $W \subseteq V(G)$, a set of vertices so that distance between any two vertices in $W$ is at least 100
  
  can any 5-colouring of $W$ be extended to a 5-colouring of $G$?
The first answer

- \( \text{dist}(W) \): minimum distance between any two vertices in \( W \)

**Theorem** (Albertson, 1998)

- \( G \) any graph with chromatic number \( \chi \)
  
  \( W \subseteq V(G) \) with \( \text{dist}(W) \geq 4 \)

  \( \implies \) any \( (\chi+1) \)-colouring of \( W \)
  
  can be extended to a \( (\chi+1) \)-colouring of \( G \)
Fractional colouring

- **fractional** \( K \)-**colouring** of graph \( G \) \( \ ( K \in \mathbb{R}_+ ) \):  
  - every vertex \( v \in V \) is assigned a subset \( \phi(v) \subseteq [0, K] \) so that:
    - every subset \( \phi(v) \) has ‘measure’ 1
    - and \( uv \in E(G) \implies \phi(u) \cap \phi(v) = \emptyset \)

- fractional chromatic number \( \chi_F(G) \):
  
  \[
  = \inf \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}
  \]
  
  \[
  = \min \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}
  \]
**Fractional colouring**

- **note:** we always have $\chi_F(G) \leq \chi(G)$
  - but the **difference** can be **arbitrarily large**

- $\chi_F(G) = 1 \iff G$ has no edges
- $\chi_F(G) = 2 \iff G$ has edges and is **bipartite**

- for all rational $K \geq 2$: there exist $G$ with $\chi_F(G) = K$
Precolouring in the fractional world

- so now suppose that for some vertices $W \subseteq V(G)$, the vertices in $W$ are already precoloured:
  - vertices $w \in W$ have been given some set $\phi(w)$ of measure 1

- when can this be extended to a fractional colouring of the whole graph $G$?

- in general we should expect to require more than $\chi_F(G)$ colours
The set-up of the problem

- $G$ a graph with fractional chromatic number $\chi_F \geq 2$
- $D \geq 3$ an integer
- $W \subseteq V(G)$ with $\text{dist}(W) \geq D$
- The vertices $w \in W$ are precoloured with $\phi(w) \subseteq [0, \chi_F + \alpha]$ of measure 1
  - for some real $\alpha \geq 0$
- and we want to extend that to a fractional colouring of the whole $G$, using colours from $[0, \chi_F + \alpha]$
- how large should $\alpha$ be to be sure this can be done?
A major part of the answer

Theorem  (Král’, Krnc, Kupec, Lužar & Volec, 2011)

extension is always possible, provided \( \alpha \) is at least:

\[
\sqrt{\left(\left\lfloor \frac{1}{4}D \right\rfloor \chi_F + 1\right)^2 - 4 \left\lfloor \frac{1}{4}D \right\rfloor - \frac{1}{4}D \chi_F + 1} - \frac{\chi_F - 1}{\left\lfloor \frac{1}{4}D \right\rfloor \chi_F},
\]

if \( D \equiv 0 \mod 4 \),

\[
\frac{\chi_F - 1}{\left\lfloor \frac{1}{4}D \right\rfloor \chi_F},
\]

if \( D \equiv 1 \mod 4 \),

\[
\sqrt{\left(\left\lfloor \frac{1}{4}D \right\rfloor \chi_F + 2\right)^2 - 4 \left(\left\lfloor \frac{1}{4}D \right\rfloor + 1\right) - \frac{1}{4}D \chi_F} - \frac{\chi_F - 1}{\left\lfloor \frac{1}{4}D \right\rfloor \chi_F + \chi_F - 1},
\]

if \( D \equiv 2 \mod 4 \),

\[
\frac{\chi_F - 1}{\left\lfloor \frac{1}{4}D \right\rfloor \chi_F + \chi_F - 1},
\]

if \( D \equiv 3 \mod 4 \).
A major part of the answer

**Theorem** (Král’, Krnc, Kupec, Lužar & Volec, 2011)

- moreover, these bounds on $\alpha$ are best possible,
  - if $D = 3$ and $\chi_F \geq 2$;
  - if $D \geq 4$ and $\chi_F \in \{2\} \cup [3, \infty)$
The picture for $D = 3$
The picture for $D = 4$
The picture for general $D \geq 4$
Almost the complete answer

- so for $D \geq 4$, we know the full answer only if $\chi_F = 2$ or $\chi_F \geq 3$

- so what happens in the gap $2 < \chi_F < 3$?

- the problem again:
  - we have some $W \subseteq V(G)$ with $\text{dist}(W) \geq D$
  - the vertices $w \in W$ are precoloured with $\phi(w) \subseteq [0, \chi_F + \alpha]$ of measure 1

- and we want to extend that to a fractional colouring of the whole $G$, using colours from $[0, \chi_F + \alpha]$
The answer for $D = 4$

**Theorem** (vdH, Král’, Kupec, Sereni & Volec, 2011)

- for $D = 4$ we need:
  
  $\alpha \geq \frac{\sqrt{(\chi_F - 1)^2 + 4(\chi_F - 1) - \chi_F + 1}}{2}$, for $\chi_F \geq 3$

  $\alpha \geq \frac{\sqrt{(\chi_F - 1)^2 + 4 - \chi_F + 1}}{2}$, for $2 \leq \chi_F < 3$

- and these bounds are best possible
The full picture for $D = 4$
Almost the answer for $D = 5$

**Theorem** (vdH, Král’, Kupec, Sereni & Volec, 2011)

- for $D = 5$ we need:
  - $\alpha \geq \frac{\chi_F - 1}{\chi_F}$, for $\chi_F \geq 3$
  - $\alpha \geq \frac{1}{\chi_F}$, for $2 \leq \chi_F < 3$

- but we don’t know if the bound for $2 \leq \chi_F < 3$ is best possible
Almost the full picture for $D = 5$
Almost the answer for $D = 6$

**Theorem** (vdH, Král’, Kupec, Sereni & Volec, 2011)

- for $D = 6$ we need:
  
  - $\alpha \geq \frac{\chi_F^2 + 4(\chi_F - 1) - \chi_F}{2}$, for $\chi_F \geq 3$
  
  - $\alpha \geq \frac{\sqrt{\chi_F^2 + 4} - \chi_F}{2}$, for $2 \frac{1}{2} \leq \chi_F < 3$
  
  - $\alpha \geq \frac{\sqrt{\chi_F^2 + 4/(\chi_F - 1)} - \chi_F}{2}$, for $2 \leq \chi_F < 2 \frac{1}{2}$

- and the bounds are best possible for $\chi_F \in \{2\} \cup [2 \frac{1}{2}, \infty)$
Almost the full picture for $D = 6$
And for $D \geq 7$

- for $D \geq 7$ we have no further precise results

- but all indications are that it gets more and more complicated when $D$ gets larger

for $2 < \chi_F < 3$
A new problem

- in all problems so far we assumed that the precoloured vertices and the extension can use the same set of available colours

- but what would happen if for the precolouring we can use a smaller colour set only?
  - for integer colouring, this would make no difference (for distance $D \geq 4$)
    (may need extra colours – one extra is always enough)
  - but for fractional precolouring one would expect a more gradual change
The set-up of the new problem

- $G$ a graph with fractional chromatic number $\chi_F \geq 2$
- $D \geq 3$ an integer
- $W \subseteq V(G)$ with $\text{dist}(W) \geq D$
- $L \geq 1$ a real number
- the vertices $w \in W$
  are precoloured with $\phi(w) \subseteq [0, L]$ with measure 1
- and we want to extend that to a fractional colouring of the whole $G$, using colours from $[0, \chi_F + \alpha]$
- how large should $\alpha$ be to be sure this can be done?
The intuition for restricted fractional precolouring

- for $L = 1$, all precoloured vertices get ‘colour’ $[0, 1)$
  - a small $\alpha$ should be enough to complete the colouring
- when we increase $L$
  - the required $\alpha$ will increase as well
- until we reach $L = \chi_F + \alpha_{\text{crit}}$
  - where $\alpha_{\text{crit}}$ is the value so that:
    - precolouring with $[0, \chi_F + \alpha_{\text{crit}}]$ can be completed with colours from $[0, \chi_F + \alpha_{\text{crit}}]$
- increasing $L$ further,
  - doesn’t require more than $[0, \chi_F + \alpha_{\text{crit}}]$ to complete
A first quarter of the answer

**Theorem** (vdH, Li & Müller, 2014+)

- If $D \equiv 2 \text{ mod } 4$, then extension is always possible, provided $\alpha$ is at least:
  
  $\frac{L(\chi_F - 1)}{L \left\lfloor \frac{1}{4}D \right\rfloor \chi_F + \chi_F - 1}$, 
  if $1 \leq L \leq \chi_F + \alpha_{\text{crit}}$

- $\alpha_{\text{crit}}$, 
  if $L \geq \chi_F + \alpha_{\text{crit}}$

- Where $\alpha_{\text{crit}}$ is given by the first Král’ el al. result

- And these bounds are best possible for $\chi_F \in \{2\} \cup [3, \infty)$
The picture for $D = 6$ and $\chi_F = 4$
A next quarter of the answer

Theorem (vdH, Li & Müller, 2014+)

- If \( D \equiv 0 \mod 4 \), then extension is always possible, provided \( \alpha \) is at least:
  - \( \frac{\chi_F - 1}{\left\lfloor \frac{1}{4} D \right\rfloor} \chi_F \), if \( 1 \leq L \leq \chi_F \)
  - \( \frac{L - 1}{\left\lfloor \frac{1}{4} D \right\rfloor} L \), if \( \chi_F \leq L \leq \chi_F + \alpha_{\text{crit}} \)
  - \( \alpha_{\text{crit}} \), if \( L \geq \chi_F + \alpha_{\text{crit}} \)

- Where \( \alpha_{\text{crit}} \) is given by the first Král’ et al. result

- And these bounds are best possible for \( \chi_F \in \{2\} \cup [3, \infty) \)
The picture for $D = 4$ and $\chi_F = 4$
And the final half of the answer

**Theorem** (vdH, Li & Müller, 2014+)

- if $D$ is odd, then extension is always possible, provided $\alpha$ is at least:
  - $\alpha_{\text{crit}}$, for any $L \geq 1$

  (i.e.: the bound doesn't depend on $L$)

- for $\chi_F \in \{2\} \cup [3, \infty)$, the best possible value of $\alpha_{\text{crit}}$ is given by the first Král’ el al. result
The end

Thank you for the attention!