Fractional Colouring and Precolouring Extension of Graphs

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Graph colouring and precolouring

- G a graph
- **chromatic number** $\chi(G)$:

  minimum k so that a vertex-colouring exists

general question :

- what can we say if some vertices are already precoloured ?
- in particular : can $\chi(G)$ colours still be enough ?
  - in general : **no**

Precolouring questions

<ins>**next best questions**</ins> :

- how many extra colours may be needed ?
- and what conditions on the precoloured vertices can make life easier ?

<ins>**Question**</ins> (Thomassen, 1997)

- G planar,

  $W \subseteq V(G)$, a set of vertices so that

  distance between any two vertices in W is at least 100

  - can any 5-colouring of W

    be extended to a 5-colouring of G ?

The first answer

- $\mathrm{dist}(W)$: minimum distance between any two vertices in W

Theorem (Albertson, 1998)

- G any graph with chromatic number χ

  $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq 4$

  $\Longrightarrow$ any $(\chi+1)$-colouring of W

  can be extended to a $(\chi+1)$-colouring of G

Fractional colouring

- **fractional K-colouring** of graph G ($K \in \mathbb{R}_+$) :
  - every vertex $v \in V$ is assigned a subset $\phi(v) \subseteq [0, K]$ so that :
    - every subset $\phi(v)$ has 'measure' 1
    - and $uv \in E(G) \implies \phi(u) \cap \phi(v) = \varnothing$

- **fractional chromatic number** $\chi_F(G)$:

$$= \inf \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

$$= \min \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

Fractional colouring

- **note** : we always have $\chi_F(G) \leq \chi(G)$
  - but the difference can be arbitrarily large

-
  - $\chi_F(G) = 1 \iff G$ has no edges
  - $\chi_F(G) = 2 \iff G$ has edges and is bipartite
  - for all rational $K \geq 2$: there exist G with $\chi_F(G) = K$

Precoluring in the fractional world

- so now suppose that for some vertices $W \subseteq V(G)$, the vertices in W are already precoloured :
  - vertices $w \in W$ have been given some set $\phi(w)$ of measure 1
- when can this be extended to a fractional colouring of the whole graph G ?
- in general we should expect to require more than $\chi_F(G)$ colours

The set-up of the problem

- - G a graph with fractional chromatic number $\chi_F \geq 2$
  - $D \geq 3$ an integer
  - $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$
- - the vertices $w \in W$
    are precoloured with $\phi(w) \subseteq [0, \chi_F + \alpha]$ of measure 1
    - for some real $\alpha \geq 0$
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$
- how large should α be to be sure this can be done ?

A major part of the answer

Theorem (Král', Krnc, Kupec, Lužar & Volec, 2011)

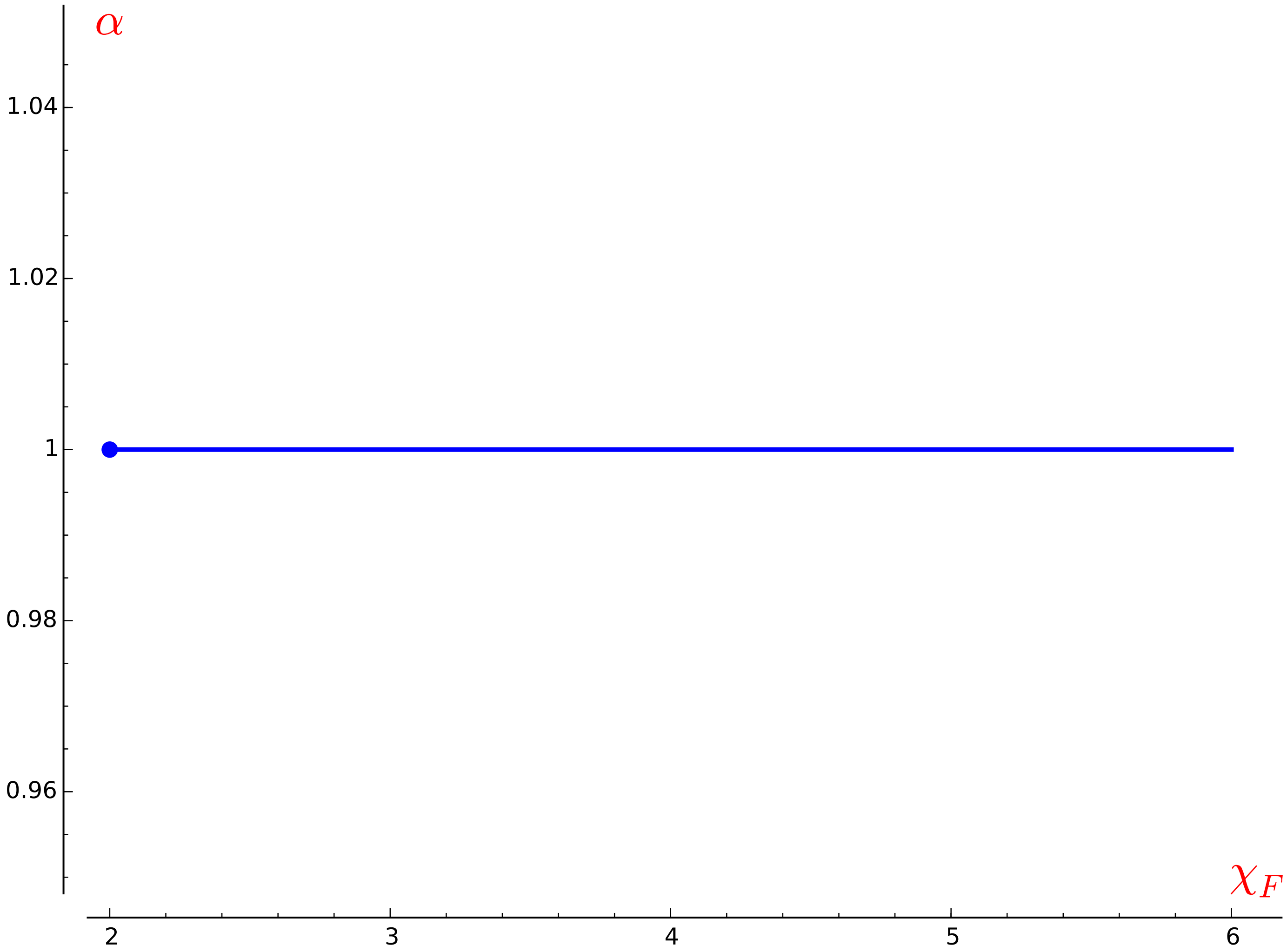
- extension is always possible, provided α is at least :
  - $$\frac{\sqrt{\left(\left\lfloor\frac{1}{4}D\right\rfloor\chi_F+1\right)^2-4\left\lfloor\frac{1}{4}D\right\rfloor}-\left\lfloor\frac{1}{4}D\right\rfloor\chi_F+1}{2\left\lfloor\frac{1}{4}D\right\rfloor},$$ if $D\equiv 0 \bmod 4$
  - $$\frac{\chi_F-1}{\left\lfloor\frac{1}{4}D\right\rfloor\chi_F},$$ if $D\equiv 1 \bmod 4$
  - $$\frac{\sqrt{\left(\left\lfloor\frac{1}{4}D\right\rfloor\chi_F+2\right)^2-4\left(\left\lfloor\frac{1}{4}D\right\rfloor+1\right)}-\left\lfloor\frac{1}{4}D\right\rfloor\chi_F}{2\left\lfloor\frac{1}{4}D\right\rfloor},$$ if $D\equiv 2 \bmod 4$
  - $$\frac{\chi_F-1}{\left\lfloor\frac{1}{4}D\right\rfloor\chi_F+\chi_F-1},$$ if $D\equiv 3 \bmod 4$

A major part of the answer

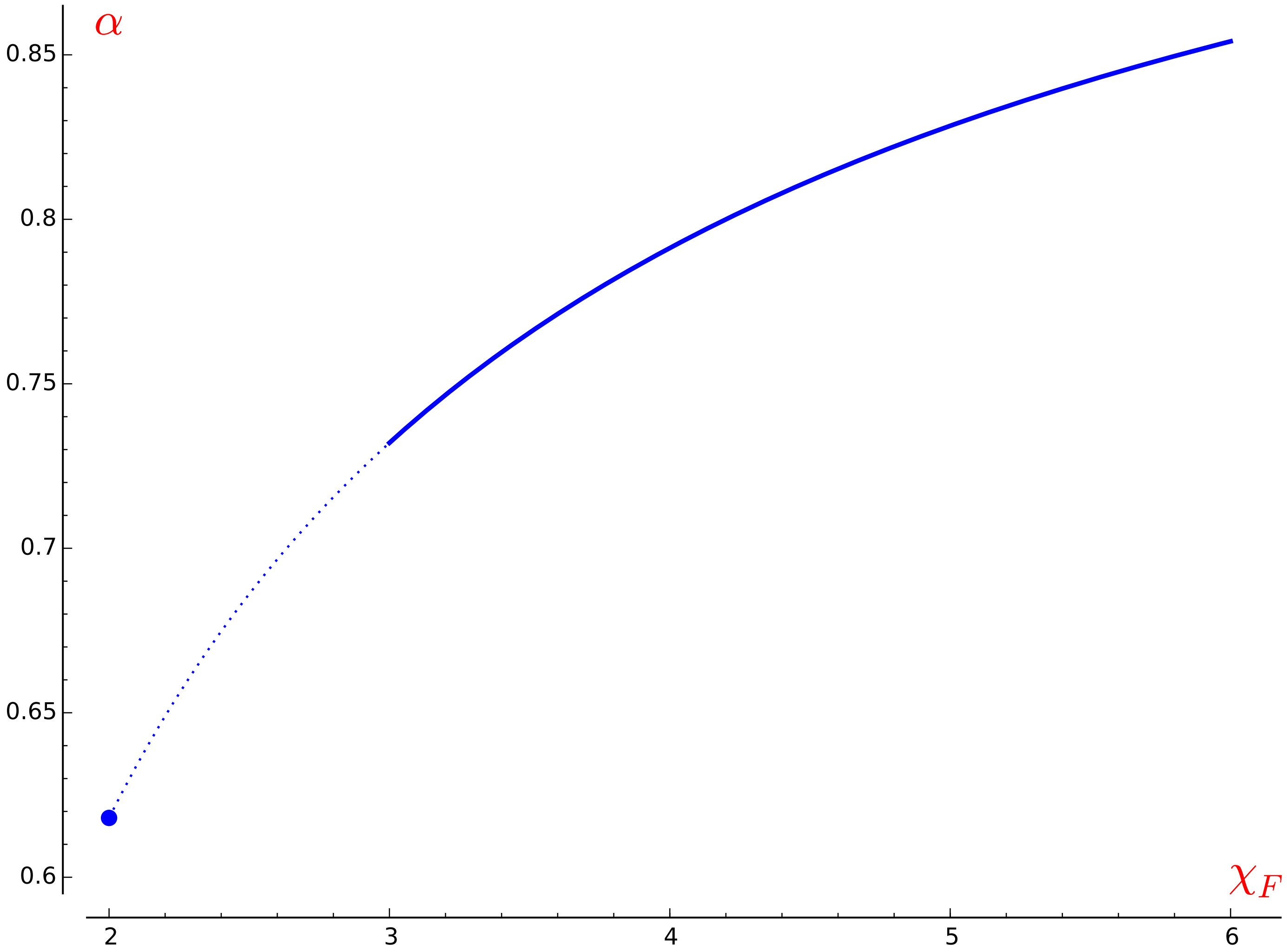
<u>**Theorem**</u> (Král', Krnc, Kupec, Lužar & Volec, 2011)

- moreover, these bounds on α are best possible,
  - if $D = 3$ and $\chi_F \geq 2$;
  - if $D \geq 4$ and $\chi_F \in \{2\} \cup [3, \infty)$

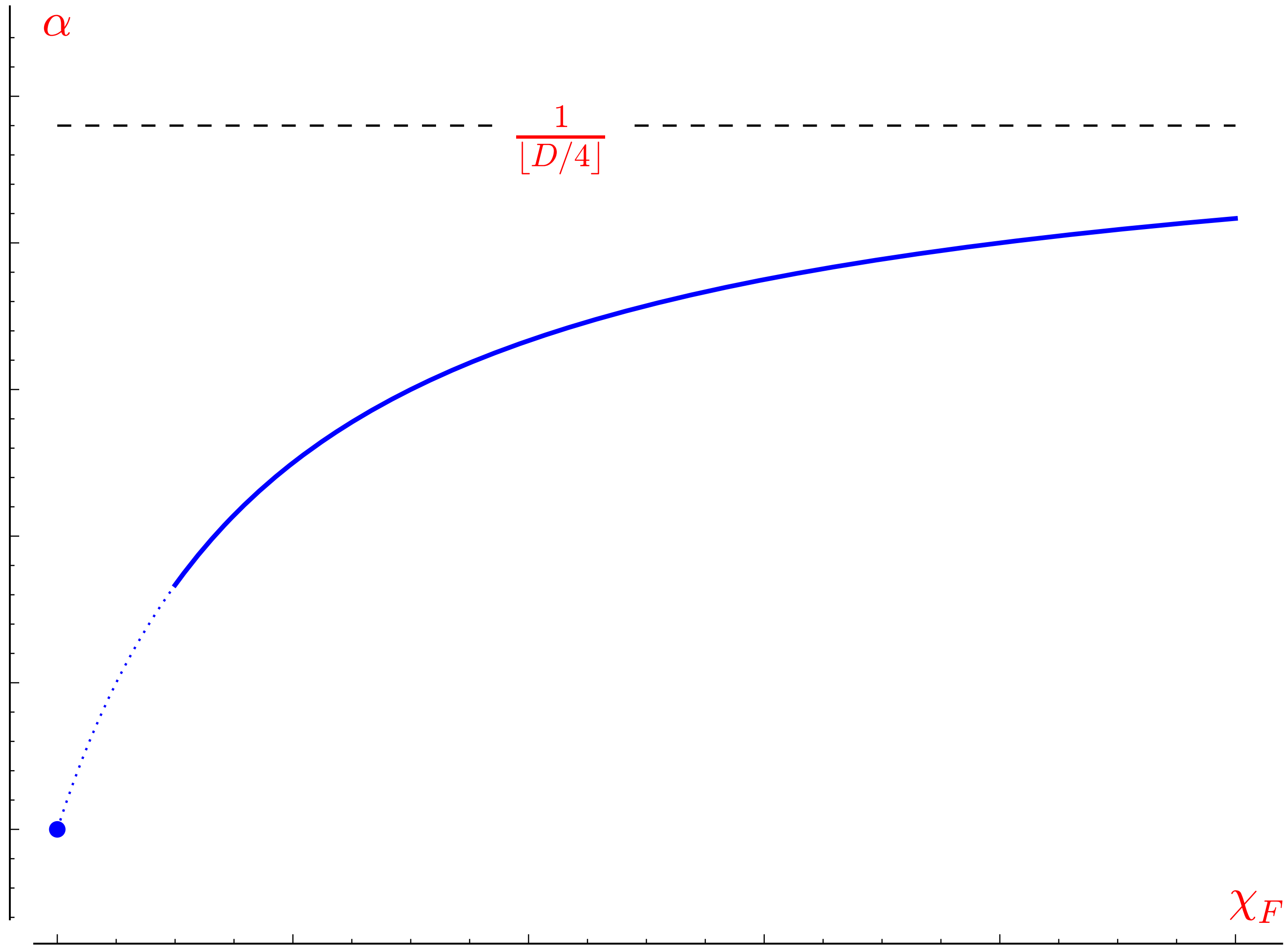
The picture for $D = 3$



The picture for $D = 4$



The picture for general $D \geq 4$



Almost the complete answer

- so for $D \geq 4$, we know the full answer
  only if $\chi_F = 2$ or $\chi_F \geq 3$
  - so what happens in the gap $2 < \chi_F < 3$?

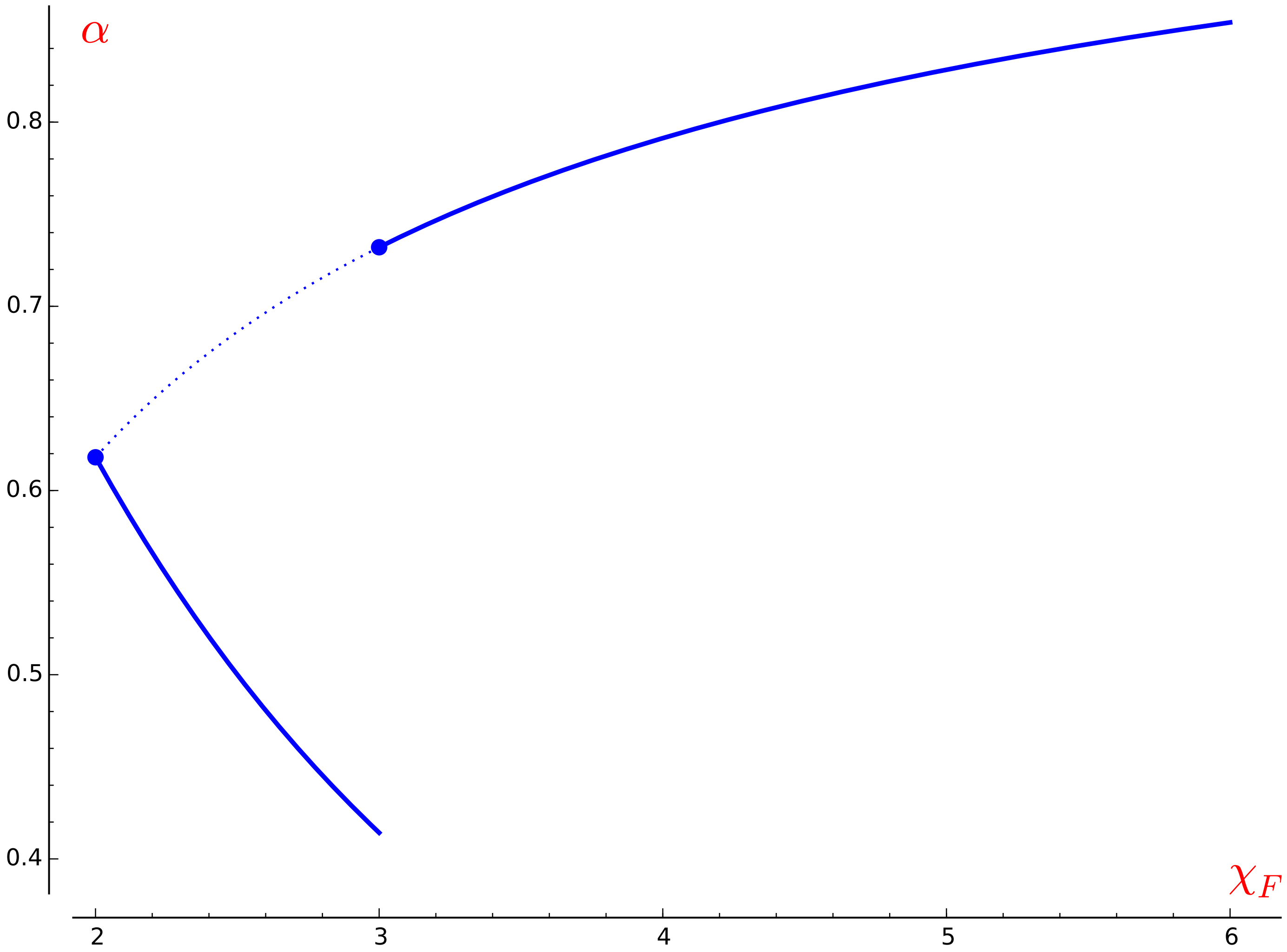
- the problem again :
  - we have some $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$
  - the vertices $w \in W$ are precoloured
    with $\phi(w) \subseteq [0, \chi_F + \alpha]$ of measure 1
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$

The answer for $D = 4$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 4$ we need :
  - $\alpha \geq \dfrac{\sqrt{(\chi_F - 1)^2 + 4\,(\chi_F - 1)} - \chi_F + 1}{2}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{\sqrt{(\chi_F - 1)^2 + 4} - \chi_F + 1}{2}$, for $2 \leq \chi_F < 3$
- and these bounds are best possible

The full picture for $D = 4$

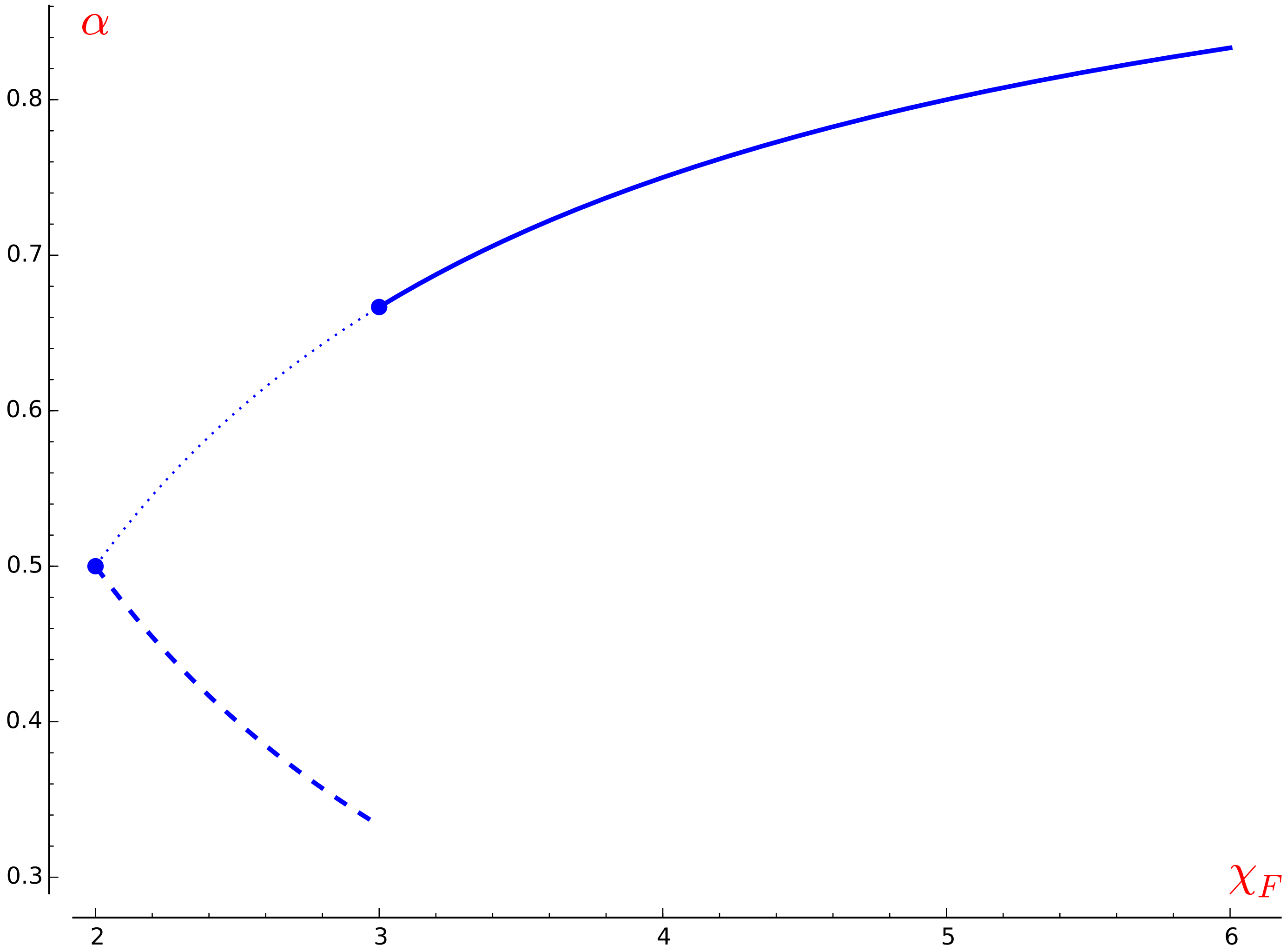


Almost the answer for $D = 5$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 5$ we need :
  - $\alpha \geq \dfrac{\chi_F - 1}{\chi_F}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{1}{\chi_F}$, for $2 \leq \chi_F < 3$
- but we don't know if the bound for $2 \leq \chi_F < 3$ is best possible

Almost the full picture for $D = 5$

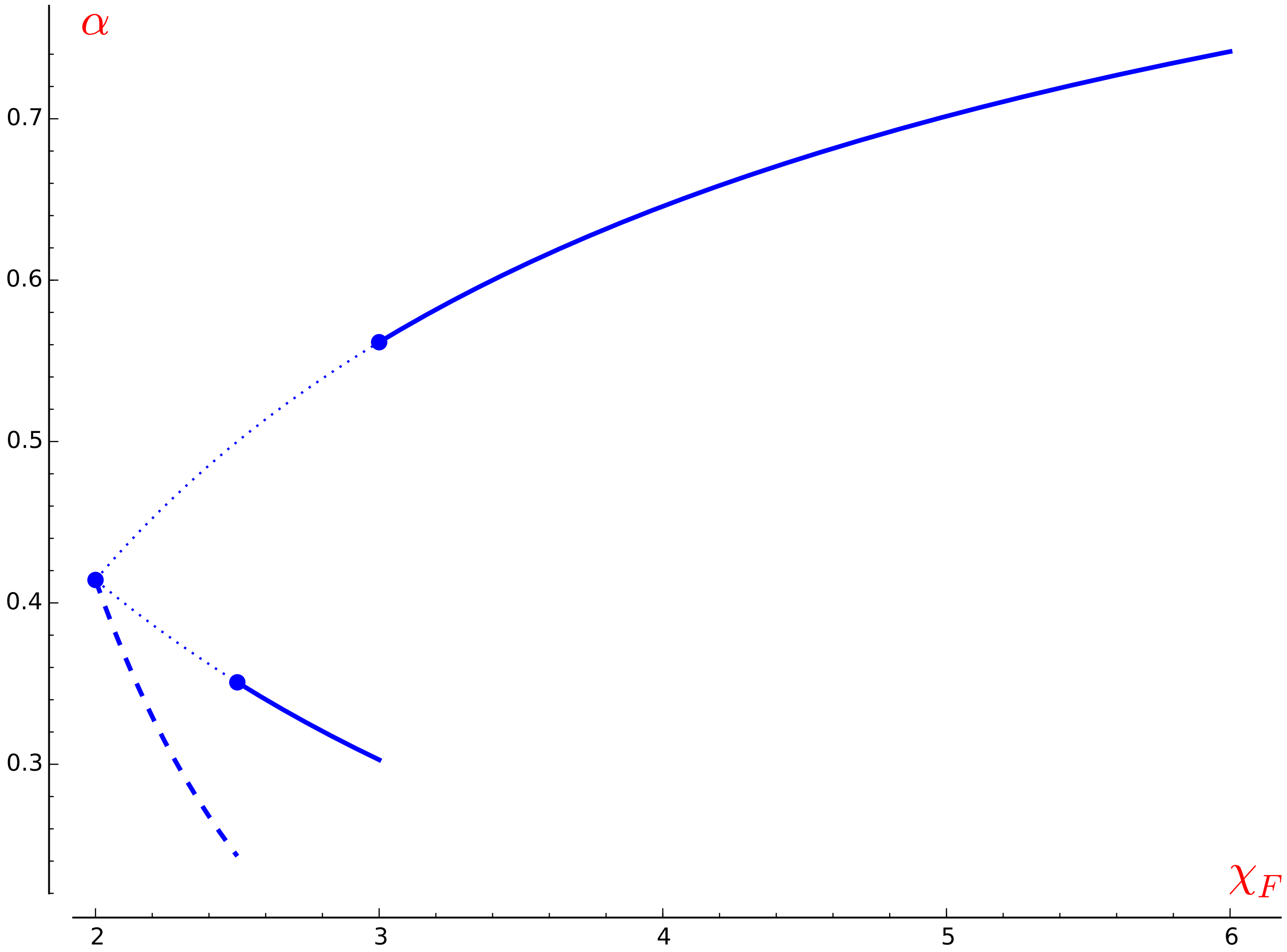


Almost the answer for $D = 6$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2011)

- for $D = 6$ we need :
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4\,(\chi_F - 1)} - \chi_F}{2}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4} - \chi_F}{2}$, for $2\frac{1}{2} \leq \chi_F < 3$
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4/(\chi_F - 1)} - \chi_F}{2}$, for $2 \leq \chi_F < 2\frac{1}{2}$
- and the bounds are best possible for $\chi_F \in \{2\} \cup [2\frac{1}{2}, \infty)$

Almost the full picture for $D = 6$



And for $D \geq 7$

- for $D \geq 7$ we have no further precise results for $2 < \chi_F < 3$
  - but all indications are that it gets more and more complicated when D gets larger

A new problem

- in all problems so far we assumed that
  the precoloured vertices and the extension
  can use the same set of available colours

- but what would happen if
  for the precolouring we can use a smaller colour set only ?
  - for integer colouring, this would make no difference
    (for distance $D \geq 4$)
    (may need extra colours – one extra is always enough)
  - but for fractional precolouring one would expect a more gradual change

The set-up of the new problem

- - G a graph with fractional chromatic number $\chi_F \geq 2$
  - $D \geq 3$ an integer
  - $W \subseteq V(G)$ with $\text{dist}(W) \geq D$
  - $L \geq 1$ a real number
- - the vertices $w \in W$ are precoloured with $\phi(w) \subseteq [0, L]$ with measure 1
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$
- how large should α be to be sure this can be done ?

The intuition for restricted fractional precolouring

- for $L = 1$, all precoloured vertices get ‘colour’ $[0, 1)$
  - a small α should be enough to complete the colouring
- when we increase L
  - the required α will increase as well
- until we reach $L = \chi_F + \alpha_{\text{crit}}$
  - where α_{crit} is the value so that :

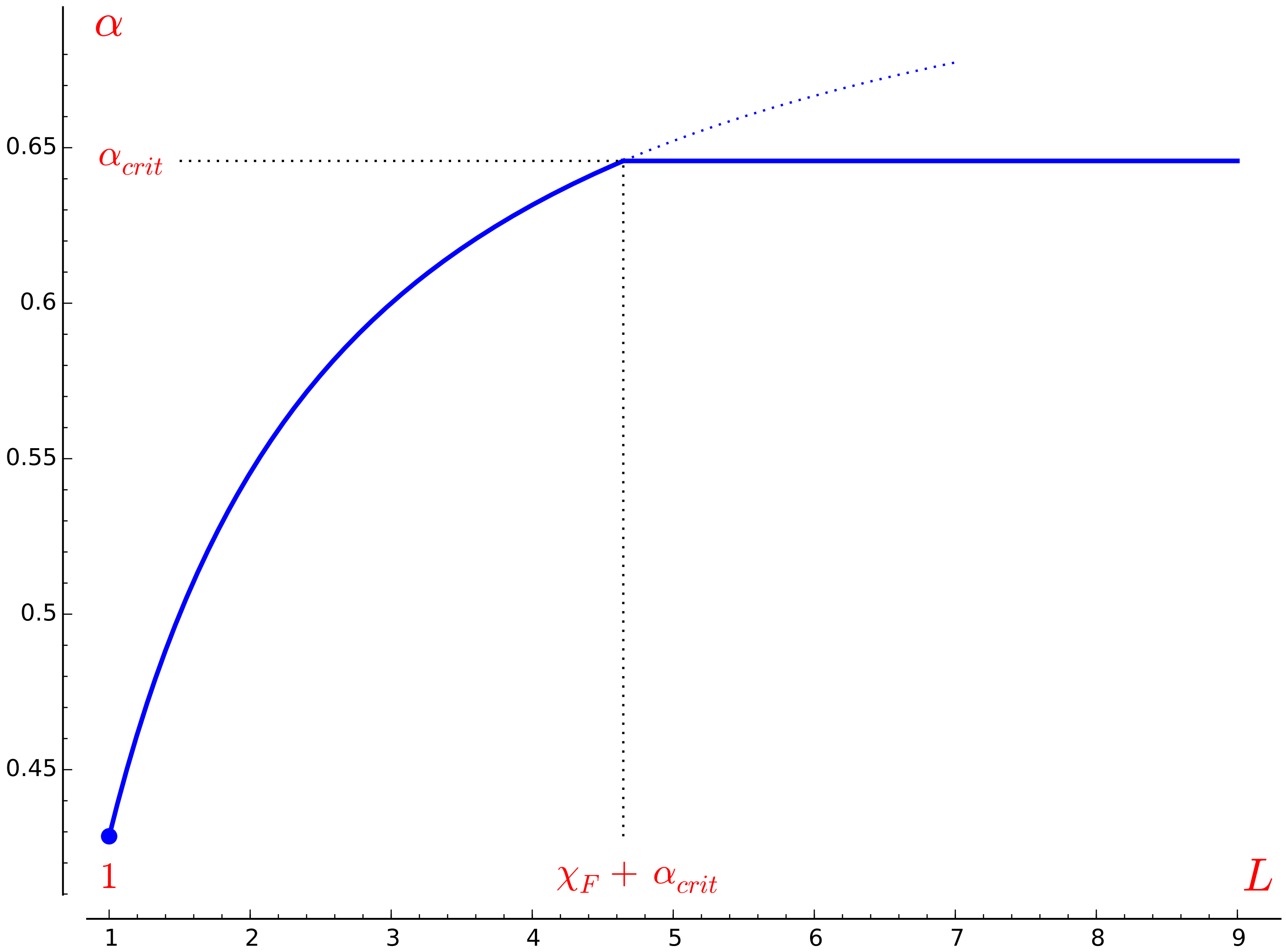
    precolouring with $[0, \chi_F + \alpha_{\text{crit}}]$
    can be completed with colours from $[0, \chi_F + \alpha_{\text{crit}}]$
- increasing L further,
  doesn’t require more than $[0, \chi_F + \alpha_{\text{crit}}]$ to complete

A first quarter of the answer

Theorem (vdH, Li & Müller, 2014+)

- if $D \equiv 2 \bmod 4$, then extension is always possible, provided α is at least :
  - $\dfrac{L\,(\chi_F - 1)}{L\left\lfloor \frac{1}{4}D \right\rfloor \chi_F + \chi_F - 1}$, if $1 \le L \le \chi_F + \alpha_{\mathrm{crit}}$
  - α_{crit}, if $L \ge \chi_F + \alpha_{\mathrm{crit}}$
- where α_{crit} is given by the first Král' el al. result
- and these bounds are best possible for $\chi_F \in \{2\} \cup [3, \infty)$

The picture for $D = 6$ and $\chi_F = 4$

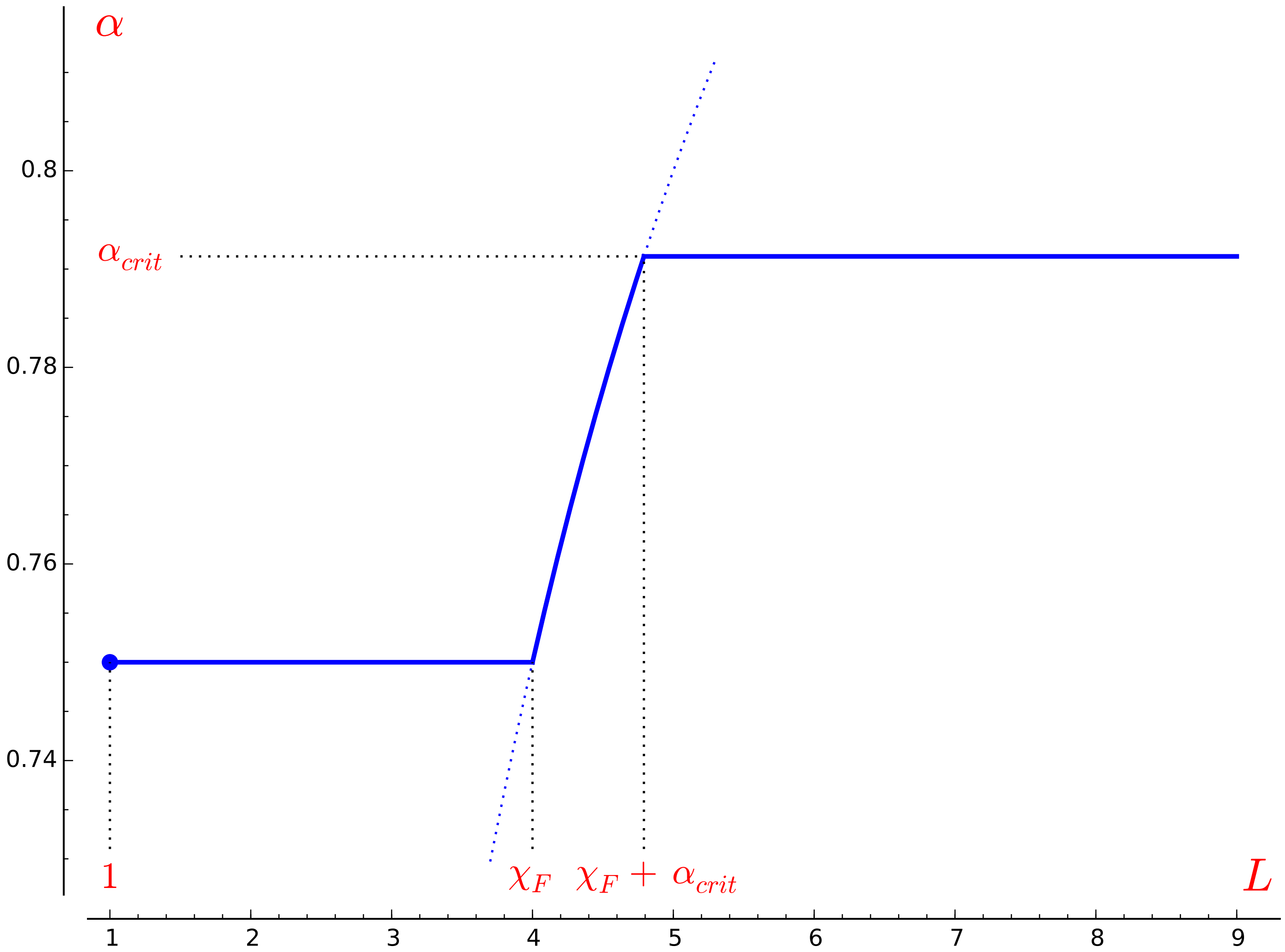


A next quarter of the answer

Theorem (vdH, Li & Müller, 2014+)

- if $D \equiv 0 \bmod 4$, then extension is always possible, provided α is at least :
  - $\dfrac{\chi_F - 1}{\lfloor \frac{1}{4} D \rfloor \, \chi_F}$, if $1 \le L \le \chi_F$
  - $\dfrac{L - 1}{\lfloor \frac{1}{4} D \rfloor \, L}$, if $\chi_F \le L \le \chi_F + \alpha_{\text{crit}}$
  - α_{crit}, if $L \ge \chi_F + \alpha_{\text{crit}}$
- where α_{crit} is given by the first Král' el al. result
- and these bounds are best possible for $\chi_F \in \{2\} \cup [3, \infty)$

The picture for $D = 4$ and $\chi_F = 4$



And the final half of the answer

Theorem (vdH, Li & Müller, 2014+)

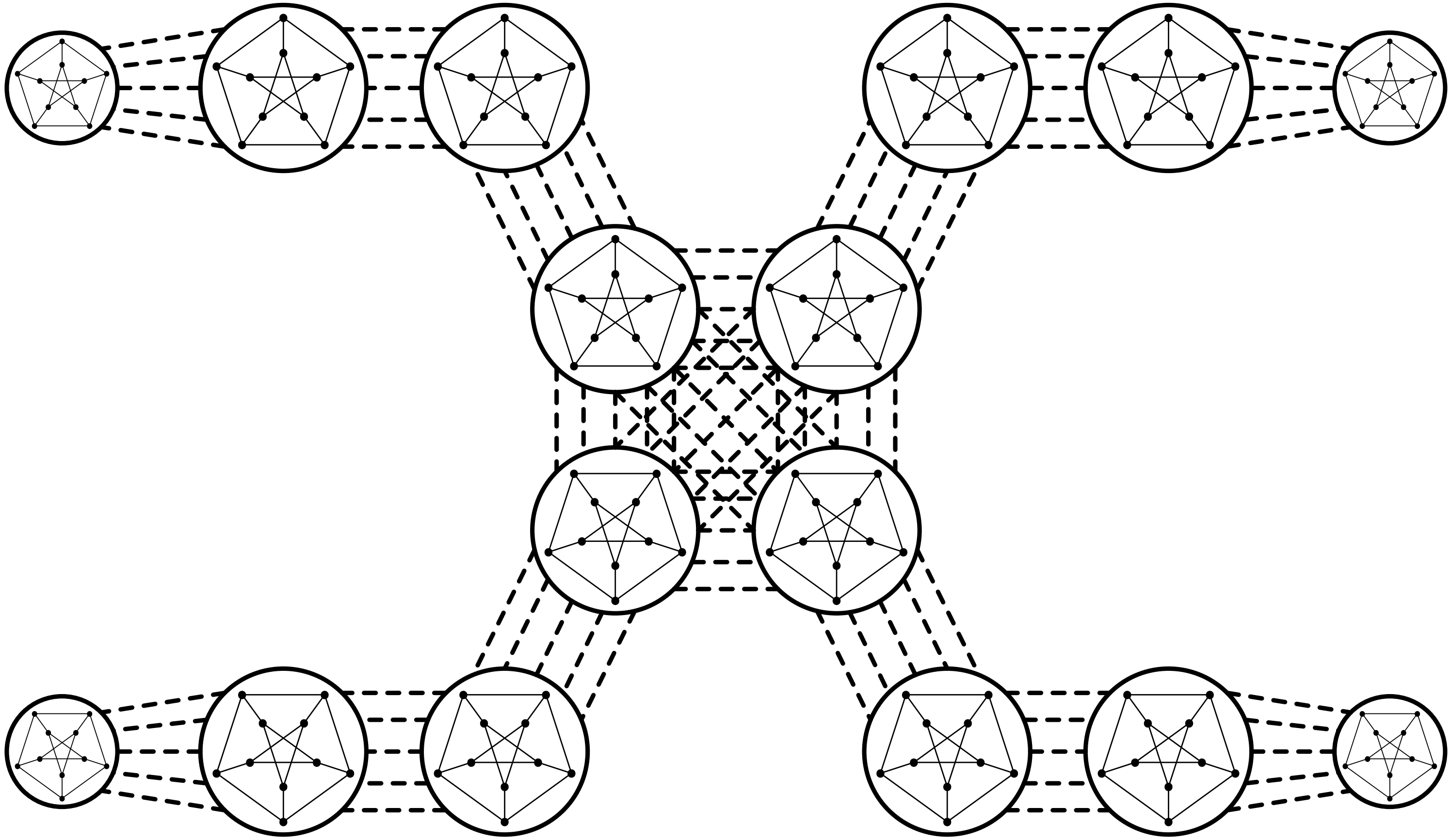
- if D is odd, then extension is always possible, provided α is at least :

  - α_{crit}, for any $L \geq 1$

  (i.e.: the bound doesn't depend on L)

- for $\chi_F \in \{2\} \cup [3, \infty)$, the best possible value of α_{crit} is given by the first Král' el al. result

The end



Thank you for the attention !