Generalised Colouring Numbers of Graphs

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joint work with :

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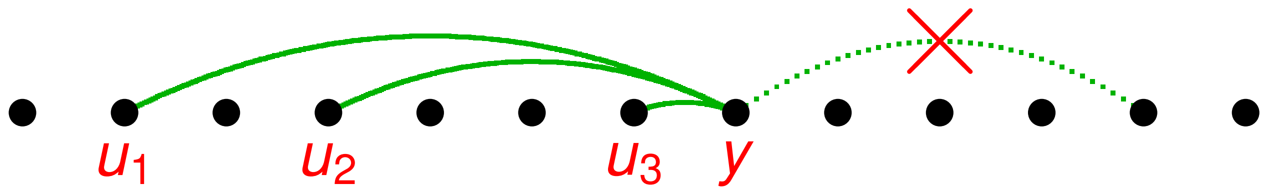
The normal colouring number

-
  - let ***L*** be a linear ordering of the vertices of a graph ***G***
  - for a vertex $y \in V(G)$,
    let $\boldsymbol{S}(\boldsymbol{L}, \boldsymbol{y})$ be the set of neighbours u of y with $u <_L y$



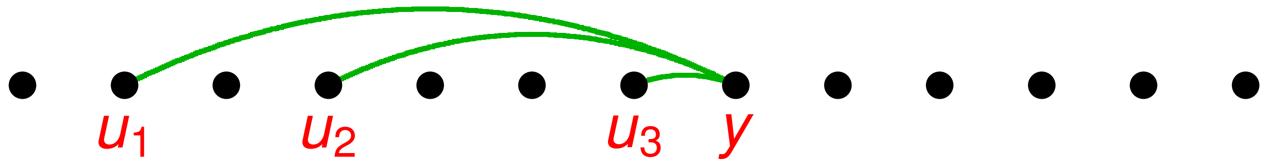
- then the **colouring number col**$(\boldsymbol{G})$ is defined as

$$\mathrm{col}(G) = \min_{L} \max_{y \in V(G)} |S(L,y)| + 1$$

- colouring from left to right gives: $\chi(G) \leq \mathrm{col}(G)$

Generalising the colouring number

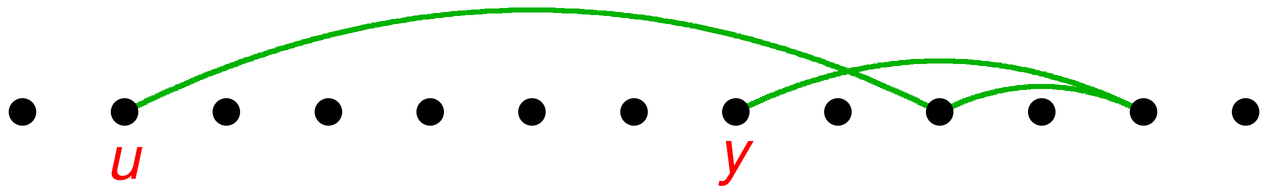
- the set $S(L, y)$ can be defined as
  "vertices $u <_L y$ for which there is an uy-path of length 1"



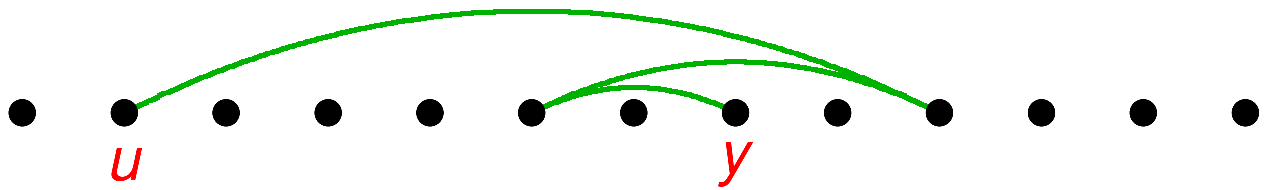
- what would happen if we allow longer paths ?

Generalising the colouring number

- the set $S(L, y)$ can be defined as
  "vertices $u <_L y$ for which there is an uy-path of length 1"
- what would happen if we allow longer paths ?
- - a **strong uy-path** has all internal vertices larger than y

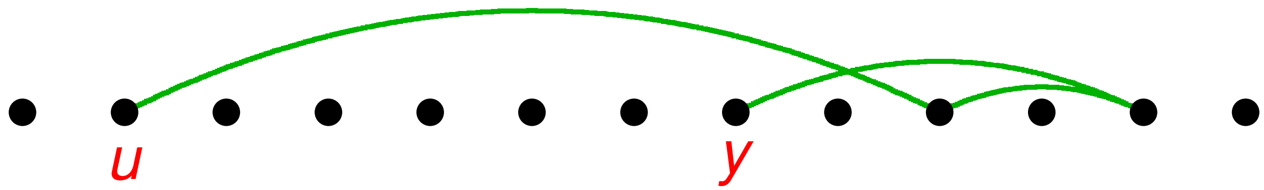


  - a **weak uy-path** has all internal vertices larger than u



Strong generalised colouring numbers

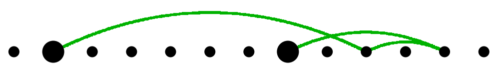
- - a **strong *uy*-path** has all internal vertices larger than y

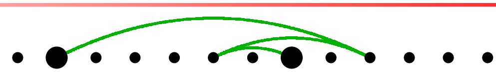


  - let $\boldsymbol{S_r(L, y)}$ be the set of vertices $u <_L y$ for which there exists a strong uv-path of length at most r

- then the **strong *r*-colouring number $\mathrm{col}_r(G)$** is defined as

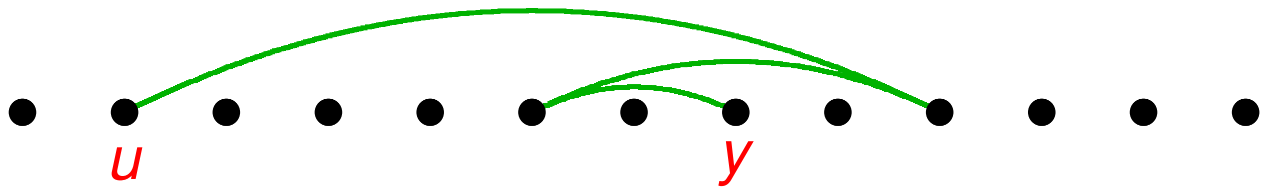
$$\mathrm{col}_r(G) = \min_L \max_{y \in V(G)} |S_r(L, y)| + 1$$

strong: 

weak: 

Weak generalised colouring numbers

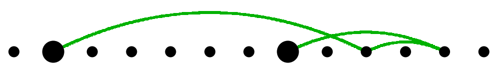
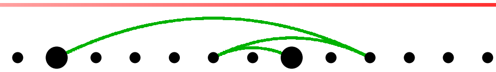
- - a **weak *uy*-path** has all internal vertices larger than u



  - let $\boldsymbol{W_r(L, y)}$ be the set of vertices $u <_L y$ for which there exists a weak uv-path of length at most r

- then the **weak *r*-colouring number** $\mathbf{wcol}_r(G)$ is defined as

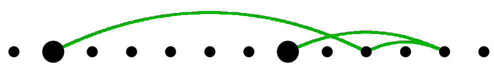
$$\mathrm{wcol}_r(G) = \min_L \max_{y \in V(G)} |W_r(L, y)| + 1$$

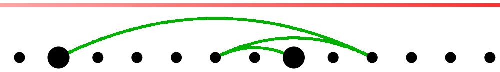
strong:  weak: 

Basic facts of generalised colouring numbers

- introduced by Kierstead & Yang, 2004
- by definition: $\mathrm{col}_1(G) = \mathrm{wcol}_1(G) = \mathrm{col}(G)$
- obviously: $\mathrm{col}_r(G) \leq \mathrm{wcol}_r(G)$
- but also: $\mathrm{wcol}_r(G) \leq (\mathrm{col}_r(G) - 1)^r + 1$

  (Proof: every weak path of length at most r is formed of at most r strong paths of length at most r.)

strong: 

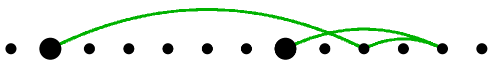
weak: 

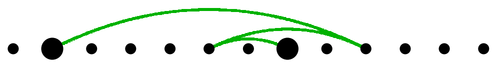
A simple application

- **acyclic chromatic number** $\chi_a(G)$:
  minimum number of colours needed to properly colour the vertices of G such that every cycle has at least 3 colours

- **Theorem** (Kierstead & Yang, 2004)

$$\chi_a(G) \;\leq\; \mathrm{col}_2(G)$$

strong: 

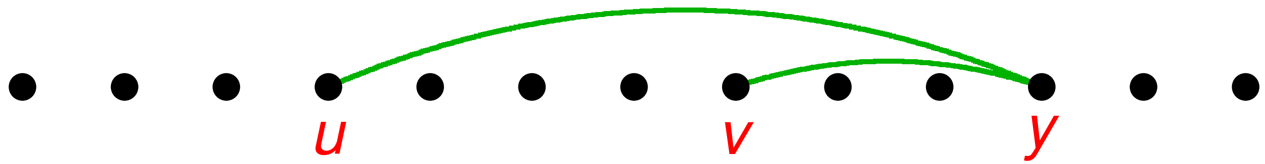
weak: 

A simple application

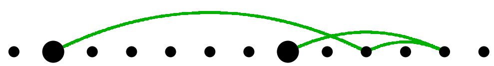
- **Theorem** (Kierstead & Yang, 2004)

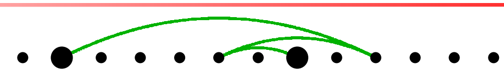
$$\chi_a(G) \;\leq\; \mathrm{col}_2(G)$$

- Proof
  - take an ordering L that gives $\mathrm{col}_2(G)$
  - colour from left to right,
    i.e. colour y different from all vertices in $S_2(L, y)$
  - for any cycle C, look at the right-most vertex y of C and its two neighbours in C



  - these 3 vertices must receive different colours $\square$

strong: 

weak: 

A structural application

- classes of graphs $\mathcal{G}$ with **bounded expansion** were introduced by Nešetřil & Ossona de Mendez in terms of "densities of shallow minors"

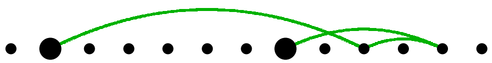
- **Theorem** (Zhu, 2009)

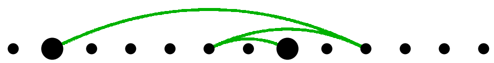
  a class of graphs $\mathcal{G}$ has bounded expansion $\iff$

  there exists constants $c_1, c_2, \ldots$ such that

  for every $G \in \mathcal{G}$ and every r we have $\mathrm{col}_r(G) \leq c_r$

  - note that we can replace $\mathrm{col}_r(G)$ by $\mathrm{wcol}_r(G)$ (with different constants $c'_1, c'_2, \ldots$)

strong: 

weak: 

Further structural applications

- we also obviously have:

$$\mathrm{wcol}_1(G) \le \mathrm{wcol}_2(G) \le \mathrm{wcol}_3(G) \le \ldots \le \mathrm{wcol}_\infty(G)$$

(where the "∞" means "any length path allowed")

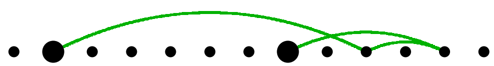
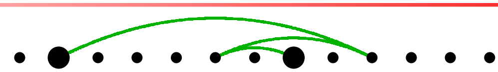
- **Property** (Nešetřil & Ossona de Mendez, $\sim$2005)

$$\mathrm{wcol}_\infty(G) \;=\; \text{tree-depth}(G) + 1$$

- similarly: $\mathrm{col}_1(G) \le \mathrm{col}_2(G) \le \mathrm{col}_3(G) \le \ldots \le \mathrm{col}_\infty(G)$

- **Property** (Grohe, Kreutzer, Rabinovich, Siebertz & Stavropoulos, 2014)

$$\text{tree-width}(G) + 1 \;=\; \mathrm{col}_\infty(G)$$

strong:  weak: 

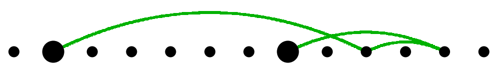
Finding generalised colouring numbers

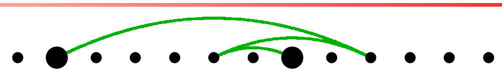
- the normal colouring number $\text{col}(G) = \text{col}_1(G) = \text{wcol}_1(G)$ can be found in polynomial time

- **Theorem** (Grohe, Kreutzer, Rabinovich, Siebertz & Stavropoulos, 2014)

  for any fixed $r \geq 3$,
  determining $\text{col}_r(G)$ and $\text{wcol}_r(G)$ is NP-complete

- **Question**
  - what is the complexity of determining $\text{col}_2(G)$ or $\text{wcol}_2(G)$?

strong: 

weak: 

Bounds on generalised colouring numbers

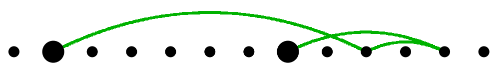
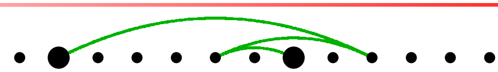
- until recently, all known upper bounds for $\mathrm{col}_r(G)$ for specific graph classes were exponential: $\mathcal{O}(c^r)$;

  while those for $\mathrm{wcol}_r(G)$ were even worse: $\mathcal{O}(r^r)$

- <u>**Theorem**</u> (vdH, Ossona de Mendez, Quiroz, Rabinovich & Siebertz, 2015+)

  G a graph without K_t -minor $\Longrightarrow$

  - $\mathrm{col}_r(G) \;\leq\; \binom{t}{2} \cdot (2r+1)$
  - $\mathrm{wcol}_r(G) \;\leq\; \binom{t}{2}^r \cdot (2r+1)$

strong:  weak: 

Generalised colouring numbers for planar graphs

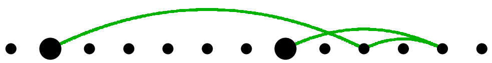
- **Theorem** (vdH, Ossona de Mendez, Quiroz, Rabinovich & Siebertz, 2015+)

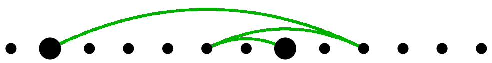
  G planar $\Longrightarrow$

  - $\mathrm{col}_r(G) \le 5r + 1$
  - $\mathrm{wcol}_r(G) \le \mathcal{O}(r^5)$

- the bound $\mathrm{col}_1(G) = \mathrm{col}(G) \le 6$ is best possible

- for $\mathrm{col}_2(G)$:
  - oldest bound: ≤ 761 (Chen & Schelp, 1993)
  - best possible: ≤ 8 (Dvořák, Kabela & Kaiser, 2015)

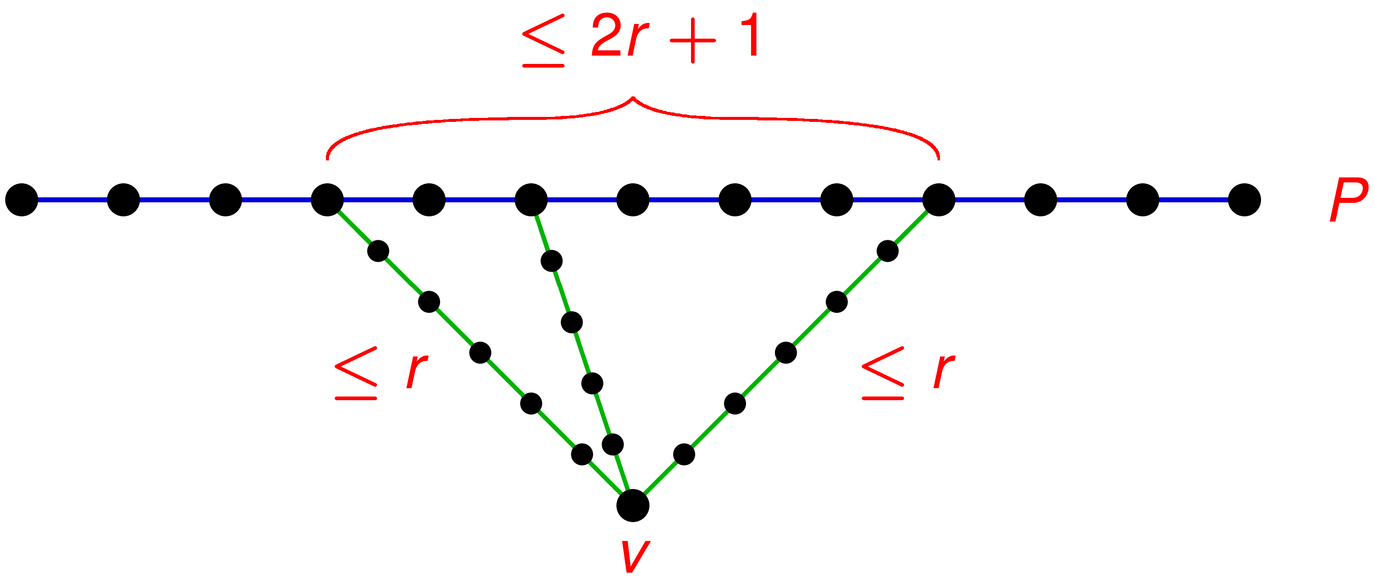
strong: 

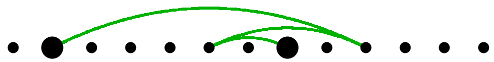
weak: 

About the proofs

- the main idea of the proofs is the following:
  - if P is a shortest path in G, then for any vertex v:

    the number of vertices on P at distance at most r from v is at most $2r+1$



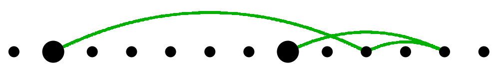
strong:  weak: 

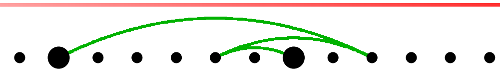
About the proofs

- - if P is a shortest path in G, then for any vertex v:

    the number of vertices on P at distance at most r from v is at most $2r + 1$

- - for planar or K_t-minor free graphs, there are separators consisting of a bounded number m of shortest paths
  - these m paths form the first part of the ordering L
    - every y outside these paths has at most $m \cdot (2r + 1)$ vertices within distance r on these paths
  - repeat the procedure for each of the components of the separator (cue: frantic handwaving)

strong: 

weak: 

And that's it for today . . .

Thanks for listening !