Fractional Colouring and Precolouring Extension of Graphs

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The basics of graph colouring

- **vertex-colouring** with $\boldsymbol{k}$ colours :
  adjacent vertices must receive different colours

- **chromatic number** $\chi(\boldsymbol{G})$:
  minimum k so that a vertex-colouring exists

general question :

- what can we say if some vertices are already precoloured ?
- in particular : are $\chi(G)$ colours still enough ?
  - **not in general**

Precolouring questions

next best questions :

- how many extra colours may be needed ?
- and what conditions on the precoloured vertices can make life easier ?

Question (Thomassen, 1997)

- G planar,

  $W \subseteq V(G)$ a set of vertices such that

  distance between any two vertices in W is at least 100

  - can any 5-colouring of W

    be extended to a 5-colouring of G ?

The first answer

- $\mathrm{dist}(W)$: minimum distance between any two vertices in W

Theorem (Albertson, 1998)

- G any graph with chromatic number χ

  $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq 4$

  $\Longrightarrow$ any $(\chi+1)$-colouring of W

  can be extended to a $(\chi+1)$-colouring of G

Some more answers

Theorem (easy)

- G any graph with chromatic number χ

  $W \subseteq V(G)$ with $\text{dist}(W) \geq 3$

  $\Longrightarrow$ any $(\chi+\chi)$-colouring of W

  can be extended to a $(\chi+\chi)$-colouring of G

Theorem (Albertson, 1998)

- G planar graph

  $W \subseteq V(G)$ with $\text{dist}(W) \geq 3$

  $\Longrightarrow$ any 6-colouring of W

  can be extended to a 6-colouring of G

A different kind of colouring

- **fractional K-colouring** of graph G $(K \in \mathbb{R}_+)$:
  - every vertex $v \in V$ is assigned a subset $\phi(v) \subseteq [0, K]$ so that :
    - every subset $\phi(v)$ has 'measure' 1
    - and $uv \in E(G) \implies \phi(u) \cap \phi(v) = \varnothing$

- **fractional chromatic number** $\chi_F(G)$:

$$= \inf \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

$$= \min \{ K \geq 0 \mid G \text{ has a fractional } K\text{-colouring} \}$$

Fractional colouring

- **note** : we always have $\chi_F(G) \le \chi(G)$
  - the difference can be arbitrarily large

-
  - $\chi_F(G) = 1 \iff G$ has no edges
  - $\chi_F(G) = 2 \iff G$ has edges and is bipartite
  - for all rational $K \ge 2$: there exist G with $\chi_F(G) = K$

Precolouring in the fractional world

- so now suppose that for some vertices $W \subseteq V(G)$, the vertices in W are already precoloured :
  - vertices $w \in W$ have been given some set $\phi(w)$ of measure 1
- when can this be extended to a fractional colouring of the whole graph G ?
- in general we should expect to require more than $\chi_F(G)$ colours

The set-up of the problem

- - G a graph with fractional chromatic number $\chi_F \geq 2$
  - $D \geq 3$ an integer
  - $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$
- - the vertices $w \in W$
    are precoloured with $\phi(w) \subseteq [0, \chi_F + \alpha]$ of measure 1
    - for some real $\alpha \geq 0$
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$
- how large should α be to guarantee this can be done ?

A major part of the answer

Theorem (Král', Krnc, Kupec, Lužar & Volec, 2011)

- extension is always possible, provided α is at least :
  - $$\frac{\sqrt{\left(\left\lfloor\frac{1}{4}D\right\rfloor\chi_F+1\right)^2-4\left\lfloor\frac{1}{4}D\right\rfloor}-\left\lfloor\frac{1}{4}D\right\rfloor\chi_F+1}{2\left\lfloor\frac{1}{4}D\right\rfloor},$$ if $D \equiv 0 \bmod 4$
  - $$\frac{\chi_F-1}{\left\lfloor\frac{1}{4}D\right\rfloor\chi_F},$$ if $D \equiv 1 \bmod 4$
  - $$\frac{\sqrt{\left(\left\lfloor\frac{1}{4}D\right\rfloor\chi_F+2\right)^2-4\left(\left\lfloor\frac{1}{4}D\right\rfloor+1\right)}-\left\lfloor\frac{1}{4}D\right\rfloor\chi_F}{2\left\lfloor\frac{1}{4}D\right\rfloor},$$ if $D \equiv 2 \bmod 4$
  - $$\frac{\chi_F-1}{\left\lfloor\frac{1}{4}D\right\rfloor\chi_F+\chi_F-1},$$ if $D \equiv 3 \bmod 4$

A major part of the answer

Theorem (Král', Krnc, Kupec, Lužar & Volec, 2011)

- moreover, these bounds on α are best possible,
  - if $D = 3$;
  - if $D \geq 4$ and $\chi_F \in \{2\} \cup [3, \infty)$

A major part of the answer – best possible

- **in other words** :
  - for all integers $D \geq 3$,

    all rational numbers $\chi_F \in \{2\} \cup [3, \infty)$,

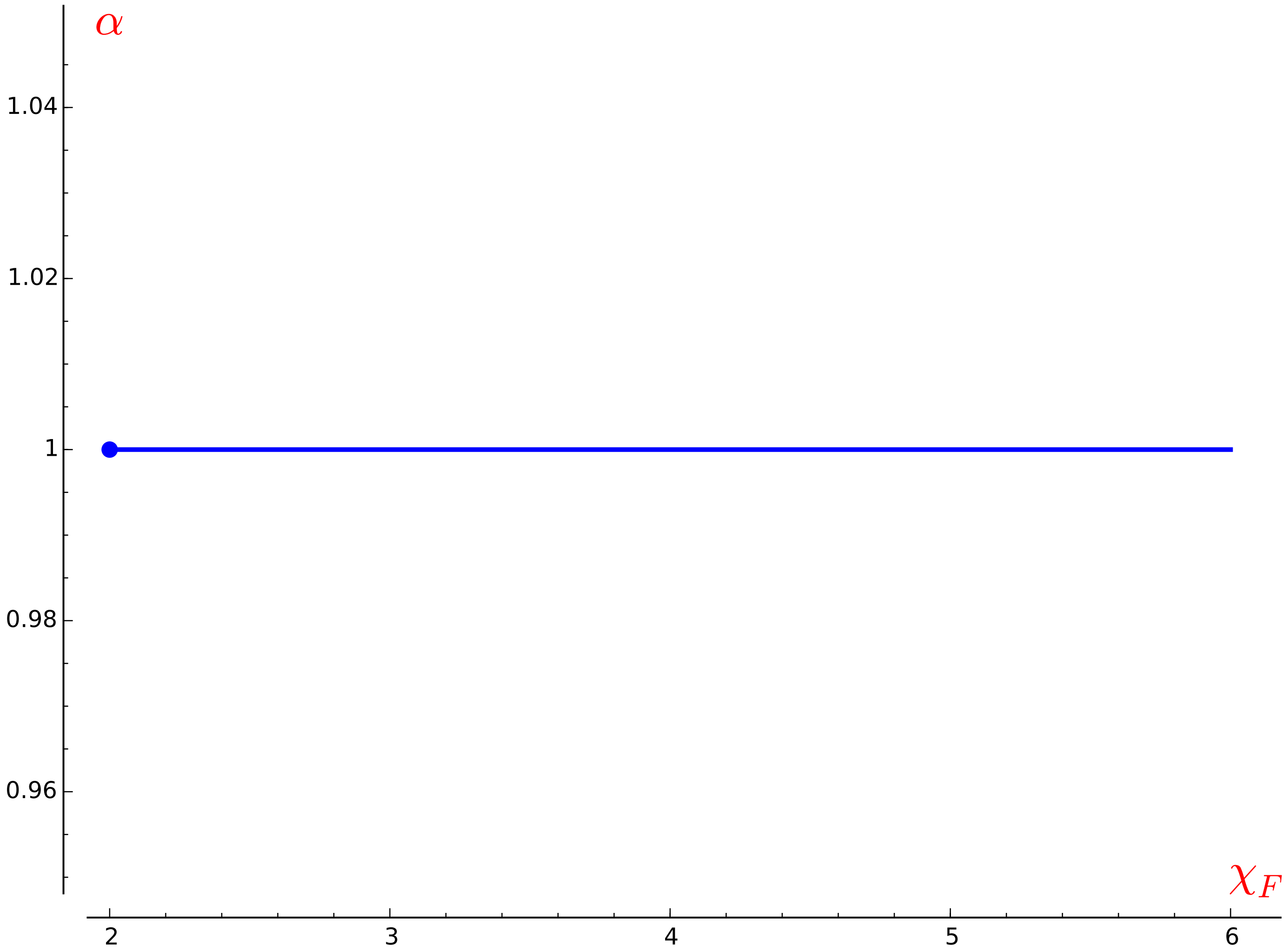
    all $\alpha \geq 0$ failing the bound for that D and χ_F
  - there is a graph G with fractional chromatic number χ_F,

    a subset $W \subseteq V(G)$ with $\text{dist}(W) \geq D$,

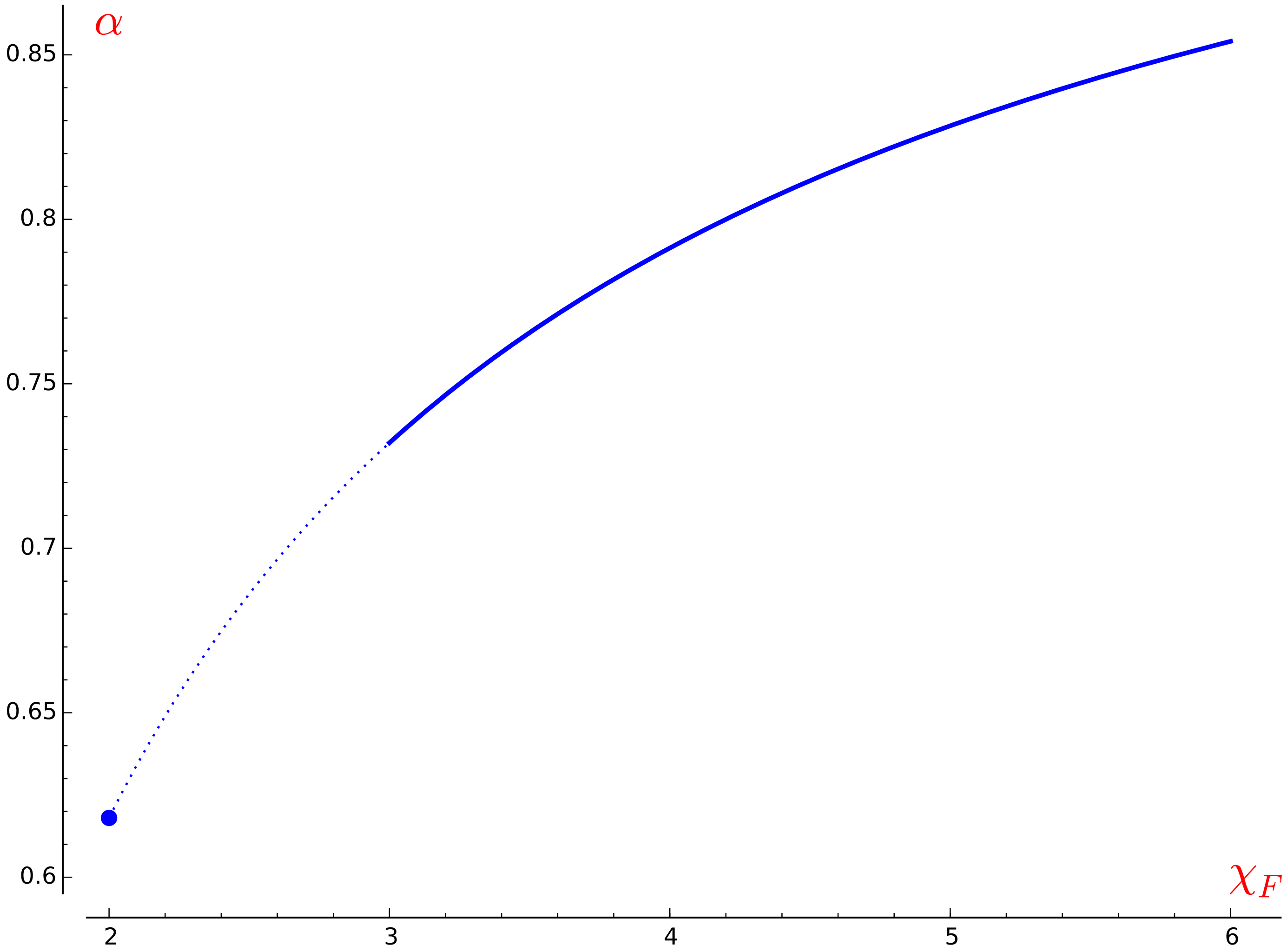
    and a fractional precolouring $\phi(w) \subseteq [0, \chi_F + \alpha]$

    for $w \in W$
  - such that ϕ cannot be extended to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$ only

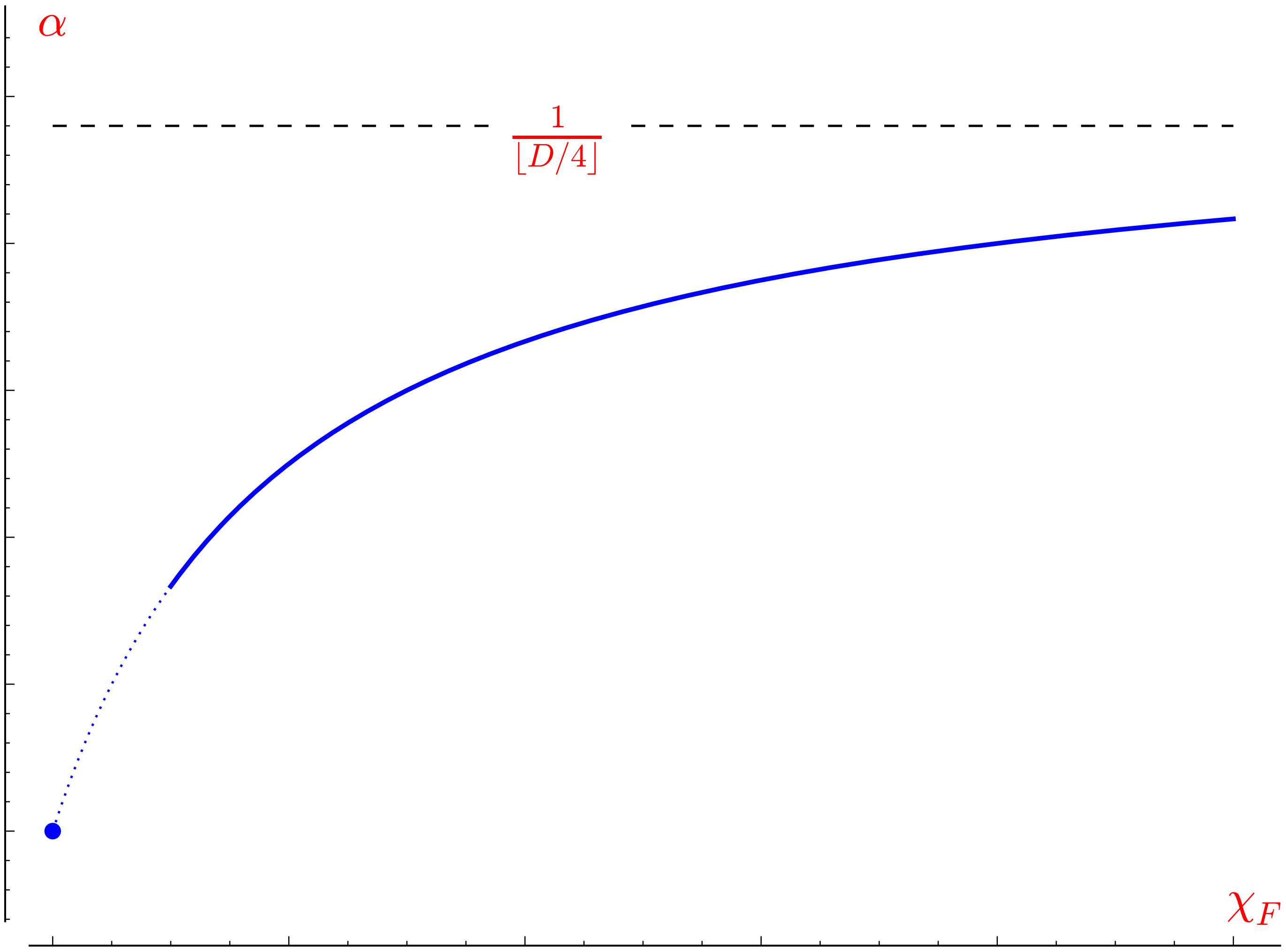
The picture for $D = 3$



The picture for $D = 4$



The picture for general $D \geq 4$



Almost the complete answer

- so for $D \geq 4$, we know the full answer only if $\chi_F = 2$ or $\chi_F \geq 3$
  - so what happens in the gap $2 < \chi_F < 3$?

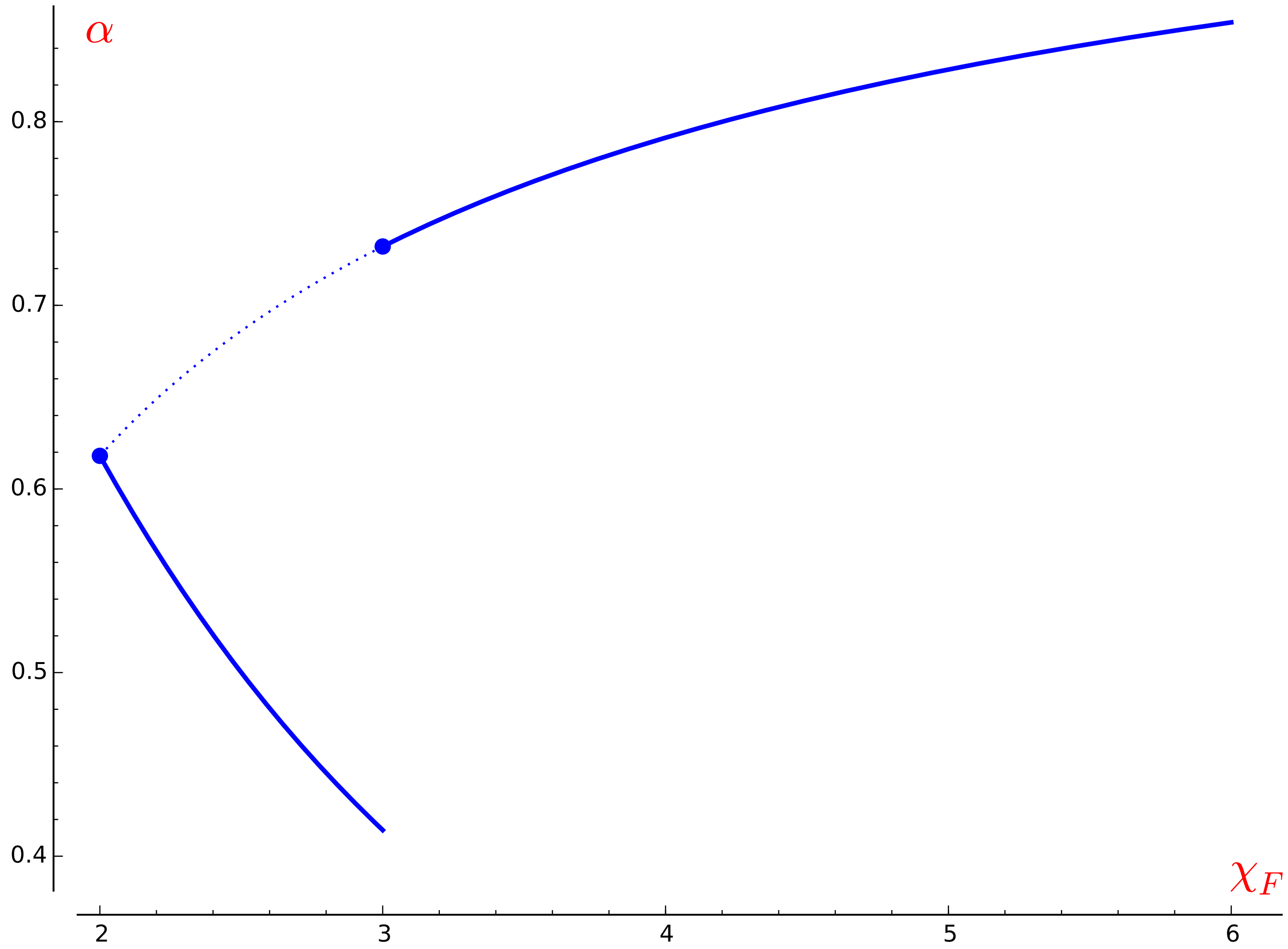
- the problem again :
  - we have some $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$
  - the vertices $w \in W$ are precoloured with $\phi(w) \subseteq [0, \chi_F + \alpha]$ of measure 1
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$

The answer for $D = 4$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2014)

- for $D = 4$ we need :
  - $\alpha \geq \dfrac{\sqrt{(\chi_F - 1)^2 + 4\,(\chi_F - 1)} - \chi_F + 1}{2}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{\sqrt{(\chi_F - 1)^2 + 4} - \chi_F + 1}{2}$, for $2 \leq \chi_F < 3$
- and these bounds are best possible

The full picture for $D = 4$

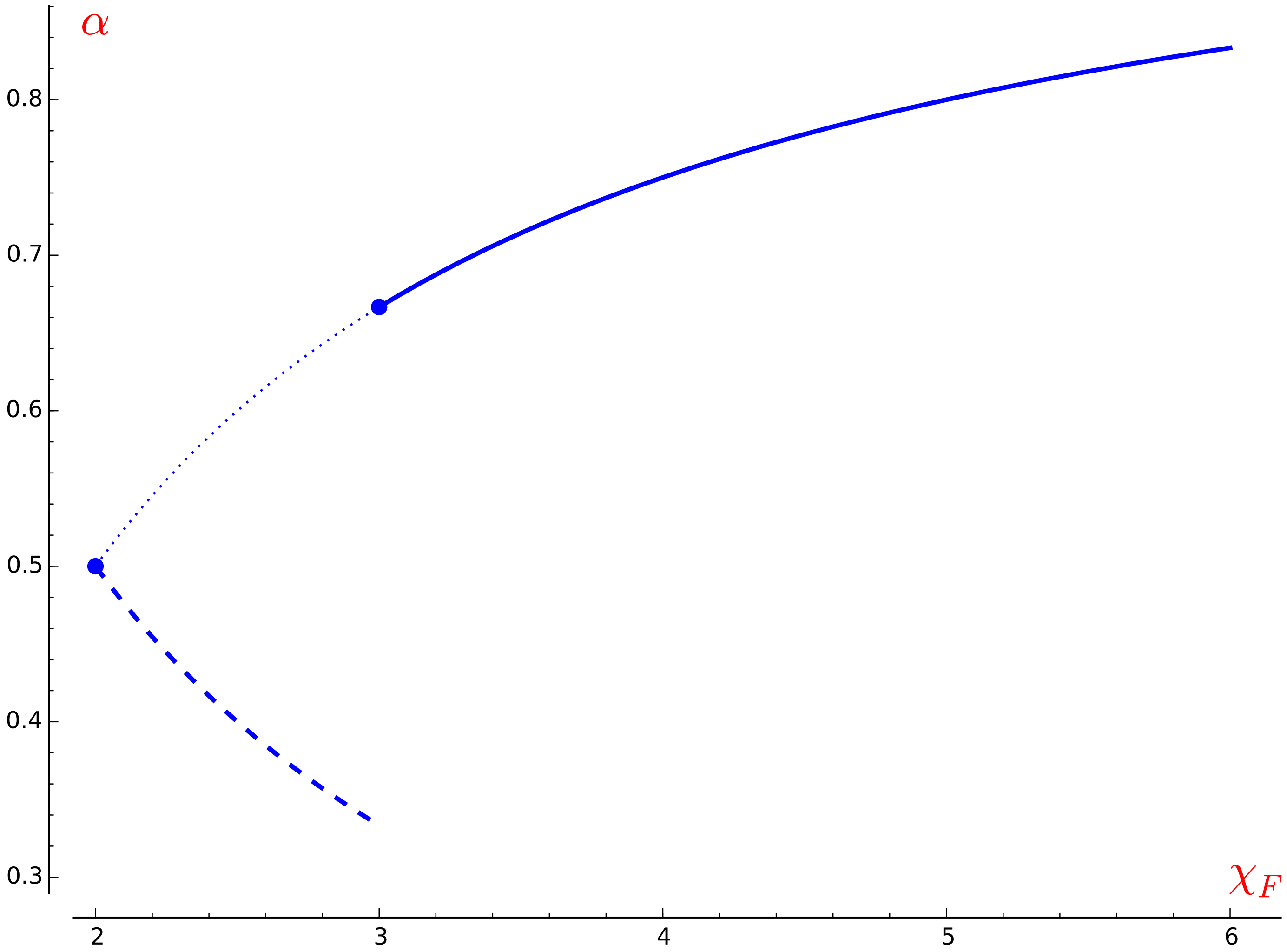


Almost the answer for $D = 5$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2014)

- for $D = 5$ we need :
  - $\alpha \geq \dfrac{\chi_F - 1}{\chi_F}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{1}{\chi_F}$, for $2 \leq \chi_F < 3$
- but we don't know if the bound for $2 \leq \chi_F < 3$ is best possible

Almost the full picture for $D = 5$

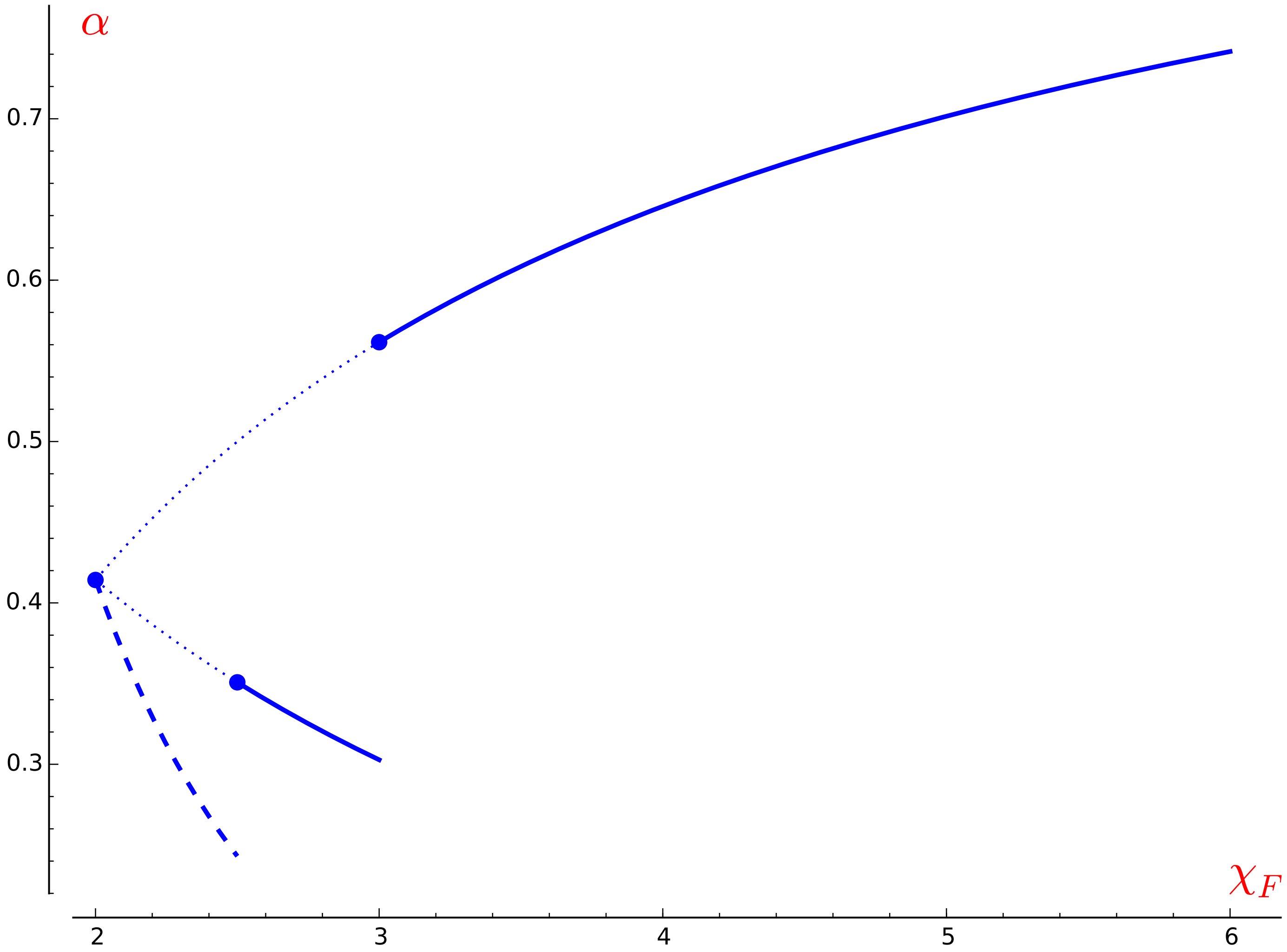


Almost the answer for $D = 6$

Theorem (vdH, Král', Kupec, Sereni & Volec, 2014)

- for $D = 6$ we need :
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4\,(\chi_F - 1)} - \chi_F}{2}$, for $\chi_F \geq 3$
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4} - \chi_F}{2}$, for $2\frac{1}{2} \leq \chi_F < 3$
  - $\alpha \geq \dfrac{\sqrt{\chi_F^2 + 4/(\chi_F - 1)} - \chi_F}{2}$, for $2 \leq \chi_F < 2\frac{1}{2}$
- and the bounds are best possible for $\chi_F \in \{2\} \cup [2\frac{1}{2}, \infty)$

Almost the full picture for $D = 6$



And for $D \geq 7$

- for $D \geq 7$ we have no further precise results for $2 < \chi_F < 3$
  - but all indications are that it gets more and more complicated when D gets larger

A new problem

- in all problems so far we assumed that
  the precoloured vertices and the extension
  can use the same set of available colours

- but what would happen if
  for the precolouring we can use a smaller colour set only ?
  - for integer colouring, this would make no difference
    (for distance $D \geq 4$)
    (may need extra colours – but 1 extra is always enough)
  - but for fractional precolouring one would expect a more
    gradual change

The set-up of the new problem

- - G a graph with fractional chromatic number $\chi_F \geq 2$
  - $D \geq 3$ an integer
  - $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$
  - $L \geq 1$ a real number
- - the vertices $w \in W$
    are precoloured with $\phi(w) \subseteq [0, L]$ of measure 1
  - and we want to extend that to a fractional colouring of the whole G, using colours from $[0, \chi_F + \alpha]$
- how large should α be to guarantee this can be done ?

The intuition for restricted fractional precolouring

- for $L = 1$, all precoloured vertices get 'colour' $[0, 1)$
  - a small α should be enough to complete the colouring
- when we increase L
  - the required α will increase as well
- until we reach $L = \chi_F + \alpha_{\text{crit}}$
  - where α_{crit} is the value so that :

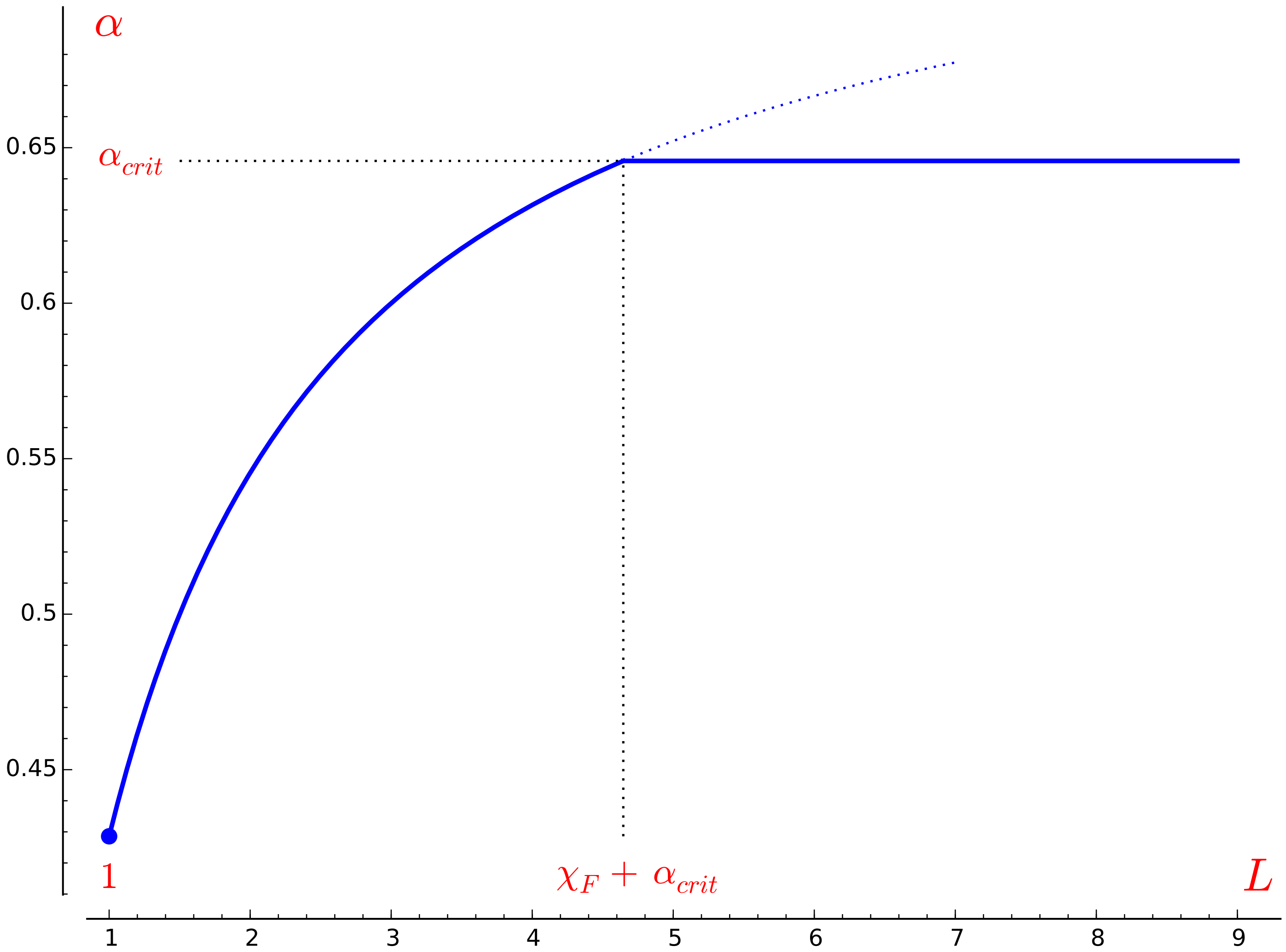
    precolouring with $[0, \chi_F + \alpha_{\text{crit}}]$
    can be completed with colours from $[0, \chi_F + \alpha_{\text{crit}}]$
- increasing L further,
  doesn't require more than $[0, \chi_F + \alpha_{\text{crit}}]$ to complete

A first quarter of the answer

Theorem (vdH, Li & Müller, 2014+)

- if $D \equiv 2 \bmod 4$, then extension is always possible, provided α is at least :
  - $\dfrac{L\,(\chi_F - 1)}{L\,\left\lfloor \frac{1}{4}D \right\rfloor \chi_F + \chi_F - 1}$, if $1 \le L \le \chi_F + \alpha_{\text{crit}}$
  - α_{crit}, if $L \ge \chi_F + \alpha_{\text{crit}}$
- where α_{crit} is given by the first Král' el al. results
- and these bounds are best possible for $\chi_F \in \{2\} \cup [3, \infty)$

The picture for $D = 6$ *and* $\chi_F = 4$

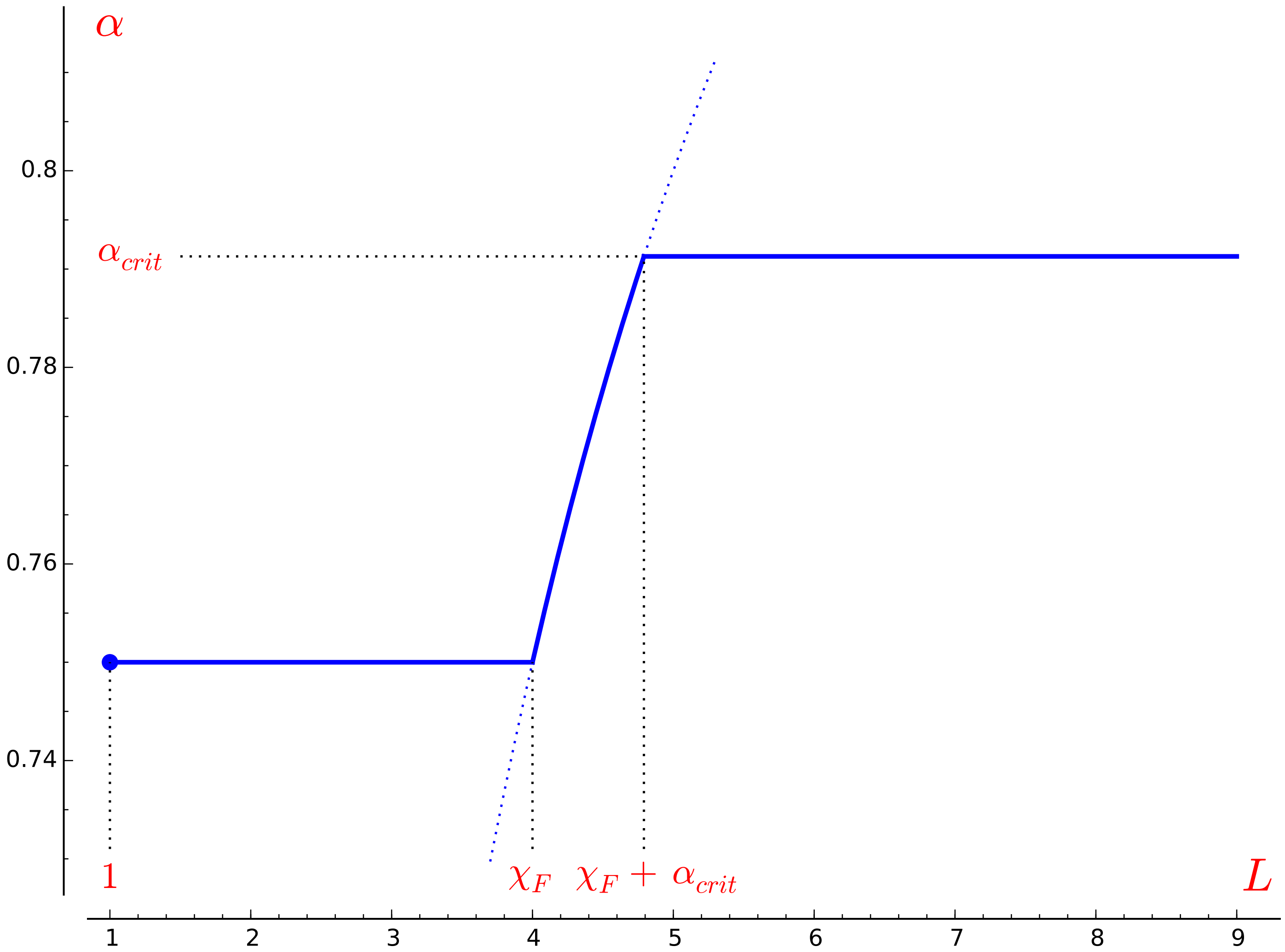


A next quarter of the answer

Theorem (vdH, Li & Müller, 2014+)

- if $D \equiv 0 \bmod 4$, then extension is always possible, provided α is at least :
  - $\dfrac{\chi_F - 1}{\lfloor \frac{1}{4} D \rfloor \, \chi_F}$, if $1 \le L \le \chi_F$
  - $\dfrac{L - 1}{\lfloor \frac{1}{4} D \rfloor \, L}$, if $\chi_F \le L \le \chi_F + \alpha_{\text{crit}}$
  - α_{crit}, if $L \ge \chi_F + \alpha_{\text{crit}}$
- where α_{crit} is given by the first Král' el al. result
- and these bounds are best possible for $\chi_F \in \{2\} \cup [3, \infty)$

The picture for $D = 4$ and $\chi_F = 4$



And the final half of the answer

Theorem (vdH, Li & Müller, 2014+)

- if D is odd, then extension is always possible, provided α is at least :
  - α_{crit}, for any $L \geq 1$

  (i.e.: the bound doesn't depend on L)

- for $\chi_F \in \{2\} \cup [3, \infty)$, the best possible value of α_{crit} is given by the first Král' el al. result

Why all this strange behaviour ?

- although we don't understand all the behaviour in those results
- the strange behaviour for $2 < \chi_F < 3$ can be explained by some aspects of the proof
  - for that we need to have a further look at fractional colouring first

Fractional colouring again

- fractional K-colouring of graph G :
  - assignment of subsets $\phi(v) \subseteq [0, K]$ to $v \in V$ so that :
    - every subset $\phi(v)$ has 'measure' 1
    - and $uv \in E(G) \implies \phi(u) \cap \phi(v) = \varnothing$

Alternative definition for fractional colouring

- notation : $[m]$: the set $\{1, \ldots, m\}$

  $\binom{[m]}{q}$: the collection of q-subsets of $[m]$

- **(m, q)-colouring** of graph G $(1 \le q \le m)$:
  - every $v \in V$ is assigned a subset $\psi(v) \in \binom{[m]}{q}$, so that :
    - $uv \in E(G) \implies \psi(u) \cap \psi(v) = \varnothing$

- and then : $\chi_F(G) = \min\left\{ \dfrac{m}{q} \;\middle|\; G \text{ has an } (m, q)\text{-colouring} \right\}$

And another definition

- **Kneser graph** $\boldsymbol{Kn(m,q)}$:
  - vertices : all of $\binom{[m]}{q}$, edge $uv \iff u \cap v = \varnothing$

- G has an (m,q)-colouring

  $\iff$ there is a homomorphism $G \longrightarrow Kn(m,q)$

- $\chi_F(G) = \dfrac{m}{q} \iff \dfrac{m}{q}$ is minimal, so that

  there is a homomorphism $G \longrightarrow Kn(m,q)$

  - we can interpret this as just a

    labelling of the vertices of G, using labels from $\binom{[m]}{q}$

Fractional colouring and Kneser graphs

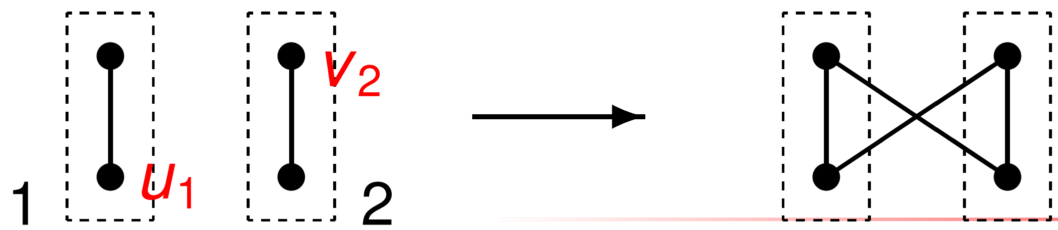
- so to understand fractional colouring,
  we can use Kneser graphs
- but we want to deal with precolouring
  of vertex sets with a minimum distance D
  - for that we need to build more complicated graphs
- in the rest of this talk we only look at the case D is even

Armed Kneser graphs

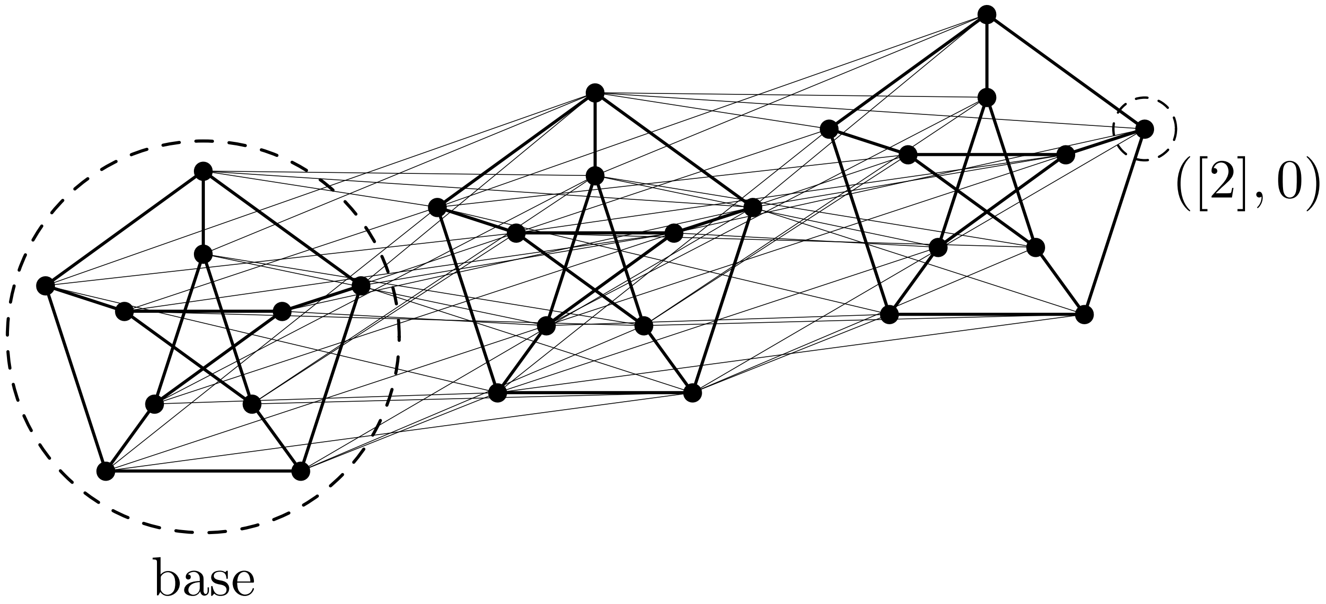
- given some m, q, D even, and an integer M
  - we start with a single $Kn(m, q)$ as a **base**
  - out of the base we grow M disjoint **arms**,

    each consisting of $D/2$ disjoint copies of $Kn(m, q)$
  - we link two consecutive copies of $Kn(m, q)$ in each arm as follows :
    - u_1 in copy 1 and v_2 in copy 2 :

$$u_1 \sim v_2 \iff uv \text{ is an edge in } Kn(m, q)$$



Armed Kneser graphs



one arm of an armed Kneser graph

with $m = 5$, $q = 2$; $D = 4$

Armed Kneser graphs

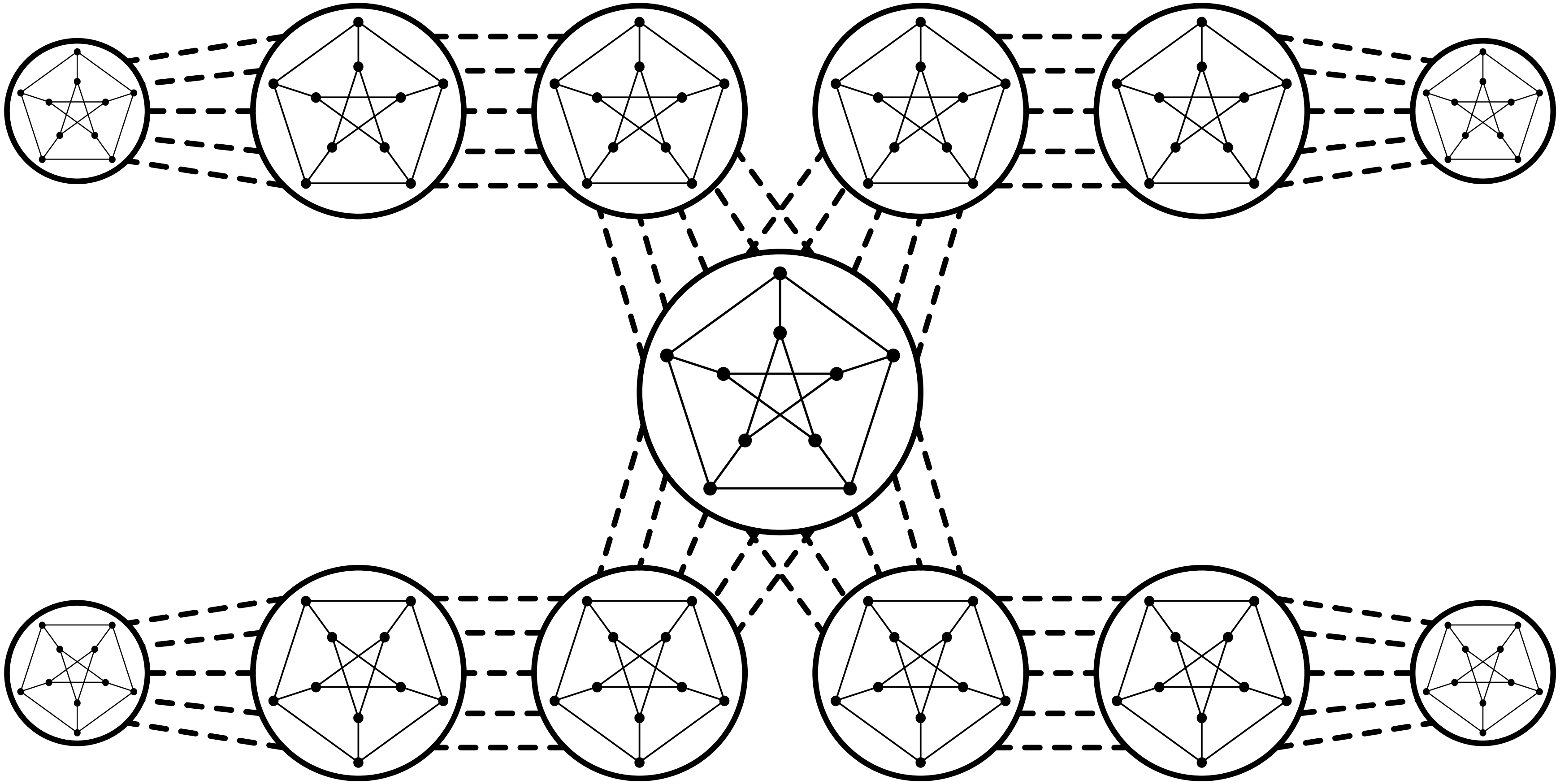
- given some m, q, D even, and an integer M
  - we start with a single $Kn(m, q)$ as a **base**
  - out of the base we grow M disjoint **arms**,

    each consisting of $D/2$ disjoint copies of $Kn(m, q)$
  - we link two consecutive copies of $Kn(m, q)$ in each arm as follows :
    - u_1 in copy 1 and v_2 in copy 2 :

$$u_1 \sim v_2 \quad \Longleftrightarrow \quad uv \text{ is an edge in } Kn(m, q)$$

- call the result the **armed Kneser graph** $\boldsymbol{a\text{-}Kn(m, q; D, M)}$

Armed Kneser graphs



the armed Kneser graph *a-Kn*$(5, 2; 6, 4)$

Using armed Kneser graphs

- - now suppose we have a graph G with $\chi_F(G) = \chi_F$

    so : $\quad G \longrightarrow Kn(m,q)$, for some m, q with $\dfrac{m}{q} = \chi_F$

  - and a set $W \subseteq V(G)$ with $\mathrm{dist}(W) \geq D$

- take the armed Kneser graph $a\text{-}Kn(m,q;D,|W|)$

  with $|W|$ arms

- we will map G to this armed Kneser graph

  - using the labels given by the homomorphism

    $$G \longrightarrow Kn(m,q)$$

Using armed Kneser graphs

- each $w \in W$ gets its own arm
  - map w in the copy of $Kn(m,q)$ at the end of its arm
  - map the neighbours of w in G

    in the copy of $Kn(m,q)$ one step closer to the base
  - map the neighbours of the neighbours of w in G

    in the next copy (closer to the base) of $Kn(m,q)$
  - etc.
- map all vertices at distance at least $D/2$ from W in G

  in the base of the armed Kneser graph

Using armed Kneser graphs

- this mapping of G in the armed Kneser graph satisfies :
  - images of elements of W have distance D
  - a precolouring of W

    gives a precolouring of the images of W
- but also :
  - a fractional colouring of the armed Kneser graph

    can be mapped back to a fractional colouring of G

in other words :

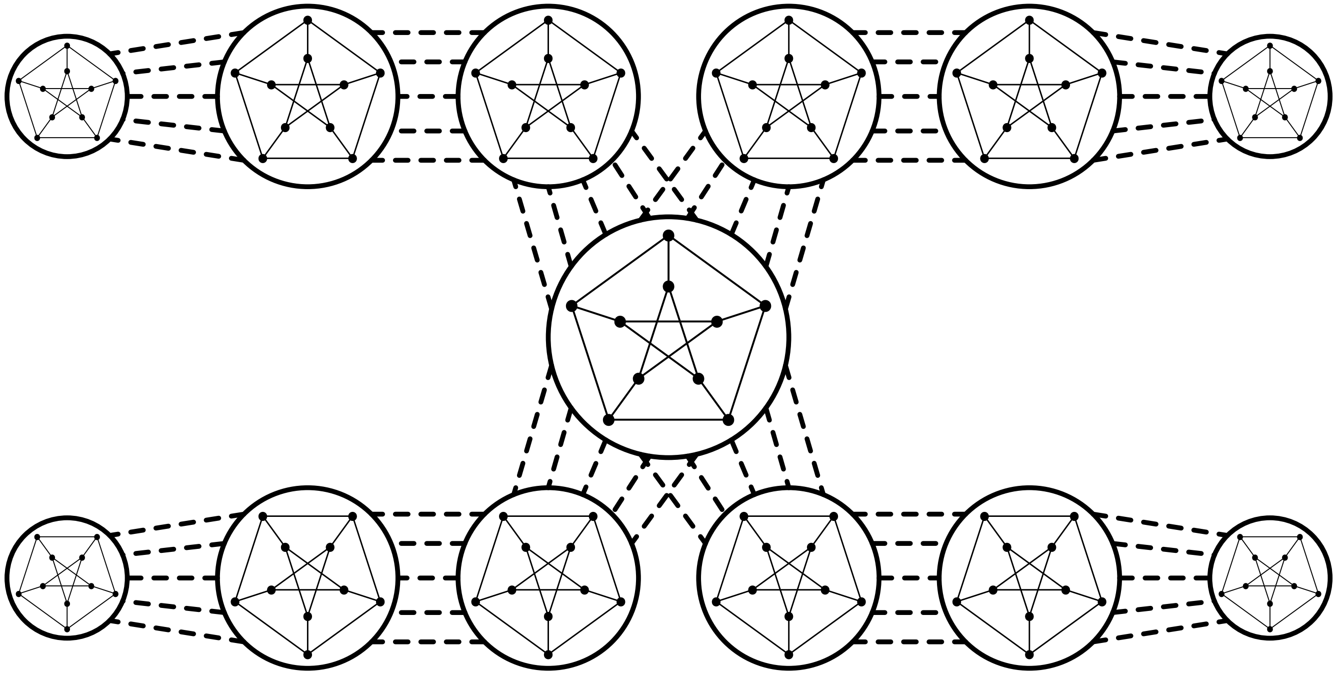
- all aspects of fractional precolouring extensions of graphs are determined by fractional precolouring extensions of armed Kneser graphs !

Precolouring extensions of armed Kneser graphs

- - suppose we have an armed Kneser graph

  $a\text{-}Kn(m, q; D, M)$

  - and one precoloured vertex in the end of each arm



Precolouring extensions of armed Kneser graphs

- - suppose we have an armed Kneser graph

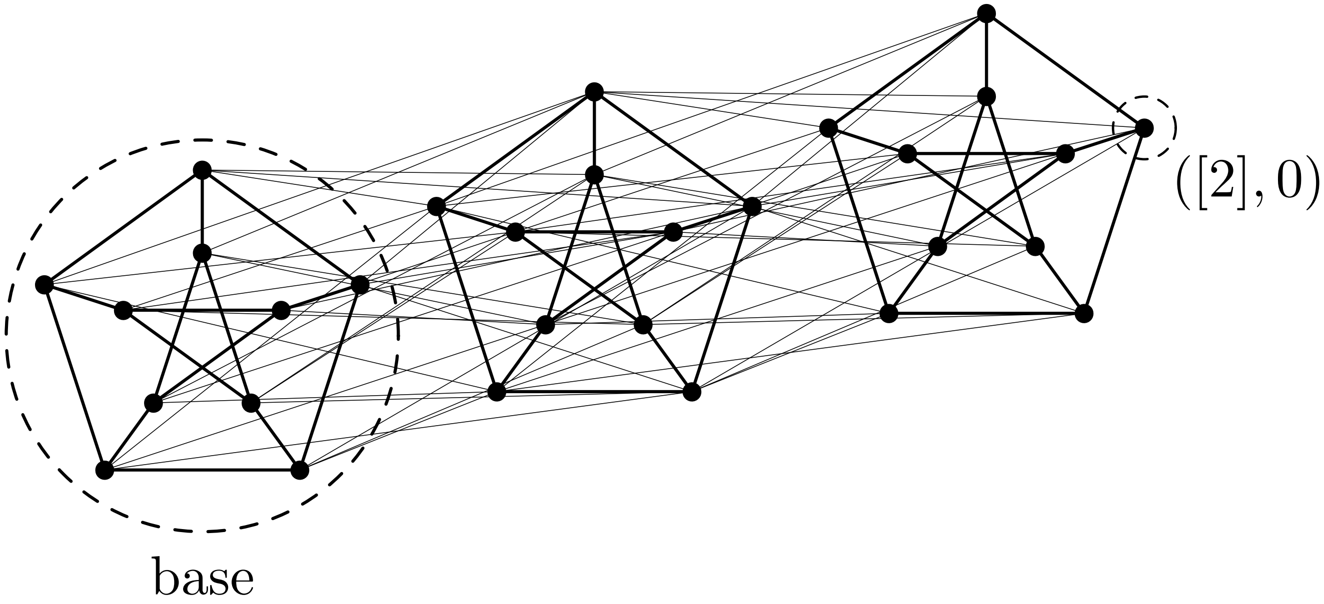
    $a\text{-}Kn(m, q; D, M)$

  - and one precoloured vertex in the end of each arm

- in a fractional colouring extending that precolouring :
  - the base must 'accommodate' all arms
    - so will have to be given some 'average' colouring
  - so along the arms, the colouring extension must connect
    - the precoloured vertex in the end
    - with some 'average' colouring of the base

Colouring along an arm of an armed Kneser graph

- so consider an arm of an armed Kneser graph
  - with one precoloured vertex w' in its end



Colouring along an arm of an armed Kneser graph

- so consider an arm of an armed Kneser graph
  - with one precoloured vertex w' in its end
- the colouring along the arm is mostly determined by :
  - w' itself, in the end copy of $Kn(m,q)$
  - then by vertices in the 2nd copy of $Kn(m,q)$ that are neighbours of w'
  - and then by vertices in the 3rd copy of $Kn(m,q)$ that are neighbours of neighbours of w'
  - etc.

Now things get interesting

- w' is a vertex in $Kn(m,q)$, i.e., a q-subset of $[m]$
- its neighbours are the q-subsets of $[m]$ disjoint from w'
  - those neighbours together form a subgraph that is isomorphic to the Kneser graph $Kn(m-q,q)$
- for $\chi_F = \dfrac{m}{q} \geq 3$ we have
$$\chi_F\big(Kn(m-q,q)\big) = \frac{m-q}{q} = \chi_F - 1$$
- for $2 \leq \chi_F = \dfrac{m}{q} < 3$, $Kn(m-q,q)$ has just isolated vertices
  - hence in those cases : $\chi_F\big(Kn(m-q,q)\big) = 1$

Now things get interesting

- so for $\chi_F \geq 3$ we have

$$\chi_F(\text{set of neighbours of } w') = \chi_F - 1$$

- while for $2 \leq \chi_F < 3$ we have

$$\chi_F(\text{set of neighbours of } w') = 1$$

- this causes the difference between the two cases when $D = 4$

  (then the armed Kneser graph has arms of length 2, with the set of neighbours of w' in the middle)

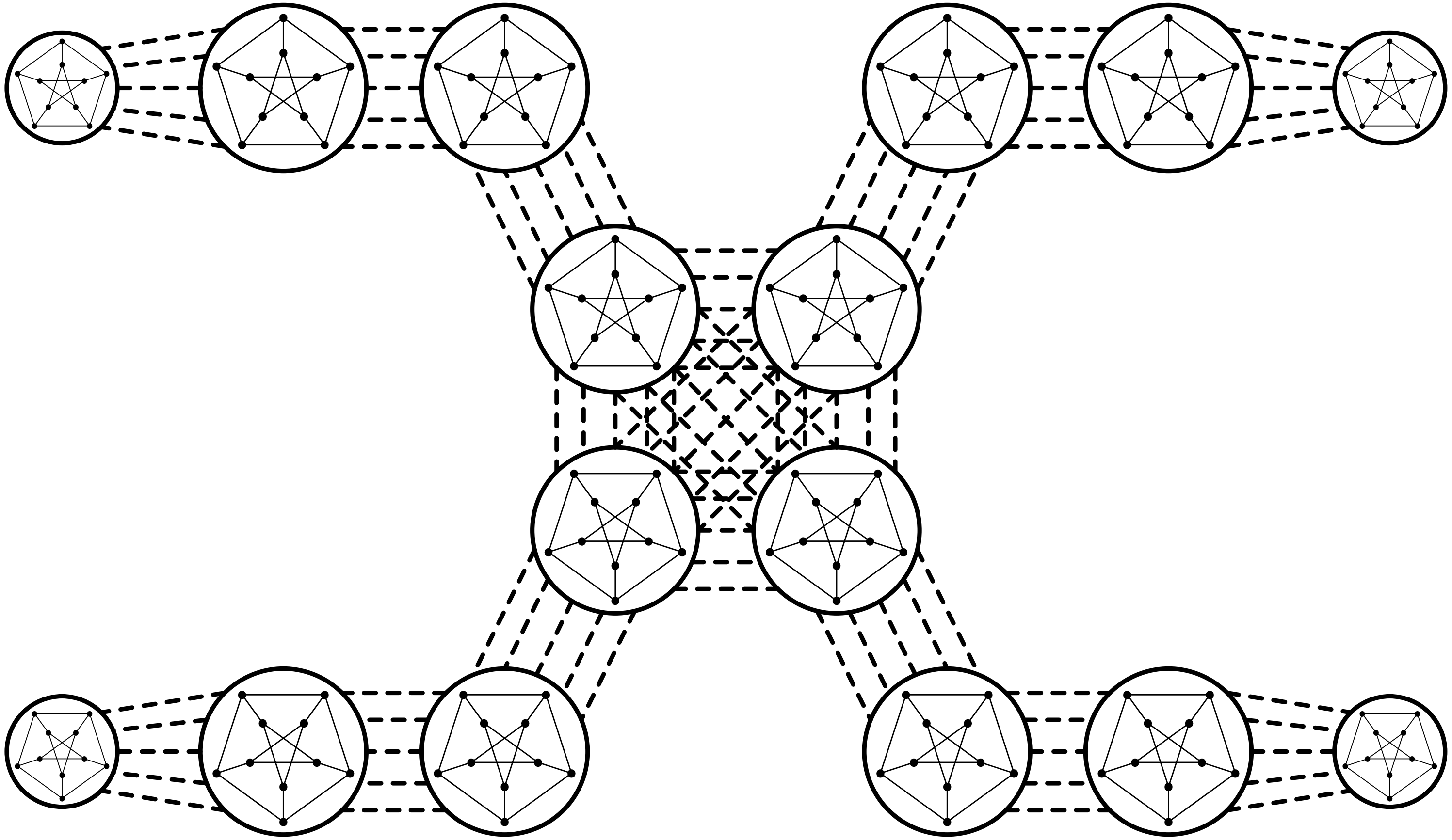
And things get even more interesting

- next consider the set of neighbours of neighbours of w'
- for $\chi_F = \dfrac{m}{q} \geq 3$, this is the whole Kneser graph $Kn(m, q)$
- for $2 \leq \chi_F = \dfrac{m}{q} < 2\frac{1}{2}$,
  this is again a collection of isolated vertices
- but for $2\frac{1}{2} \leq \chi_F = \dfrac{m}{q} < 3$, it gets complicated
  - the structure is not a Kneser graph
  - its structure can vary even in cases where $\dfrac{m}{q} = \dfrac{m'}{q'}$

To summarise these findings

- when colouring along an arm of an an armed Kneser graph :
  - for $\chi_F = \frac{m}{q} \geq 3$, we are always dealing with structures that are Kneser graphs itself
  - for $2 \leq \chi_F = \frac{m}{q} < 3$, we encounter structures that are not Kneser graphs
- we just seem to lack an understanding of the internal structure of Kneser graphs to deal with those latter cases in general

The end



Thank you for the attention !