The Complexity of Change

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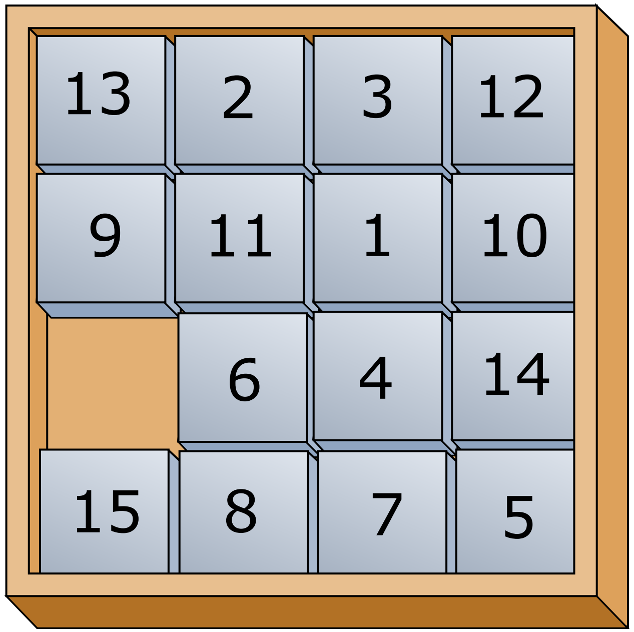
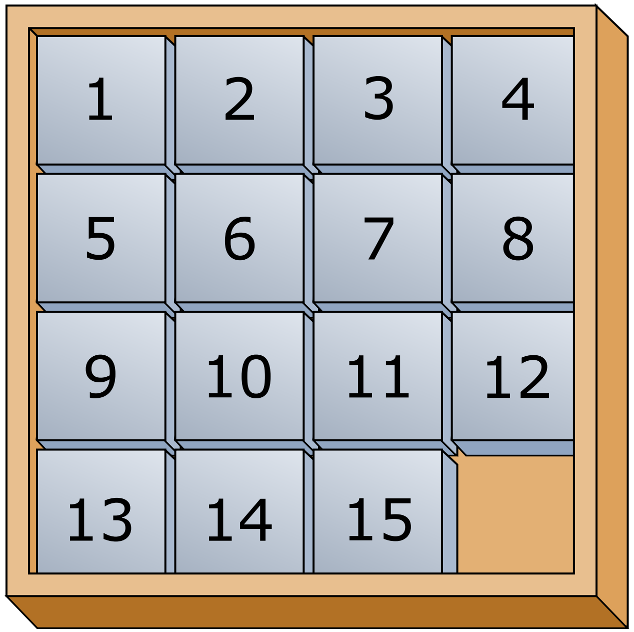
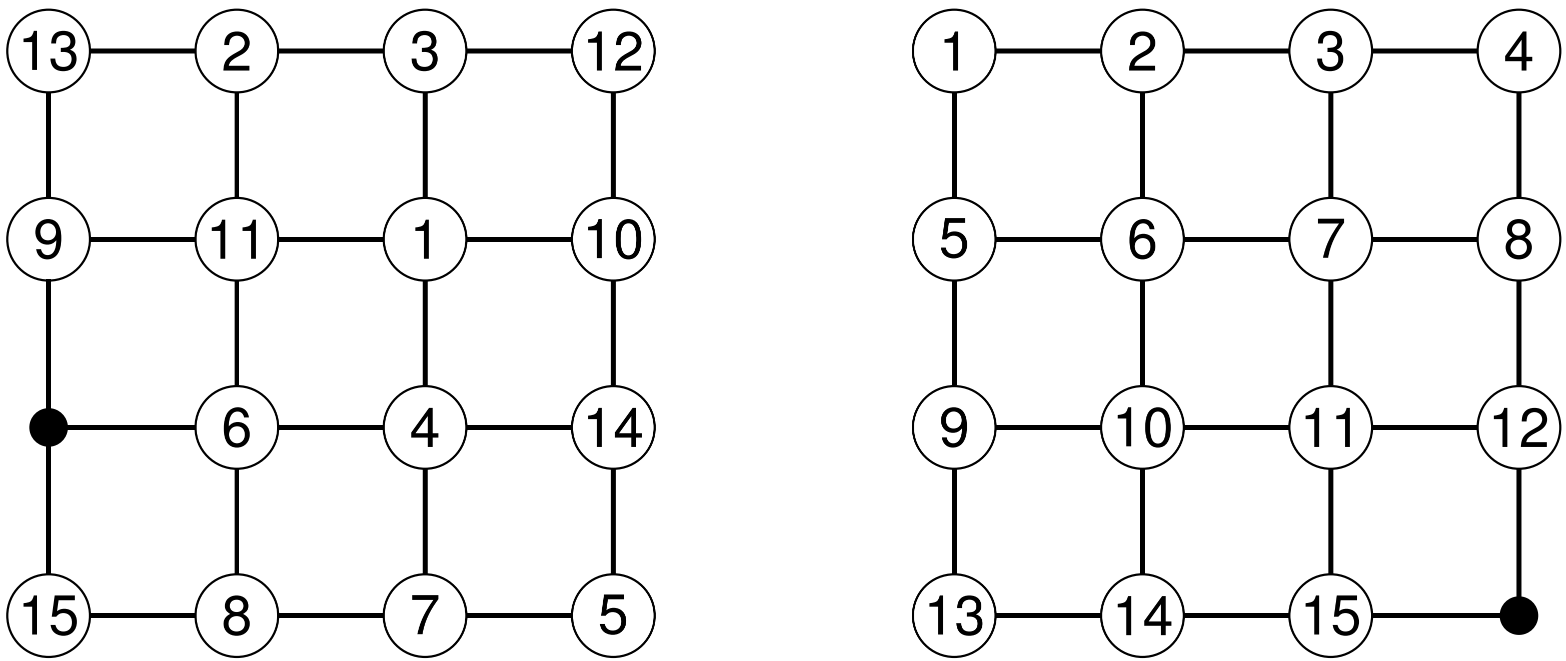
A classical puzzle: the 15-Puzzle





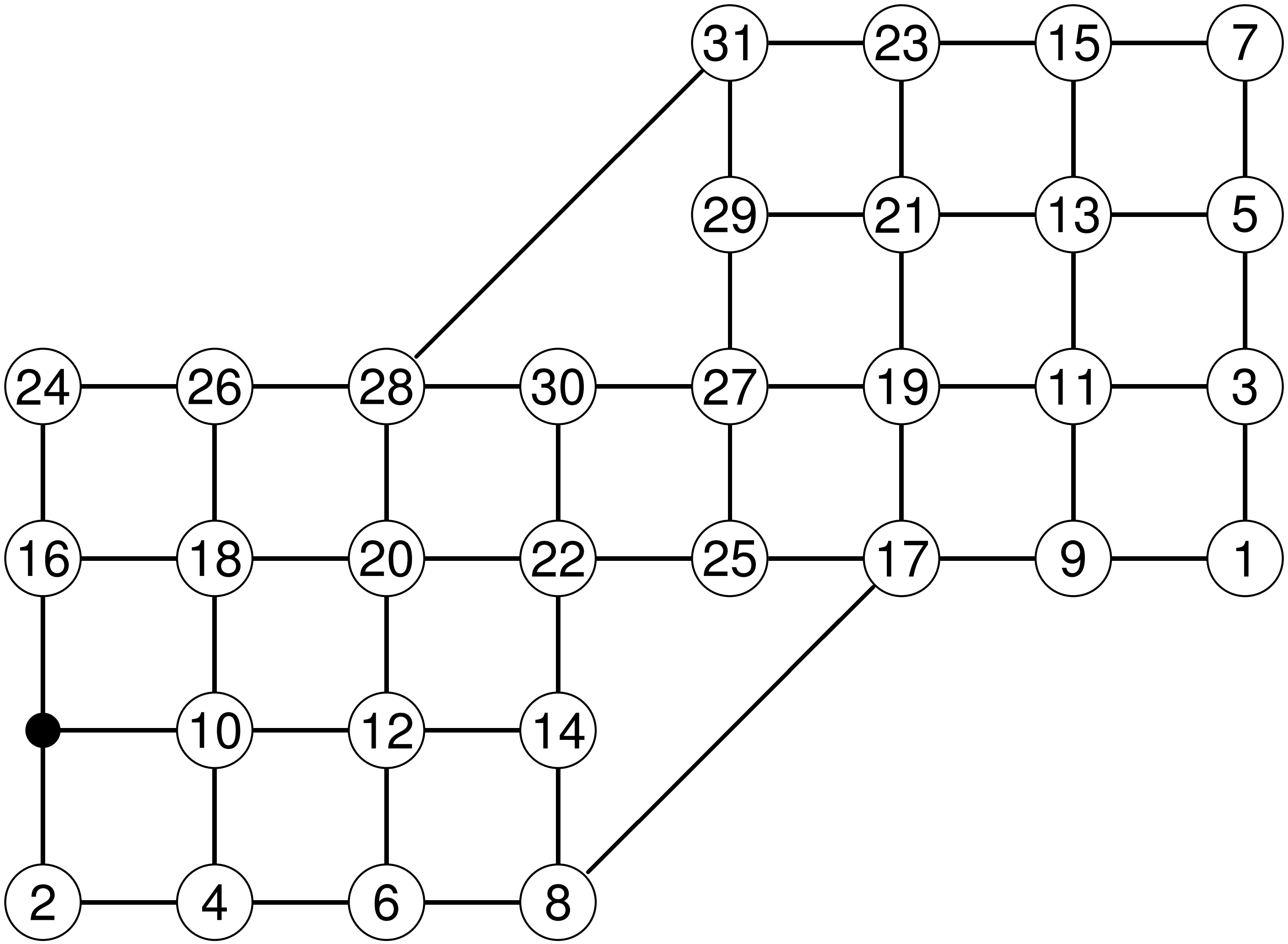
- can you always solve it?

Another way to look at the 15-Puzzle

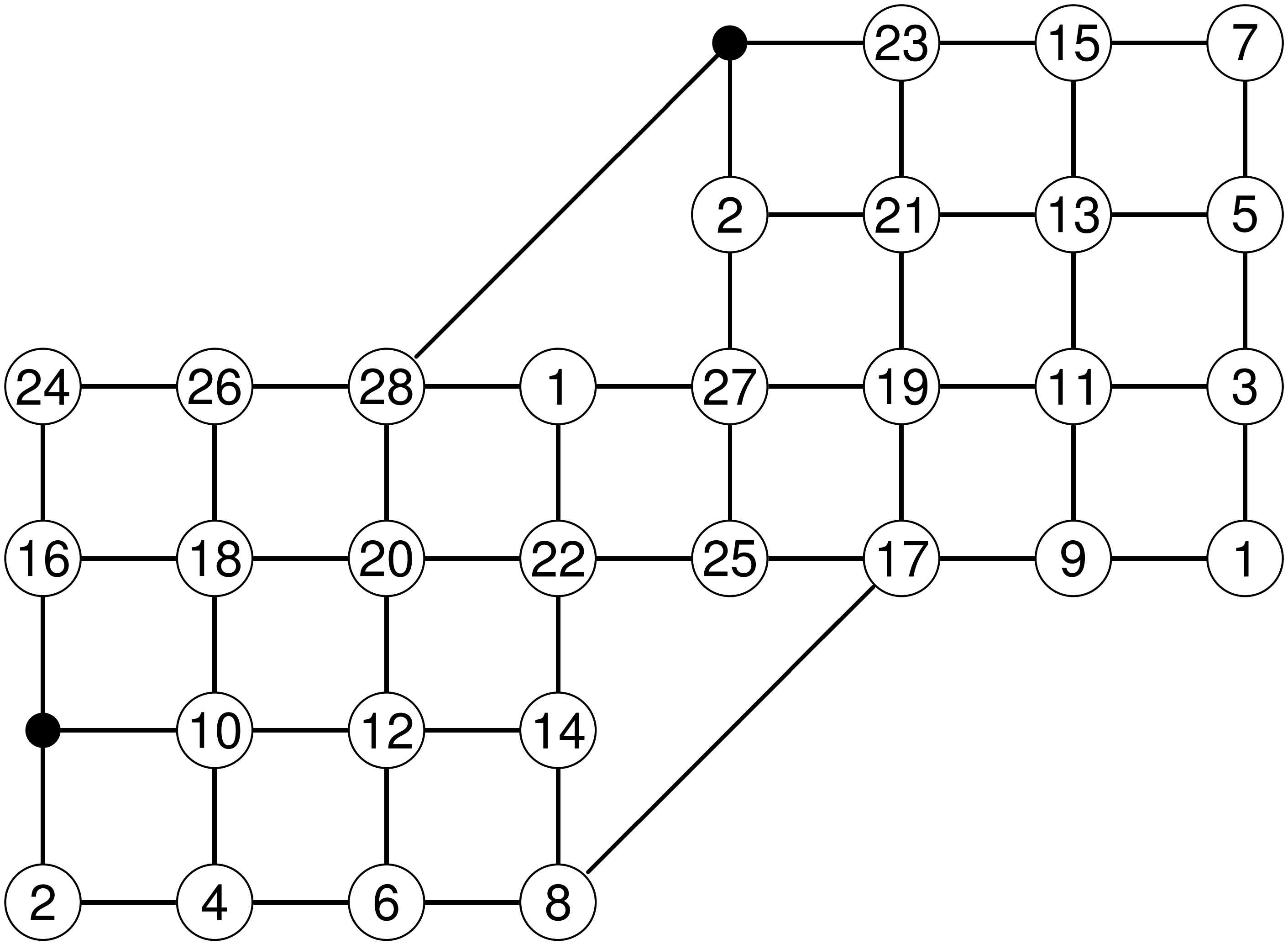


- we slide labelled tokens on some graph
- and want to go from one configuration to another one

What if we would play on a different graph?



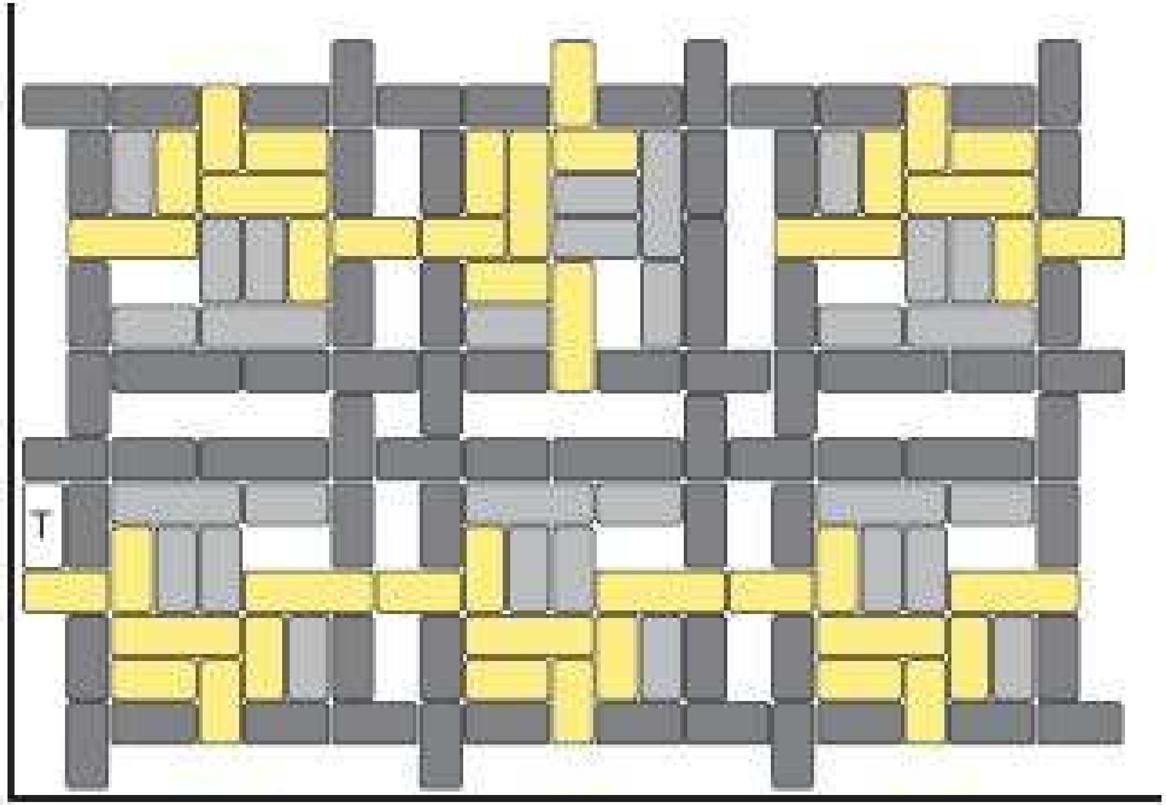
And maybe more empty spaces and/or repeated tokens?



Another moving items game: Rush Hour™

- can you free the red car?

And we can make that more challenging . . .



- can you make any move with car **T**?

Reconfiguration of satisfiability problems

- consider some Boolean formula with n variables
  - e.g.: $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2)$

    whose set of satisfying assignments is

    $\{ (F, F, F), (F, T, F), (F, T, T), (T, F, F), (T, F, T) \}$

    which we write as

    $\{ (0, 0, 0), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1) \}$

Reconfiguration of satisfiability problems

- consider some Boolean formula with n variables
  - e.g.: $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2)$

    whose set of satisfying assignments is

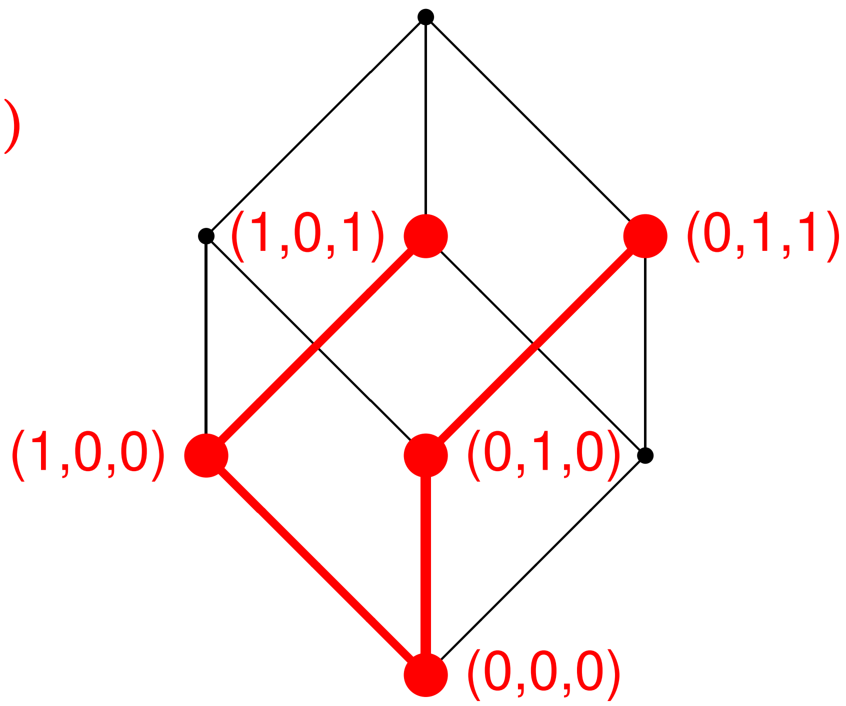
    $\{ (0,0,0),\ (0,1,0),\ (0,1,1),\ (1,0,0),\ (1,0,1) \}$
- the allowed transformation is: change one bit x_i at the time
- natural questions:
  - is the set of all satisfying assignments connected?
  - given two satisfying assignments, can you go from one to the other, changing one bit at the time?

Reconfiguration of satisfiability problems

- for a Boolean formula φ, the set of satisfying assignments is an induced subgraph of the n-dimensional hypercube

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2)$$

corresponds to:

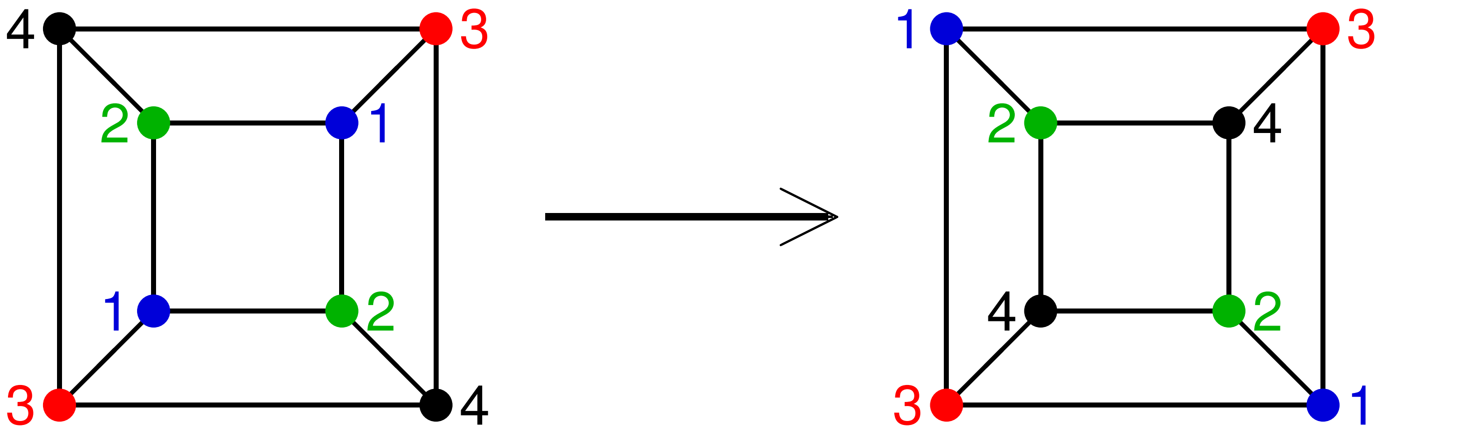


One more example: recolouring planar graphs

- *Input*: a planar graph G,
  and two proper 4-colourings of G

  Question: can we change one 4-colouring to the other one,
  by recolouring one vertex at the time,
  while always maintaining a proper 4-colouring?

- sometimes we can:



One more example: recolouring planar graphs

- *Input*: a planar graph G,
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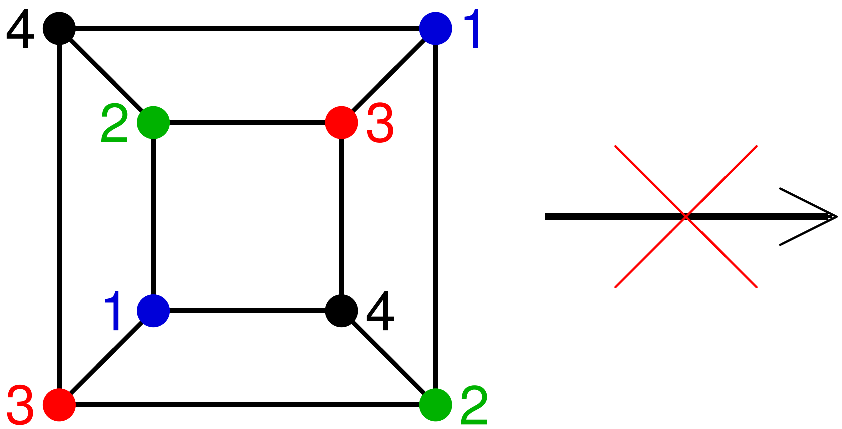
  Question: can we change one 4-colouring to the other one,
  by recolouring one vertex at the time,
  while always maintaining a proper 4-colouring?

- but not always:



Connections

single-vertex recolouring of graph colourings is

- related to work in theoretical physics on

  Glauber dynamics

  of the k-state anti-ferromagnetic Potts model at zero temperature

- related to work in theoretical computer science on

  - Markov chain Monte Carlo methods

    for generating random k-colourings

  - Markov chain Monte Carlo methods

    for approximately counting the number of k-colourings

The Markov chain for k-colourings

define the Markov chain $\mathcal{M}(G;k)$ as follows :

- the states are all k-colourings of G
- transitions from a state (= colouring) α:
  - choose a vertex v uniformly at random
  - choose a colour $c \in \{1,\ldots,k\}$ uniformly at random
  - try to recolour vertex v with colour c
    - if it remains a proper colouring:
      $\Longrightarrow$ make this new k-colouring the new state
    - otherwise: the state remains the same colouring α

A bit of Markov chain theory

- the chain $\mathcal{M}(G;k)$ is aperiodic (since $\text{Prob}(\alpha,\alpha) > 0$)
- the chain is irreducible $\iff$

  all k-colourings are connected via single-vertex recolourings

- hence if all k-colourings are connected:
  - $\mathcal{M}(G;k)$ is ergodic
  - with the unique stationary distribution $\pi \equiv 1/_{\#\ k\text{-colourings}}$
  - **which means**: starting at some k-colouring α,
    walking through the Markov chain long enough,
    the final state can be any k-colouring
    with (almost) equal probability

The main interests for today

- how easy or hard is it to **decide** questions about the connectedness of configurations with certain allowed transformations?
- **in other words**:
  what is the (computational) complexity of these decision problems?

The two kinds of reconfiguration problems

- **A-TO-B-PATH**

  Input: some collection of feasible configurations, some collection of allowed transformations, and two feasible configurations A, B

  Question: can we go from A to B by a sequence of transformations, so that each intermediate configuration is feasible as well?

- **PATH-BETWEEN-ALL-PAIRS**

  Input: some collection of feasible configurations, and some collection of allowed transformations

  Question: is it possible to do the above for any two feasible configurations A, B?

A crash course in complexity theory

- classical complexity theory studies the resources
  - time $=$ number of steps and/or
  - amount of memory

  needed to solve a decision problem for a given input

  in terms of the length of the input (in some encoding)

The complexity classes we need

we say a decision problem is in the class

- **P**: Polynomial-Time
  - if you are clever, you can find the answer in polynomial time

The complexity classes we need

we say a decision problem is in the class

- **P**: Polynomial-Time
- **NP**: Non-Deterministic Polynomial-Time
  - if the answer is "yes" and you are lucky,
    you can discover the "yes" in polynomial time

The complexity classes we need

we say a decision problem is in the class

- **P**: Polynomial-Time
- **NP**: Non-Deterministic Polynomial-Time
- **coNP**: complement of Non-Deterministic Polynomial-Time
  - if the answer is “no” and you are lucky,
    you can discover the “no” in polynomial time

The complexity classes we need

we say a decision problem is in the class

- **P**: Polynomial-Time
- **NP**: Non-Deterministic Polynomial-Time
- **coNP**: complement of Non-Deterministic Polynomial-Time
- **PSPACE**: Polynomial-Space
  - if you are clever, you can find the answer using a polynomial amount of memory

The complexity classes we need

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- **P**: Polynomial-Time
- **NP**: Non-Deterministic Polynomial-Time
- **coNP**: complement of Non-Deterministic Polynomial-Time
- **PSPACE**: Polynomial-Space
- **NPSPACE**: Non-Deterministic Polynomial-Space
  - if the answer is “yes” and you are lucky, you can discover the “yes” using a polynomial amount of memory

The complexity classes we need

- **P**: Polynomial-Time
- **NP**: Non-Deterministic Polynomial-Time
- **coNP**: complement of Non-Deterministic Polynomial-Time
- **PSPACE**: Polynomial-Space
- **NPSPACE**: Non-Deterministic Polynomial-Space
- easy: $\mathbf{P} \subseteq \begin{matrix}\mathbf{NP} \\ \mathbf{coNP}\end{matrix} \subseteq \mathbf{PSPACE} \subseteq \mathbf{NPSPACE}$
- and in fact: $\mathbf{PSPACE} = \mathbf{NPSPACE}$ (Savitch, 1970)

The complexity classes we need

- **P**: Polynomial-Time
- **NP**: Non-Deterministic Polynomial-Time
- **coNP**: complement of Non-Deterministic Polynomial-Time
- **PSPACE**: Polynomial-Space
- **NPSPACE**: Non-Deterministic Polynomial-Space
- finally:
  - a problem is **complete** in a class if it is the “**hardest type**” of problems in that class

How to describe a problem?

- when being given a particular reconfiguration problem, we don't expect to being told an exhaustive list of all feasible configurations and/or an exhaustive list of all related pairs
  - since then the input would be so large that almost any algorithm would be in **P**
- instead we assume we are told:
  - a "description" of all feasible configurations,
  - and a "description" of the allowed transformations

How to describe a problem?

- when being given a particular reconfiguration problem, we don’t expect to being told an exhaustive list of all feasible configurations and/or an exhaustive list of all related pairs
  - since then the input would be so large that almost any algorithm would be in **P**

hence:

- we assume the input is in the form of two algorithms to decide
  - if a possible configuration is feasible,
  - and if a possible transformation is allowed
- and we assume these algorithms give the correct answer in polynomial time

The complexity of all reconfiguration problems

- **under these assumptions**

  A-TO-B-PATH and PATH-BETWEEN-ALL-PAIRS are in **NPSPACE**

  (and hence in **PSPACE**)

- suppose we want to decide if we can go from A to B
  - starting from A, "guess" a next configuration A_1
    - check that A_1 is feasible
    - check that going from A to A_1 is an allowed transformation
  - if A_1 is a valid next configuration,
    "forget" A and replace it by A_1
  - repeat those steps until the target configuration B is reached

Deciding satisfiability problems

- Schaefer (1978) considered “types” of Boolean formulas that can be defined using certain logical relations
- depending on what logical relations are allowed:
  - the decision problem whether or not a Boolean formula is satisfiable is always either in **P** or **NP-complete**

Deciding satisfiability problems

- Schaefer (1978) considered "types" of Boolean formulas that can be defined using certain logical relations

- Gopalan, Kolaitis, Maneva & Papadimitriou (2009) tried to use the same set-up to prove results on:
  - given the type of logical relations allowed
    - what is the complexity of deciding A-TO-B-PATH for two satisfying assignments of some Boolean formula?
    - and what is the complexity of PATH-BETWEEN-ALL-PAIRS

      (i.e. when is the set of satisfying assignments a connected subgraph of the hypercube)?

Reconfiguration of satisfiability problems

Theorem (Gopalan, Kolaitis, Maneva & Papadimitriou, 2009)

for Boolean formulas formed from some fixed set of logical relations:

- A-TO-B-PATH for two satisfying assignments of some Boolean formula is either in **P** or **PSPACE-complete**
- for the cases that A-TO-B-PATH is **PSPACE-complete**:
  - PATH-BETWEEN-ALL-PAIRS is also **PSPACE-complete**
- in the cases that A-TO-B-PATH is in **P**:
  - PATH-BETWEEN-ALL-PAIRS can be in **P**, in **coNP**, or **coNP-complete**
  - the boundaries between the classes are far from clear

Reconfiguration of graph colourings

- **k-COLOUR-α-TO-β-PATH**

  Input: a graph G,
  and two k-colourings α and β of G

  Question: can we go from α to β
  by recolouring one vertex at the time,
  always maintaining a proper k-colouring?

- **k-COLOUR-PATH-BETWEEN-ALL-PAIRS**

  Input: a graph G

  Question: can we go between any two k-colourings of G
  in the manner above?

Reconfiguration of graph colourings

Recall

- if $k = 2$, then deciding if a graph is k-colourable is in **P**
  - a 2-colourable graph is also called **bipartite**

- if $k \geq 3$, then deciding if a graph is k-colourable is **NP-complete**
  - this means that if $k \geq 3$,
    for k-COLOUR-PATH-BETWEEN-ALL-PAIRS we already have a problem to check if at least one colouring exists!

Reconfiguration of graph colourings

Recall

- if $k = 2$, then deciding if a graph is k-colourable is in **P**
- if $k \geq 3$, then deciding if a graph is k-colourable is **NP-complete**

Theorem

- if $k = 2, 3$, then k-COLOUR-α-TO-β-PATH is in **P**

  (Cereceda, vdH & Johnson, 2011)
- if $k \geq 4$, then k-COLOUR-α-TO-β-PATH is **PSPACE-complete**

  (Bonsma, Cereceda, 2009)

Reconfiguration of graph colourings

Completely trivial

restricted to bipartite, planar graphs:

- for any $k \geq 2$, deciding if a graph is k-colourable is in **P**

Theorem

restricted to bipartite, planar graphs:

- if $k = 2, 3$, then k-COLOUR-α-TO-β-PATH is in **P**

  (Cereceda, vdH & Johnson, 2011)

- if $k = 4$, then k-COLOUR-α-TO-β-PATH is **PSPACE-complete**

  (Bonsma, Cereceda, 2009)

- if $k \geq 5$, then k-COLOUR-α-TO-β-PATH is in **P** ("print(yes)")

Reconfiguration of graph colourings

Theorem

restricted to bipartite graphs:

- if $k = 2$, then k-COLOUR-PATH-BETWEEN-ALL-PAIRS is in **P**:

  if no edges then print(yes), else print(no)

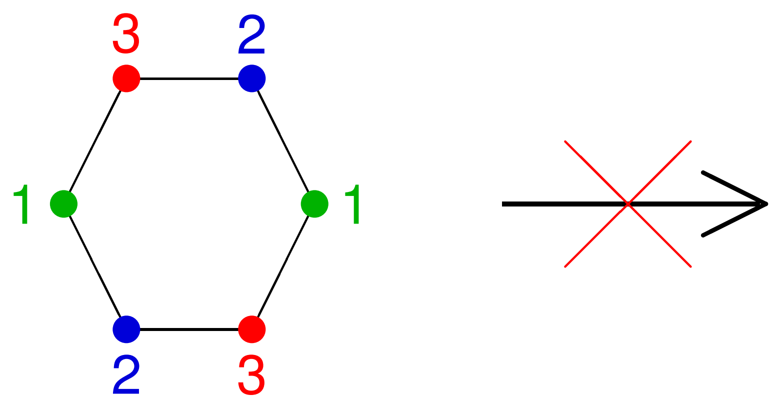
- if $k = 3$,
  then k-COLOUR-PATH-BETWEEN-ALL-PAIRS is **coNP-complete**

  (Cereceda, vdH & Johnson, 2009)

- if $k \geq 4$, then the complexity of
  k-COLOUR-PATH-BETWEEN-ALL-PAIRS is unknown

The case $k = 3$ for bipartite graphs

- the smallest bipartite graph for which not all 3-colourings are connected is the **6-cycle** $\boldsymbol{C_6}$:



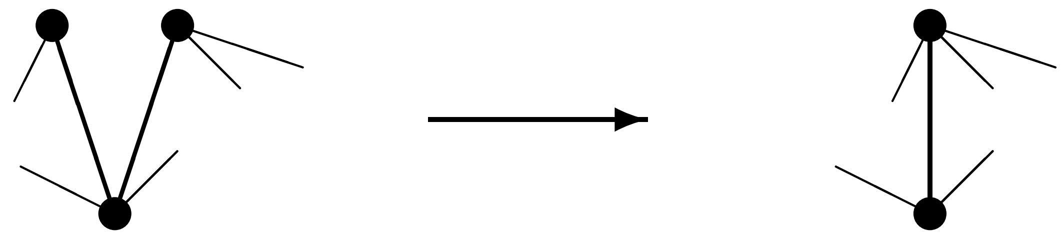
Theorem (Cereceda, vdH & Johnson, 2011)

G is a bipartite graph:

- not all 3-colourings are connected $\iff$ G "contains C_6"

Folding

- **fold** of two vertices at distance 2:



- ***G* foldable to *H***: sequence of folds changes G to H

Theorem (Cook & Evans, 1979)

G a connected graph:

- $\min\{\, k \mid G$ can be coloured with k colours $\}$

$$= \min\{\, k \mid G \text{ is foldable to complete graph } K_k \,\}$$

Folding and 3-colouring

- **fold** of two vertices at distance 2:
- ***G* foldable to *H***: sequence of folds changes G to H

Theorem (Cereceda, vdH & Johnson, 2011)

G a connected, bipartite graph:

- not all 3-colourings are connected $\iff$ G is foldable to C_6
- deciding if G is foldable to C_6 is **NP-complete**

Reconfiguration of graph colourings

Theorem

restricted to bipartite, planar graphs:

- if $k = 2, 3$,
  then k-COLOUR-PATH-BETWEEN-ALL-PAIRS is in **P**

  (Cereceda, vdH & Johnson, 2009)

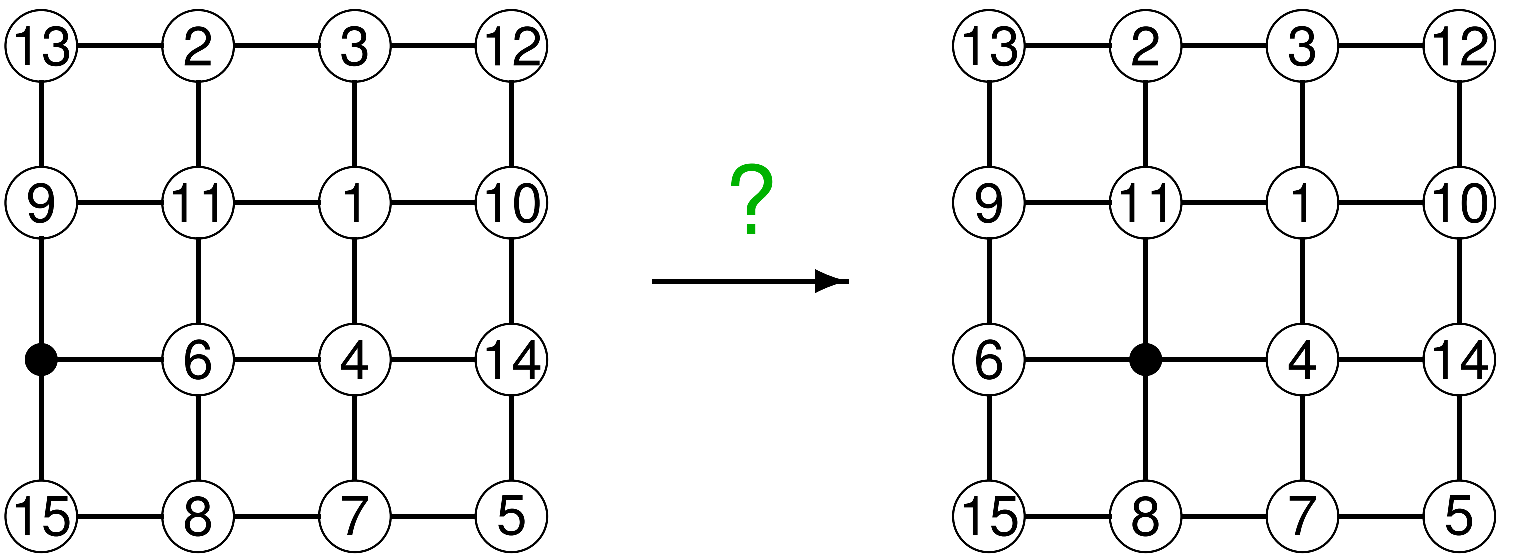
- if $k = 4$, then the complexity of
  k-COLOUR-PATH-BETWEEN-ALL-PAIRS is unknown

- if $k \geq 5$,
  then k-COLOUR-PATH-BETWEEN-ALL-PAIRS is in **P**:

  "print(yes)"

Sliding token puzzles

- as seen already, we can interpret the 15-puzzle as a problem involving moving tokens on a given graph:



Sliding token puzzles

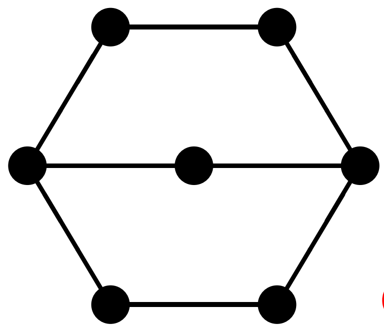
- so what happens if we would play this on other graphs?
- for a given graph G on n vertices,

  define **puz(*G*)** as the graph that has:
  - nodes: all possible placements of $n - 1$ tokens on G
  - adjacency: sliding one token along an edge of G to an empty vertex
- and our standard decision problems become:
  - are two token configurations in one component of puz(G)?
  - is puz(G) connected?

Sliding token puzzles

Theorem (Wilson, 1974)

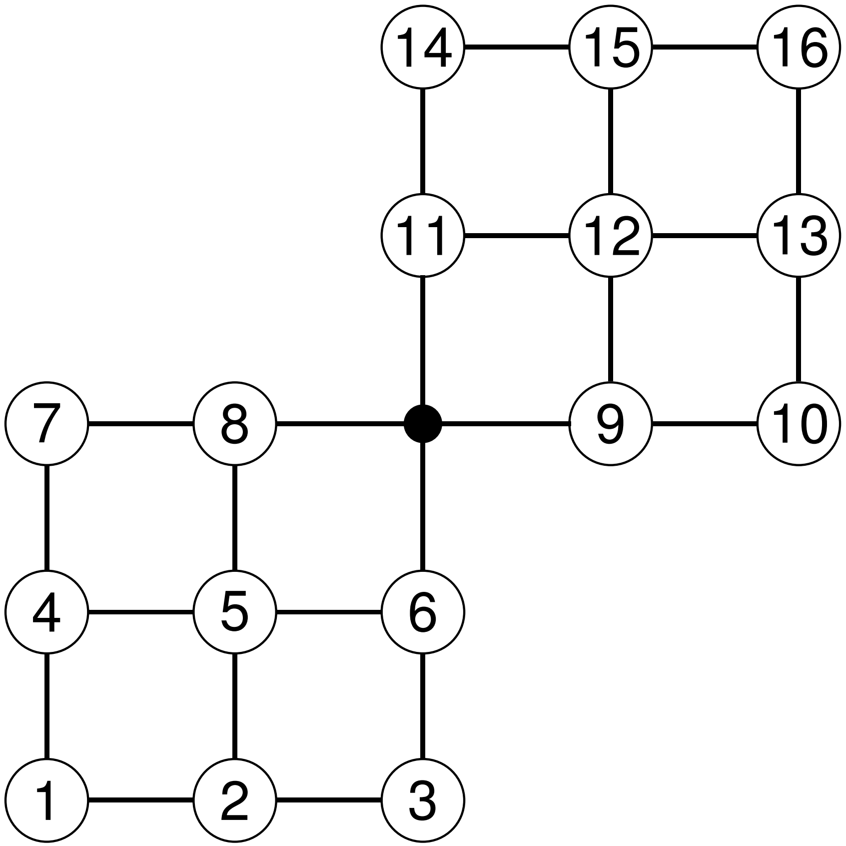
- if G is a 2-connected graph, then puz(G) is connected, except if:
  - G is a cycle on $n \geq 4$ vertices
    (then puz(G) has $(n-2)!$ components)
  - G is bipartite different from a cycle
    (then puz(G) has 2 components)
  - G is the exceptional graph Θ_0 (puz(Θ_0) has 6 components)



Θ_0

Why does Wilson only consider 2-connected graphs?

- since puz(G) is never connected if G has connectivity below 2:



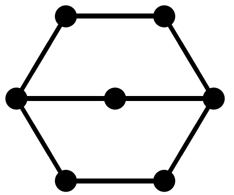
Generalised sliding token puzzles

- **what would happen if**:
  - we have fewer than $n - 1$ tokens (i.e. more empty vertices)?
  - and/or not all tokens are the same?
- so suppose we have a set $(k_1, k_2, \ldots, k_p)$ of labelled tokens
  - meaning: k_1 tokens with label 1, k_2 tokens with label 2, etc.
  - tokens with the same label are indistinguishable
  - we can assume that $k_1 \geq k_2 \geq \cdots \geq k_p$
    and their sum is at most $n - 1$
- the corresponding graph of all token configurations on G is denoted by $\mathbf{puz}(G; k_1, \ldots, k_p)$

Generalised sliding token puzzles

Theorem (Brightwell, vdH & Trakultraipruk, 2013)

- G a graph on n vertices, $(k_1, k_2, \ldots, k_p)$ a token set, then $\mathrm{puz}(G; k_1, \ldots, k_p)$ is connected, except if:
  - G is not connected
  - G is a path and $p \geq 2$
  - G is a cycle, and $p \geq 3$, or $p = 2$ and $k_2 \geq 2$
  - G is a 2-connected, bipartite graph with token set $(1^{(n-1)})$
  - G is the exceptional graph Θ_0 with token set $(2, 2, 2)$, $(2, 2, 1, 1)$, $(2, 1, 1, 1, 1)$ or $(1, 1, 1, 1, 1, 1)$



Generalised sliding token puzzles

Theorem (Brightwell, vdH & Trakultraipruk, 2013)

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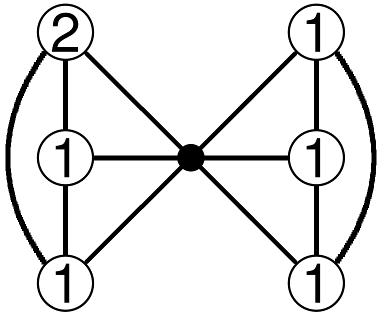
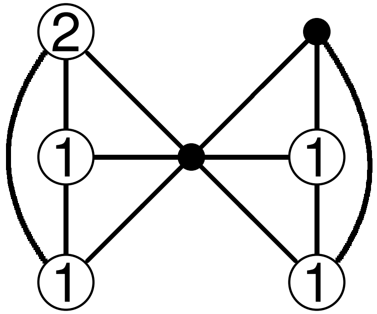
  then $\mathrm{puz}(G; k_1, \ldots, k_p)$ is connected, except if:

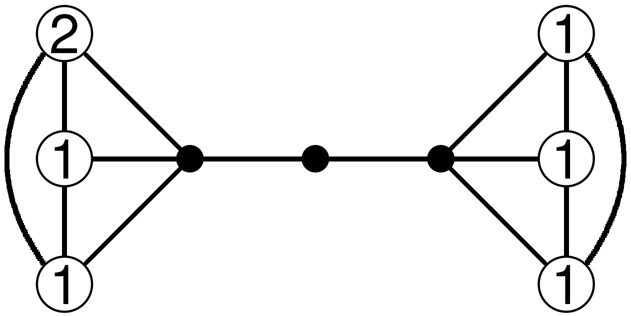
  - G is not connected
  - G is a path and $p \geq 2$
  - G is a cycle, and $p \geq 3$, or $p = 2$ and $k_2 \geq 2$
  - G is a 2-connected, bipartite graph with token set $(1^{(n-1)})$
  - G is the exceptional graph Θ_0 with some “bad” token sets
  - G has connectivity 1, $p \geq 2$ and there is a “separating path preventing tokens from moving between blocks”

Generalised sliding token puzzles

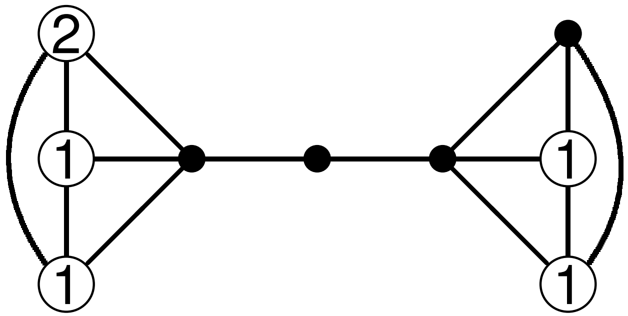
- “separating paths” in graphs of connectivity one:







bad



good

Generalised sliding token puzzles

- we can also characterise:
  - given a graph G, token set $(k_1, \ldots, k_p)$, and two token configurations on G,
  - are the two configurations in the same component of $\mathrm{puz}(G; k_1, \ldots, k_p)$?
- so recognising connectivity properties of $\mathrm{puz}(G; k_1, \ldots, k_p)$ is easy
- so can we say something about the number of steps we would need?

The length of sliding token paths

- **SHORTEST-A-TO-B-TOKEN-MOVES**

  Input: a graph G, a token set $(k_1, \ldots, k_p)$,
  two token configurations A and B on G,
  and a positive integer N

  Question: can we go from A to B in at most N steps?

The length of sliding token paths

Theorem (Goldreich, 1984-2011)

- restricted to the case that there are $n-1$ different tokens,

  SHORTEST-A-TO-B-TOKEN-MOVES is **NP-complete**

Theorem (vdH & Trakultraipruk, 2013; probably others earlier)

- restricted to the case that all tokens are the same,

  SHORTEST-A-TO-B-TOKEN-MOVES is in **P**

Theorem (vdH & Trakultraipruk, 2013)

- restricted to the case that there is just one special token and all others are the same:

  SHORTEST-A-TO-B-TOKEN-MOVES is already **NP-complete**

Robot motion

- the proof of that last result uses ideas of the proof of

Theorem (Papadimitriou, Raghavan, Sudan & Tamaki, 1994)

- SHORTEST-ROBOT-MOTION-WITH-ONE-ROBOT is **NP-complete**

- **Robot Motion** problems on graphs are sliding token problems,
  - with some special tokens (the robots)
    - that have to end in specified positions
  - all other tokens are just obstacles
    - and it is not important where those are at the end

A final puzzle: Rush Hour™

- **RUSH-HOUR**

  Input: some rectangular board,
  a configuration of cars on that board,
  and one special car

  Question: is it possible to get the special car moving?

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  Input: some rectangular board,
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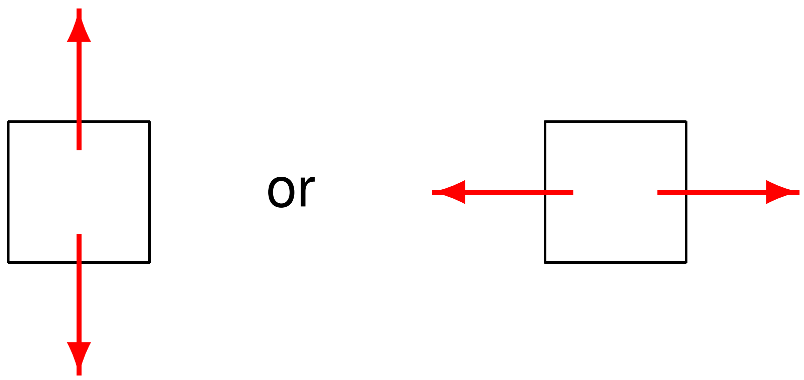
Theorem

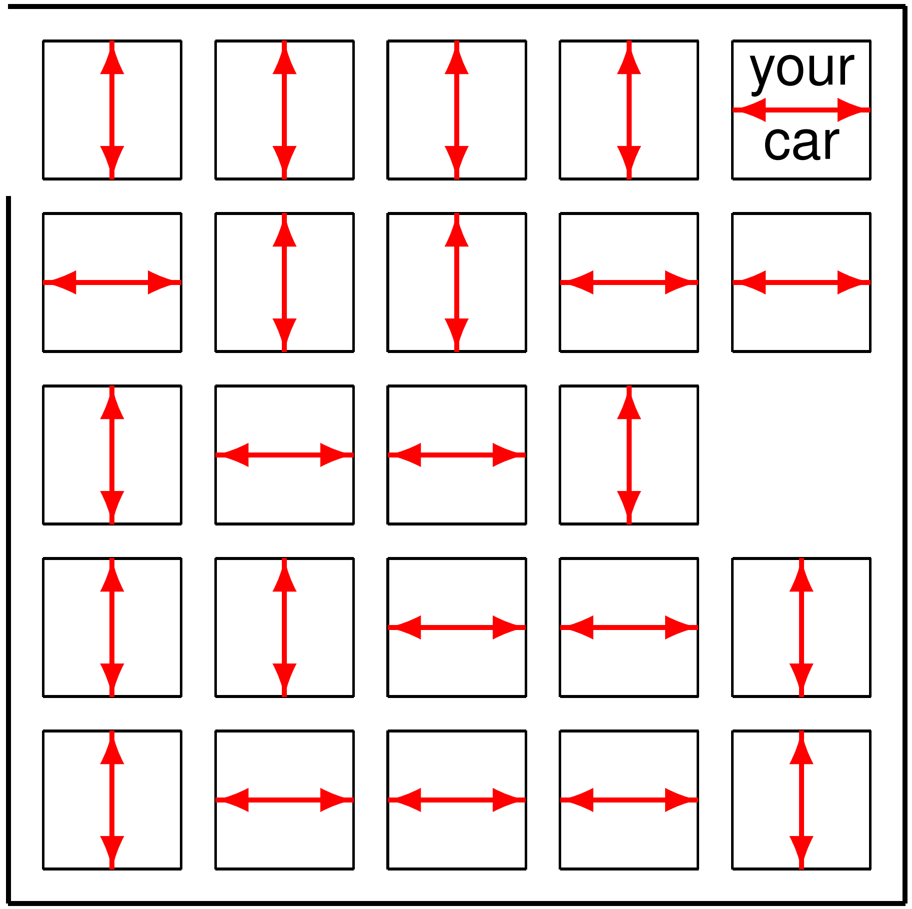
- RUSH-HOUR is **PSPACE-complete** (Flake & Baum, 2002)
- RUSH-HOUR remains **PSPACE-complete** even if all cars have length two (Tromp & Cilibrasi, 2005)

A final puzzle: Rush Hour™

- what is the complexity if all cars have length one?
  - i.e. each car is a 1×1 block,
    but can move in only one direction



Can you move your city car out of the garage?



How to prove a decision problem is PSPACE-complete?

standard method:

- reduction to the basic PSPACE-complete problem:

  QUANTIFIED-SAT:

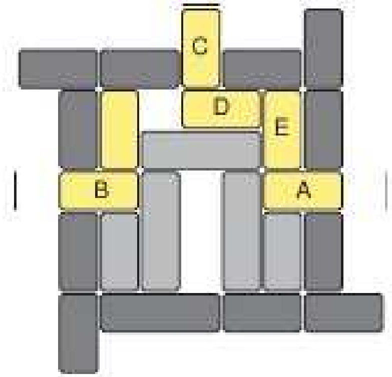
  $$\forall x_1\, \exists y_1\, \forall x_2\, \exists y_2\, \cdots\, \forall x_n\, \exists y_n\, \varphi(x_1, y_1, \ldots, x_n, y_n)$$

  for some Boolean formula $\varphi(x_1, y_1, \ldots, x_n, y_n)$

- Hearn & Demaine (2005) developed an approach that is often much easier to use
  - main idea: show that a QUANTIFIED-SAT formula can be represented by certain logical circuits

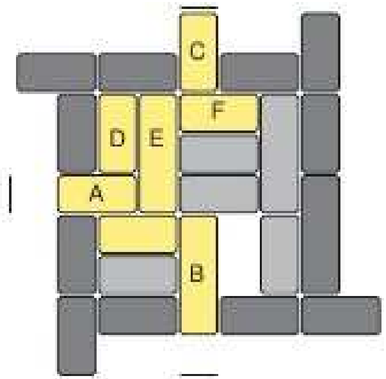
Logical Gates in Rush Hour

- an OR-like collection of cars:



C can only move in, if at least one of **A**, **B** moves out

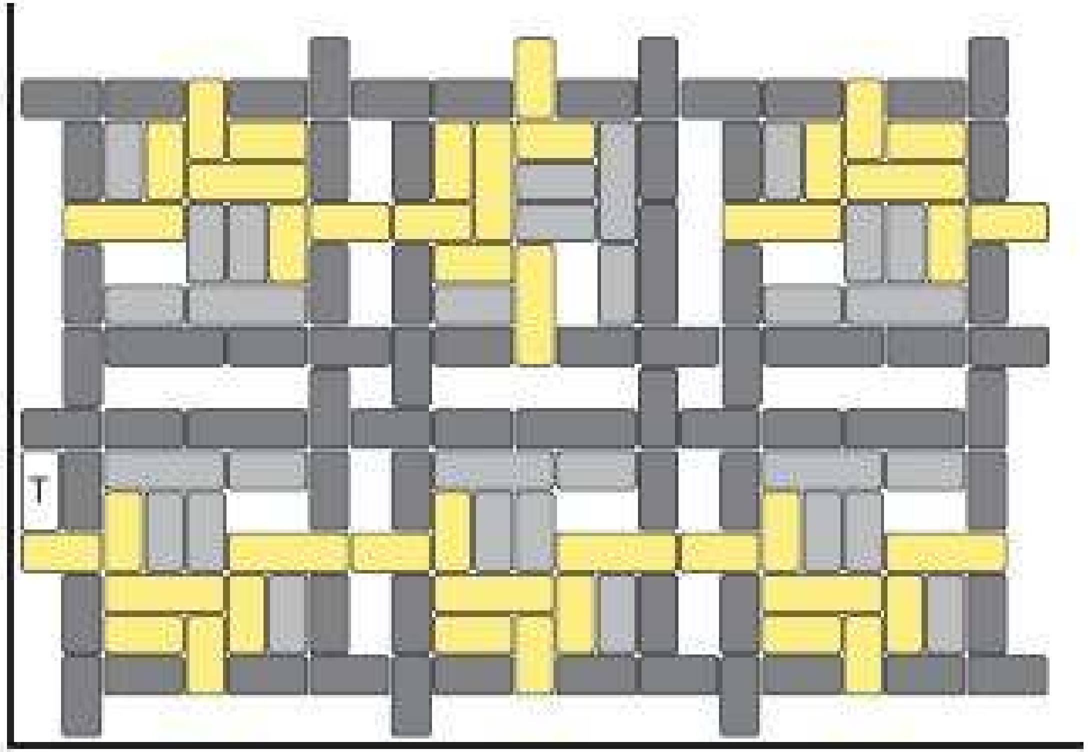
- an AND-like collection of cars:



C can only move in, if both **A** and **B** move out

Logical Gates in Rush Hour

- and then combine it all in big tableaux:



But what to do with small cars?