Generalised Colouring Numbers of Graphs

applications, bounds, structural aspects, algorithms

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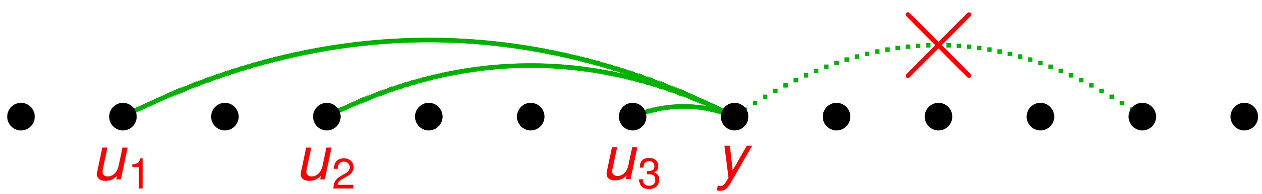
The normal colouring number

- - let L be a linear ordering of the vertices of a graph G
  - for a vertex $y \in V(G)$,
    let $S(L, y)$ be the set of neighbours u of y with $u <_L y$



- then the **colouring number col(G)** is defined as

$$\mathrm{col}(G) = \min_{L} \max_{y \in V(G)} |S(L, y)| + 1$$

- greedily colouring from left to right gives: $\chi(G) \leq \mathrm{col}(G)$

Bounding the colouring number

- for a planar graph G with n vertices and m edges we know

$$m \leq 3n - 6$$

- hence:
  - a planar graph G has a vertex of degree at most 5
  - there is an ordering L of the vertices of G such that

$$\text{for all vertices } y\text{:} \quad |S(L, y)| \leq 5$$

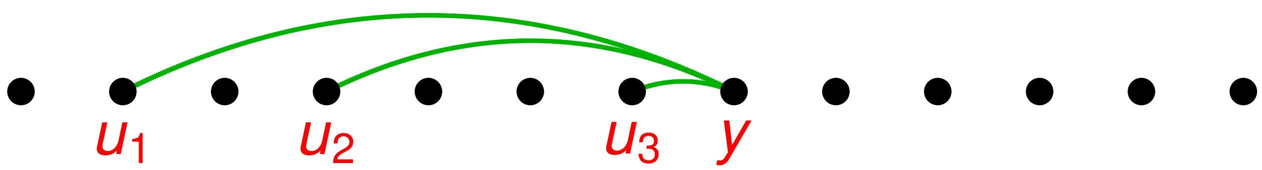
- **Corollary**

$$G \text{ planar} \implies \mathrm{col}(G) \leq 6$$

Generalising the colouring number

- the set $S(L,y)$ can be defined as

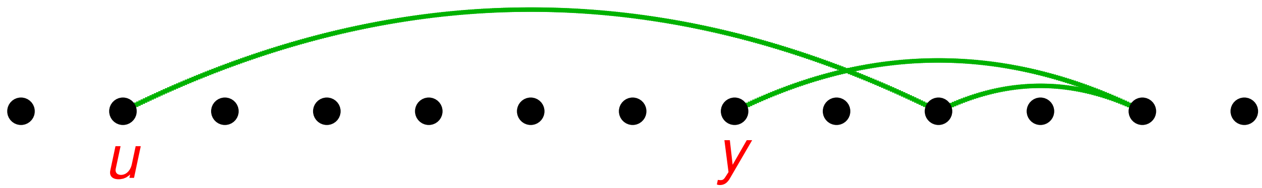
  "vertices $u <_L y$ for which there is an uy-path of length 1"



- what would happen if we allow longer paths ?

Generalising the colouring number

- a **strong *uy*-path** has all internal vertices larger than y



- let $\boldsymbol{S_r(L, y)}$ be the set of vertices $u <_L y$ for which there exists a strong uy-path with length at most r

- then the **strong *r*-colouring number $\mathrm{col}_r(G)$** is defined as

$$\mathrm{col}_r(G) = \min_L \max_{y \in V(G)} |S_r(L, y)| + 1$$

A simple application

- **acyclic chromatic number** $\chi_a(G)$:
  minimum number of colours needed to properly colour the vertices of G such that every cycle has at least 3 colours

- **Theorem** (Kierstead & Yang, 2004)

$$\chi_a(G) \;\leq\; \mathrm{col}_2(G)$$

Bounding the colouring numbers

- for a planar graph G with n vertices and m edges we know

$$m \leq 3n - 6$$

- this is enough to prove:
  - there is an ordering L of the vertices of G such that

    for all $y \in V(G)$: $|S_1(L,y)| \leq 5$ and $|S_2(L,y)| \leq 14$

- **Corollary**

  G planar $\implies \mathrm{col}_2(G) \leq 15$

- best possible bound: $\mathrm{col}_2(G) \leq 8$

  (Dvořák, Kabela & Kaiser, 2015)

A structural application

- classes of graphs $\mathcal{G}$ with **bounded expansion** were introduced by Nešetřil & Ossona de Mendez in terms of "densities of shallow minors"

- **equivalent Definition** (Zhu, 2009)

  a class of graphs $\mathcal{G}$ has **bounded expansion**:

  there exists constants $c_1, c_2, \ldots$ such that

  for every $G \in \mathcal{G}$ and every r we have $\mathrm{col}_r(G) \le c_r$

Further structural aspects

- we obviously have:

  $$\mathrm{col}_1(G) \le \mathrm{col}_2(G) \le \mathrm{col}_3(G) \le \ldots \le \mathrm{col}_\infty(G)$$

  (where the "∞" means "any length strong path allowed")

- **Property** (Grohe et al., 2014)

  $$\mathrm{col}_\infty(G) \;=\; \text{tree-width}(G) + 1$$

- so the concept "class of graphs with bounded expansion" generalises "class of graphs with bounded tree-width"

  (and "graphs with bounded genus", "graphs with forbidden minors", "graphs with bounded cop number", etc., etc.)

An algorithmic aspect

- **Theorem** (Courcelle, 1990)

  let $\mathcal{G}$ be a class of graphs with bounded tree-width

  - then any graph property that can be described using monadic second-order logic is decidable for $G \in \mathcal{G}$ in linear time

- **Theorem** (Dvořák, Král' & Thomas, 2010)

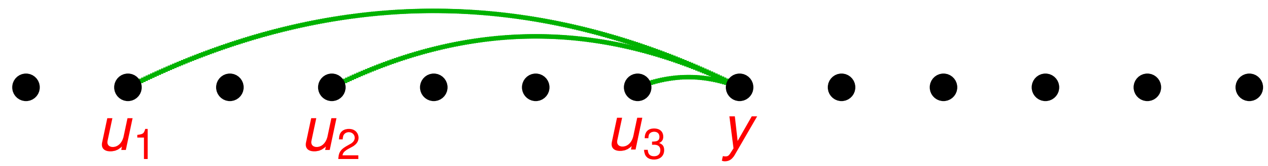
  let $\mathcal{G}$ be a class of graphs with bounded expansion

  - then any graph property that can be described using first-order logic is decidable for $G \in \mathcal{G}$ in linear time

Back to generalising the colouring number

- - the set $S(L,y)$ can be defined as

    "vertices $u <_L y$ for which there is an uy-path of length 1"

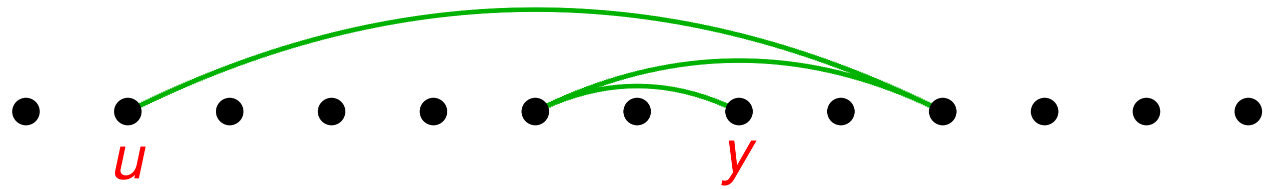


  - what would happen if we allow longer paths ?

- but several choices for such paths are possible
  - strong paths (leading to $\mathrm{col}_r(G)$) is just one of them

The weak colouring number

- - a **weak $u y$-path** has all internal vertices larger than u



  - let $W_r(L,y)$ be the set of vertices $u <_L y$ for which there exists a weak uy-path of length at most r

- then the **weak r-colouring number** $\mathrm{col}_r(G)$ is defined as

$$\mathrm{wcol}_r(G) \;=\; \min_L \max_{y \in V(G)} |W_r(L,y)| + 1$$

Yet another generalisation: admissibility

- - we use strong uy-paths again, so all internal vertices larger than y
  - let $\boldsymbol{a_r(L,y)}$ be the maximal size of a set of vertices $u <_L y$ for which there exists strong uy-paths of length at most r, that are disjoint apart from all starting at y
- then the ***r*-admissibility** $\mathbf{adm}_r(\boldsymbol{G})$ is defined as

$$\mathrm{adm}_r(G) \;=\; \min_{L}\; \max_{y\in V(G)}\; a_r(L,y)+1$$

Some basic facts of these generalised colouring numbers

- by definition: $\mathrm{col}(G) = \mathrm{col}_1(G) = \mathrm{wcol}_1(G) = \mathrm{adm}_1(G)$

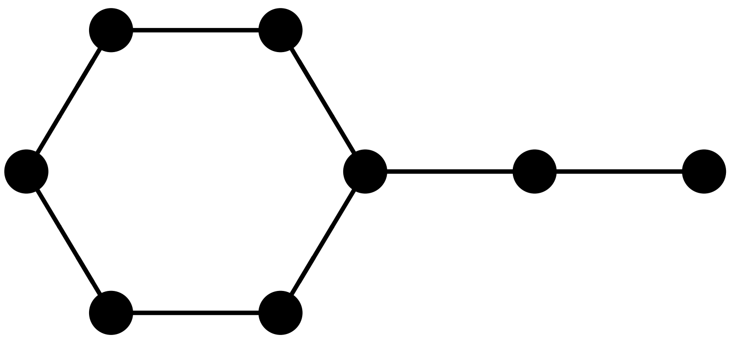
- - obviously: $\mathrm{adm}_r(G) \leq \mathrm{col}_r(G) \leq \mathrm{wcol}_r(G)$
  - in fact, also: $\mathrm{wcol}_r(G) \leq \big(\mathrm{adm}_r(G)\big)^r$ (Dvořák, 2013)

- - hence:
    if one of col_r, wcol_r, adm_r is bounded on some class of graphs, then all are bounded on that class
  - in particular:
    classes of bounded expansion can be defined using any of col_r, wcol_r, adm_r

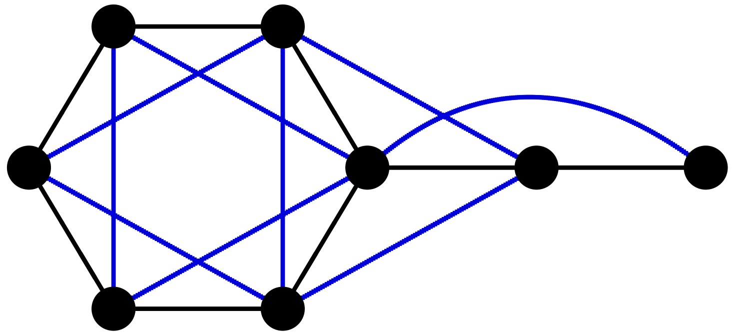
Another application: colouring at a distance

- - **vertex-colouring** with k colours : adjacent vertices must receive different colours
  - **chromatic number** $\chi(G)$: minimum k so that a vertex-colouring exists
- now suppose we want vertices at larger distances (say, up to distance d) to receive different colours as well
- can be modelled using the **d-th power G^d of a graph** :
  - same vertex set as d
  - edges between vertices with distance at most d in G

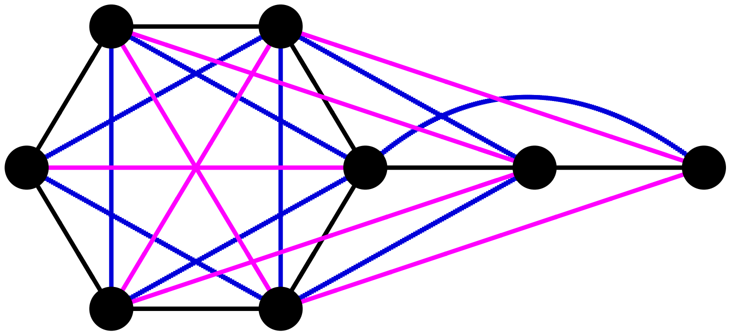
Powers of a graph



G



G^2



G^3

Colouring at a distance

- - easy: $d \geq 2 \implies \chi(G^d) \geq \Delta(G) + 1$
    ($\Delta(G)$: maximum degree of G)
  - so for most classes of graphs, $\chi(G^d)$ is not bounded

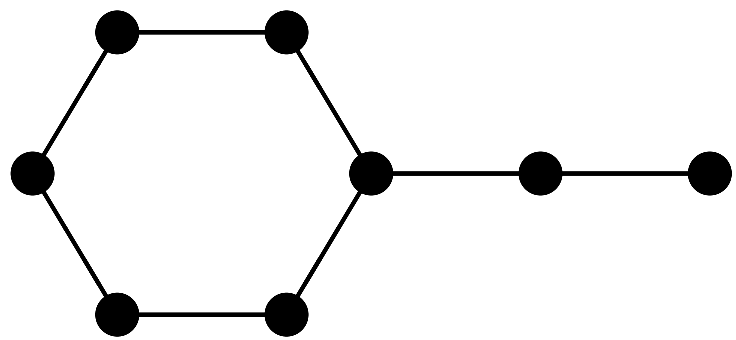
- **Theorem** (Agnarsson & Halldórsson, 2003)
  there exist functions $C_d(\cdot)$
  such that for all graphs G and all $d \geq 1$:

$$\chi(G^d) \;\leq\; C_d(\mathrm{col}(G)) \cdot \Delta(G)^{\lfloor d/2 \rfloor}$$

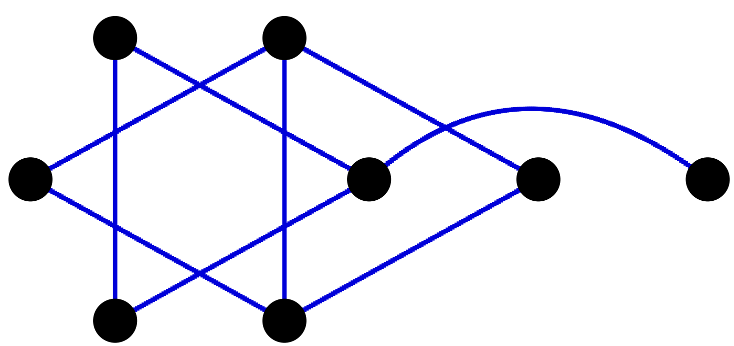
A variant with exact distances

- suppose we only want
  vertices at distance exactly d to have different colours
- can be modelled using the **exact distance graph** $G^{[\sharp d]}$:
  - same vertex set as G
  - edges between vertices with distance exactly d in G

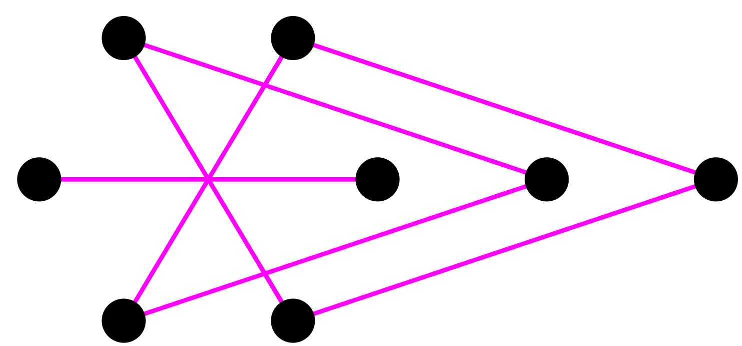
Exact distance graphs



G



$G^{[\sharp 2]}$



$G^{[\sharp 3]}$

Colouring at an exact distance

- for even d, for most classes of graphs
  the chromatic number $\chi(G^{[\sharp d]})$ is not bounded

- but for odd d, the situation is quite different

  Theorem (Nešetřil & Ossona de Mendez, 2008)

  d odd, $\mathcal{G}$ a class of graphs with bounded expansion

  $\Longrightarrow$ there exists a constant N such that:

  $$\chi(G^{[\sharp d]}) \leq N, \quad \text{for all } G \in \mathcal{G}$$

A very, very special case

- **Corollary**

  there exists a constant C such that

  $$G \text{ planar} \implies \chi(G^{[\sharp 3]}) \le C$$

  - proof of Nešetřil & Ossona de Mendez is long, complicated, and gives little idea what is going on
  - until recently, best known bounds on C:

    $$6 \le C \le 5 \cdot 2^{10{,}241}$$

A very, very simple result

- **Theorem** (vdH, Kierstead & Quiroz, 2016)
  - d odd, then for every graph G:

$$\chi(G^{[\sharp d]}) \ \leq \ \mathrm{wcol}_{2d-1}(G)$$

-
  - by being a bit more careful, we can prove:

$$G \text{ planar} \quad \Longrightarrow \quad \chi(G^{[\sharp 3]}) \ \leq \ 143$$

  - we also constructed a planar H with $\chi(H^{[\sharp 3]}) \ = \ 7$

A very, very simple result

- **Theorem** (vdH, Kierstead & Quiroz, 2016)
  - d odd, then for every graph G:

$$\chi(G^{[\sharp d]}) \;\leq\; \mathrm{wcol}_{2d-1}(G)$$

  - d even, then for every graph G:

$$\chi(G^{[\sharp d]}) \;\leq\; \mathrm{wcol}_{2d}(G) \cdot \Delta(G)$$

Thanks for your attention!

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