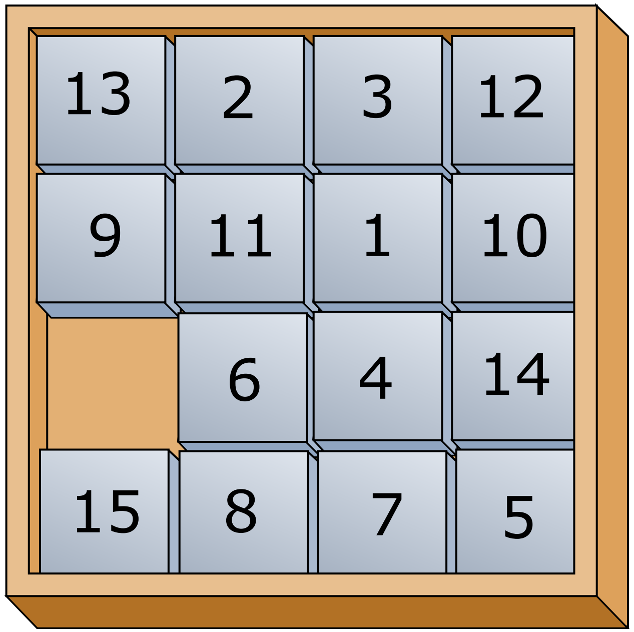
Generalisations of the 15-Puzzle
(Sliding Tokens on Graphs)

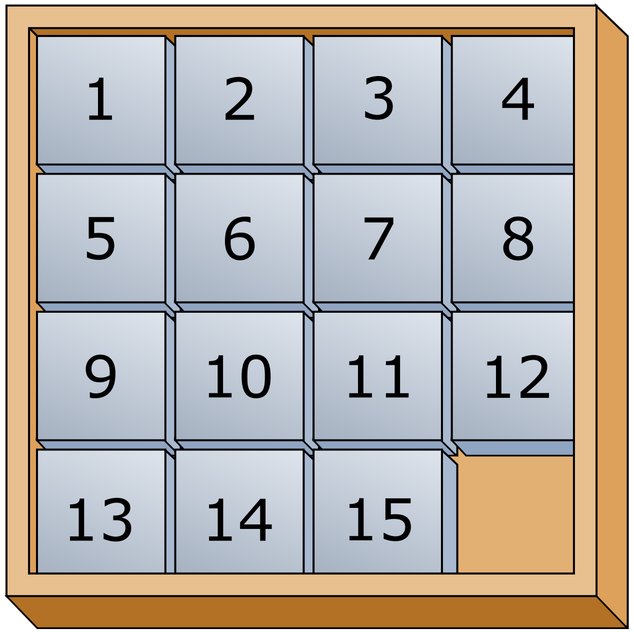
JAN VAN DEN HEUVEL

BIRS, Banff, 26 January 2017

Department of Mathematics
London School of Economics and Political Science

A classical puzzle: the 15-Puzzle

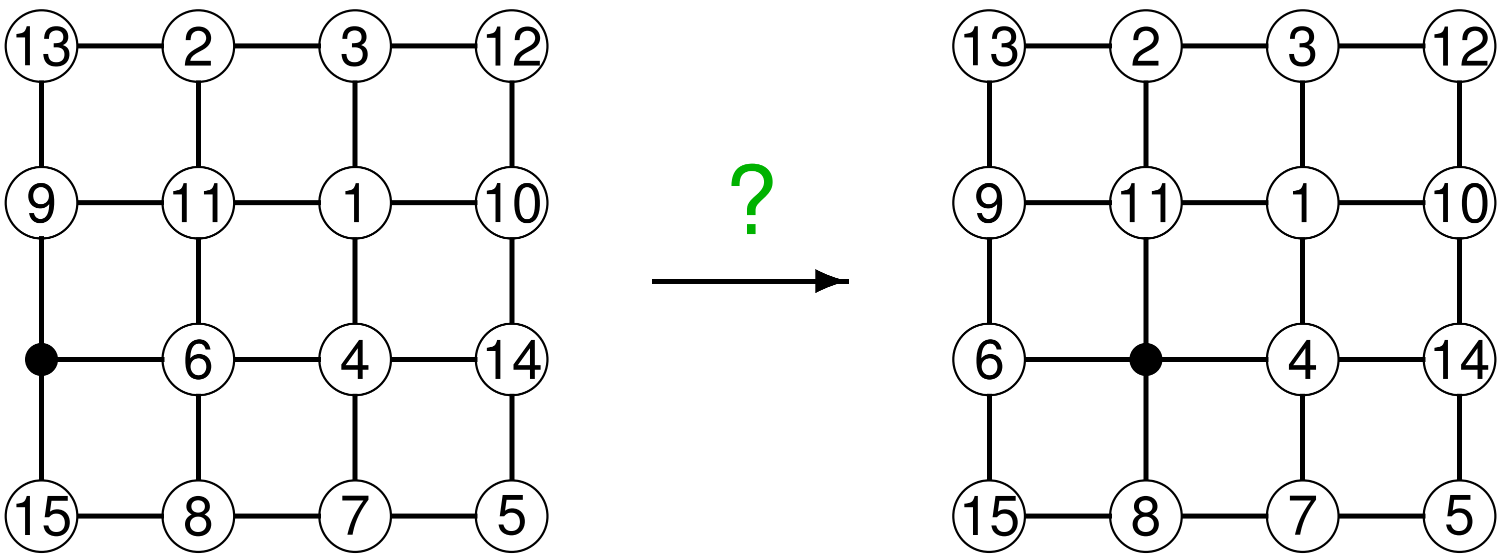




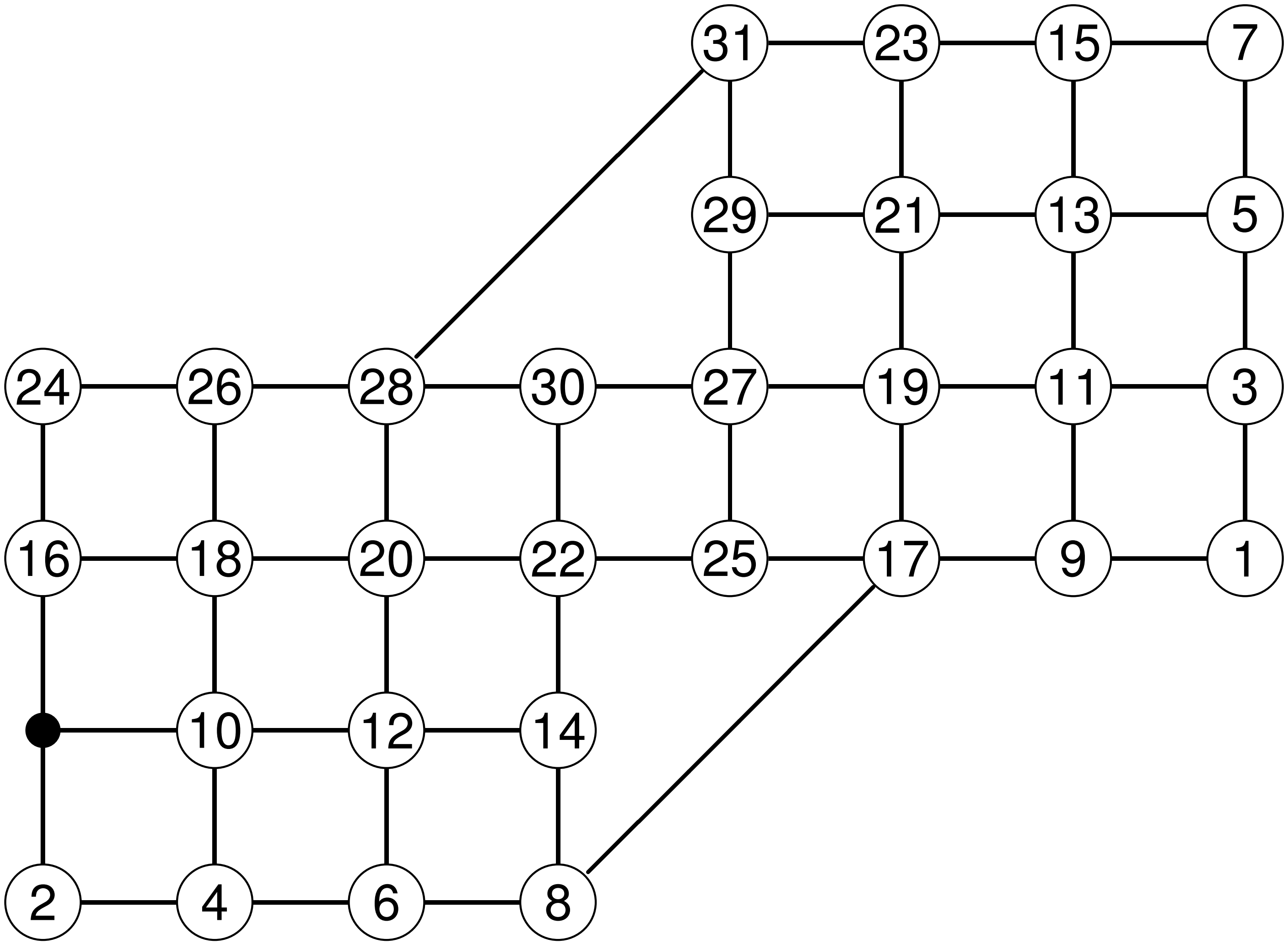
- can you always solve it?

Sliding token puzzles

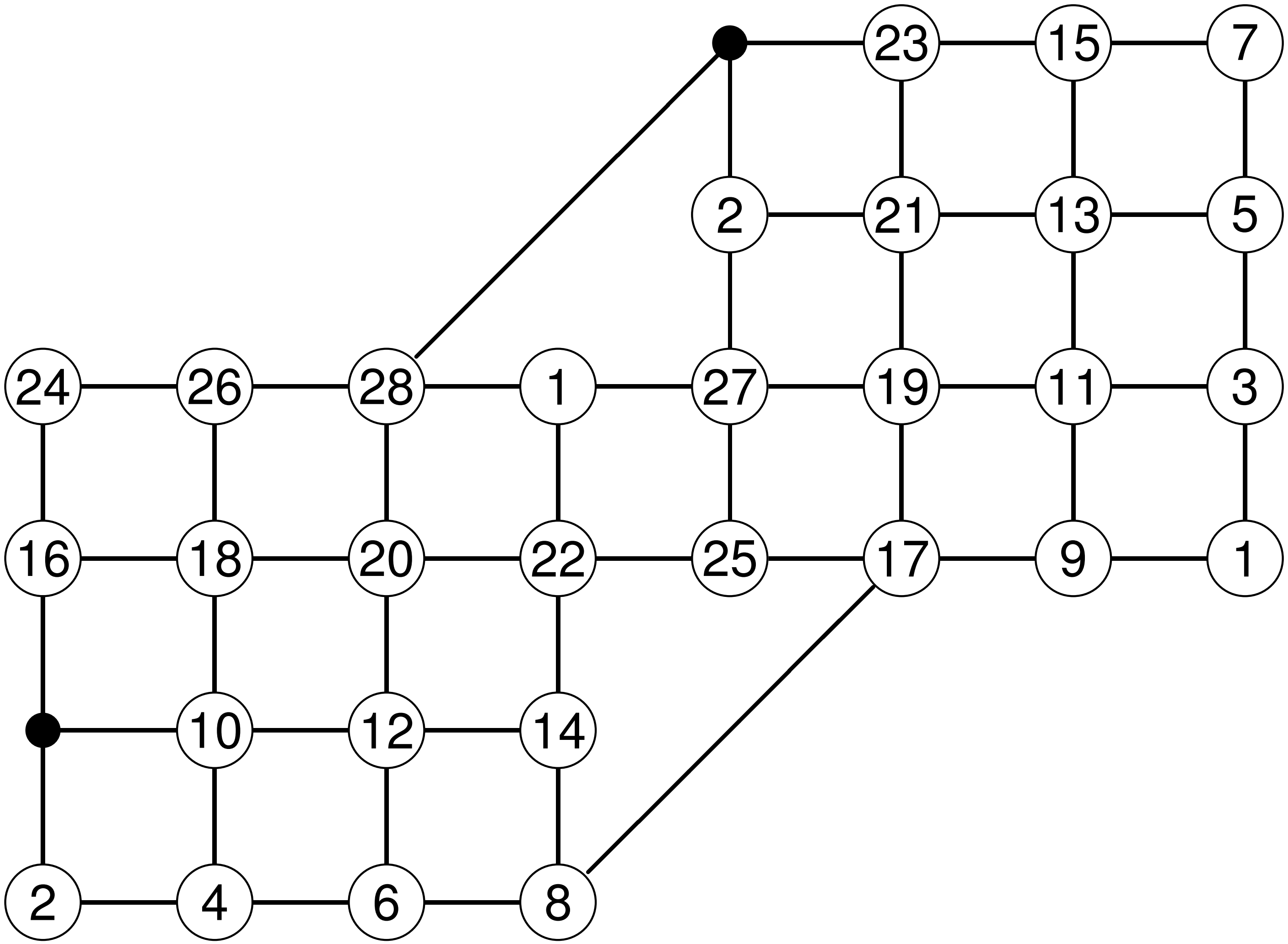
- we can interpret the 15-puzzle as a problem involving moving tokens on a given graph:



What if we would play on a different graph?



And maybe more empty spaces and/or repeated tokens?



Sliding token puzzles

- for a given graph G on n vertices,

  define **puz(G)** as the graph that has:

  - nodes: all possible placements of $n-1$ different tokens on G
  - adjacency: sliding one token along an edge of G to an empty vertex

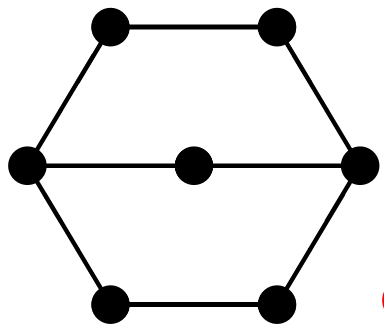
- and our standard decision problems become:

  - are two token configurations in one component of puz(G)?
  - is puz(G) connected?

Sliding token puzzles

Theorem (Wilson, 1974)

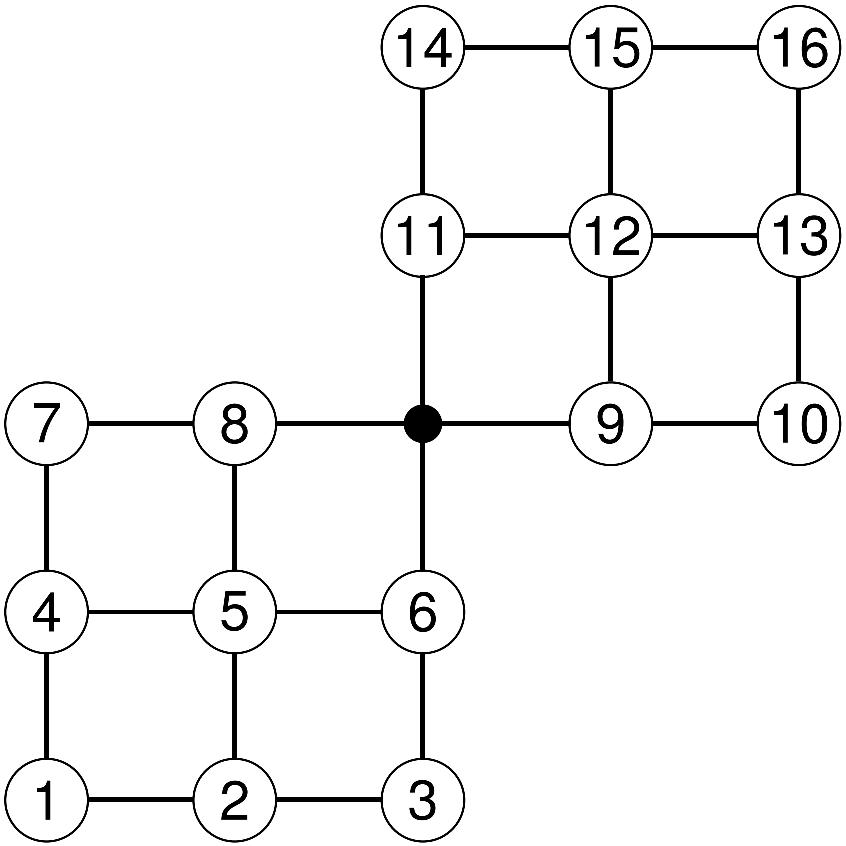
- if G is a 2-connected graph, then $\mathrm{puz}(G)$ is connected, except if:
  - G is a cycle on $n \geq 4$ vertices
    (then $\mathrm{puz}(G)$ has $(n-2)!$ components)
  - G is bipartite different from a cycle
    (then $\mathrm{puz}(G)$ has 2 components)
  - G is the exceptional graph Θ_0 ($\mathrm{puz}(\Theta_0)$ has 6 components)



Θ_0

Why does Wilson only consider 2-connected graphs?

- since puz(G) is never connected if G has connectivity below 2:



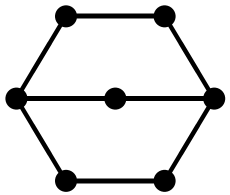
Generalised sliding token puzzles

- **what would happen if**:
  - we have fewer than $n - 1$ tokens (i.e. more empty vertices)?
  - and/or not all tokens are the same?
- so suppose we have a set $(k_1, k_2, \ldots, k_p)$ of labelled tokens
  - meaning: k_1 tokens with label 1, k_2 tokens with label 2, etc.
  - tokens with the same label are indistinguishable
  - we can assume that $k_1 \geq k_2 \geq \cdots \geq k_p$ and their sum is at most $n - 1$
- the corresponding graph of all token configurations on G is denoted by **puz**$(G; k_1, \ldots, k_p)$

Generalised sliding token puzzles

Theorem (Brightwell, vdH & Trakultraipruk, 2013)

- G a graph on n vertices, $(k_1, k_2, \ldots, k_p)$ a token set, then $\mathrm{puz}(G; k_1, \ldots, k_p)$ is connected, except if:
  - G is not connected
  - G is a path and $p \geq 2$
  - G is a cycle, and $p \geq 3$, or $p = 2$ and $k_2 \geq 2$
  - G is a 2-connected, bipartite graph with token set $(1^{(n-1)})$
  - G is the exceptional graph Θ_0 with token set $(2, 2, 2)$, $(2, 2, 1, 1)$, $(2, 1, 1, 1, 1)$ or $(1, 1, 1, 1, 1, 1)$



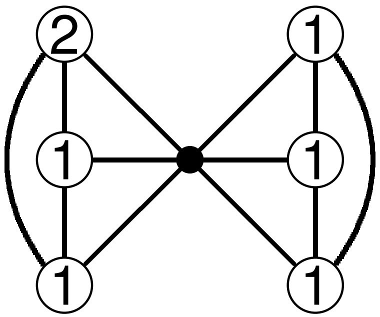
Generalised sliding token puzzles

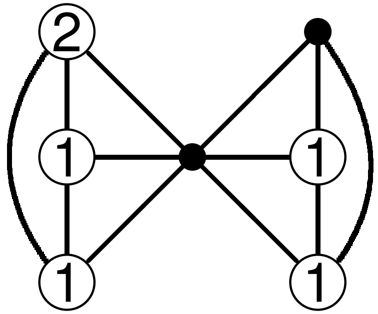
Theorem (Brightwell, vdH & Trakultraipruk, 2013)

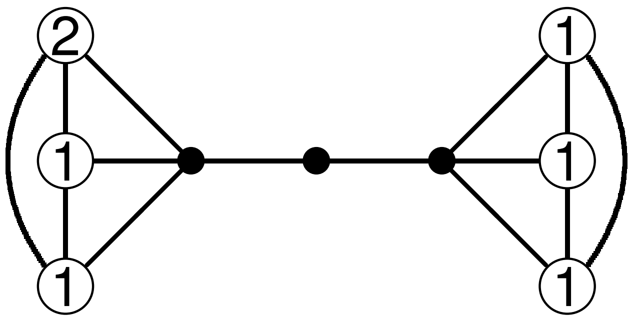
- G a graph on n vertices, $(k_1, k_2, \ldots, k_p)$ a token set, then $\mathrm{puz}(G; k_1, \ldots, k_p)$ is connected, except if:
  - G is not connected
  - G is a path and $p \geq 2$
  - G is a cycle, and $p \geq 3$, or $p = 2$ and $k_2 \geq 2$
  - G is a 2-connected, bipartite graph with token set $(1^{(n-1)})$
  - G is the exceptional graph Θ_0 with some “bad” token sets
  - G has connectivity 1, $p \geq 2$ and there is a “separating path preventing tokens from moving between blocks”

Generalised sliding token puzzles

- "separating paths" in graphs of connectivity one:

bad: 

good: 



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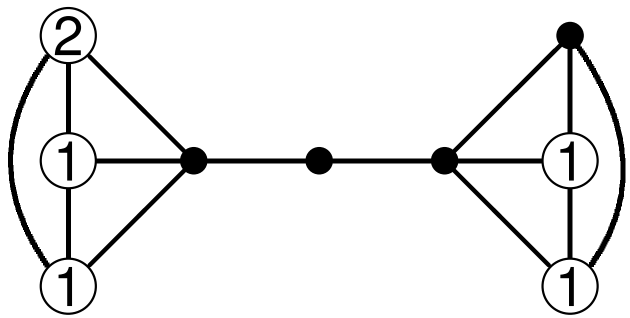

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Generalised sliding token puzzles

- we can also characterise:
  - given a graph G, token set $(k_1, \ldots, k_p)$, and two token configurations on G,
  - are the two configurations in the same component of $\mathrm{puz}(G; k_1, \ldots, k_p)$?
- so recognising connectivity properties of $\mathrm{puz}(G; k_1, \ldots, k_p)$ is easy
- can we say something about the number of steps we would need?

The length of sliding token paths

- **SHORTEST-A-TO-B-TOKEN-MOVES**

  Input: a graph G, a token set $(k_1, \ldots, k_p)$, two token configurations A and B on G, and a positive integer N

  Question: can we go from A to B in at most N steps?

The length of sliding token paths

Theorem (Goldreich, 1984-2011)

- restricted to the case that there are $n - 1$ different tokens,

  SHORTEST-A-TO-B-TOKEN-MOVES is **NP-complete**

Theorem (vdH & Trakultraipruk, 2013; probably others earlier)

- restricted to the case that all tokens are the same,

  SHORTEST-A-TO-B-TOKEN-MOVES is in **P**

Theorem (vdH & Trakultraipruk, 2013)

- restricted to the case that there is just one special token and all others are the same:

  SHORTEST-A-TO-B-TOKEN-MOVES is already **NP-complete**

Robot motion

- the proof of that last result uses ideas of the proof of

Theorem (Papadimitriou, Raghavan, Sudan & Tamaki, 1994)

- SHORTEST-ROBOT-MOTION-WITH-ONE-ROBOT is **NP-complete**

- **Robot Motion** problems on graphs are sliding token problems,
  - with some special tokens (the robots)
    - that have to end in specified positions
  - all other tokens are just obstacles
    - and it is not important where those are at the end