Graph Colouring with Distances

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The basics of graph colouring

- **vertex-colouring** with k colours:

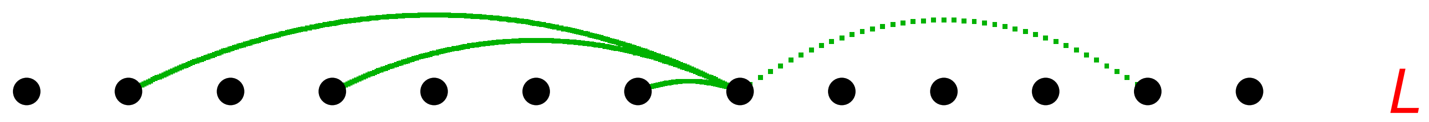
  adjacent vertices must receive different colours

- **chromatic number** $\chi(G)$:

  minimum k such that a vertex-colouring exists

Some essential graph parameters

- $\delta(G)$: **minimum vertex degree**
- $\Delta(G)$: **maximum vertex degree**
- - G is **k-degenerate**: every subgraph of G has minimum degree at most k
  - equivalent:
    there is an ordering L of the vertices of G, such that every vertex has at most k neighbours that come earlier in the ordering



Another way to look at vertex-colouring

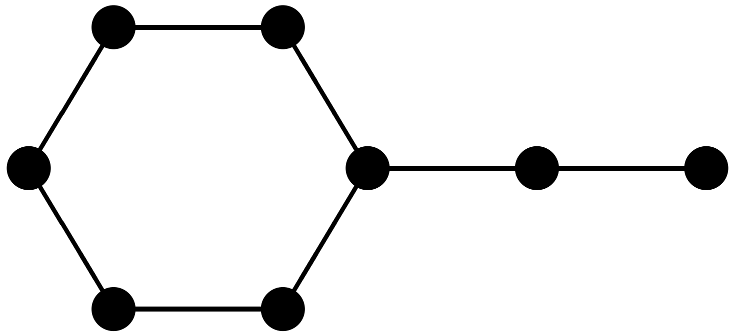
- vertex-colouring:

  vertices at distance one must receive different colours

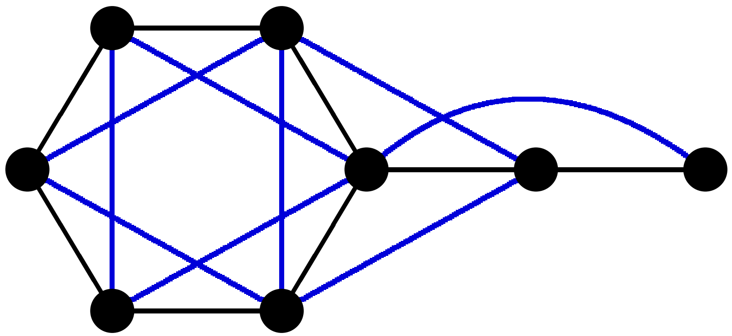
- now suppose we want vertices at larger distances (say, up to distance d) to receive different colours as well

- can be modelled using the **d-th power G^d of a graph**:
  - same vertex set as G
  - edges between vertices with distance at most d in G

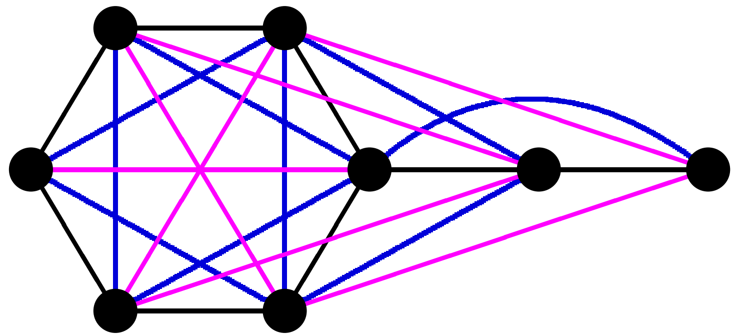
Powers of a graph



G

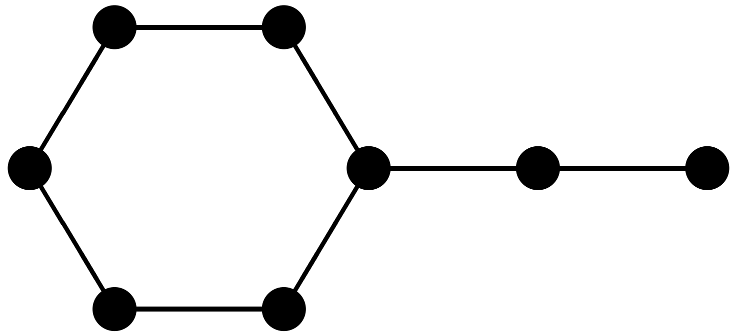


G^2

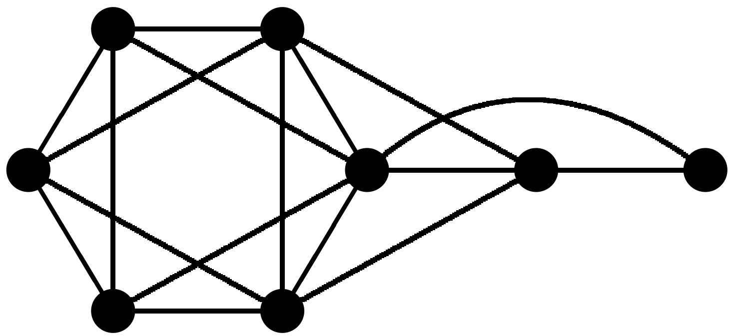


G^3

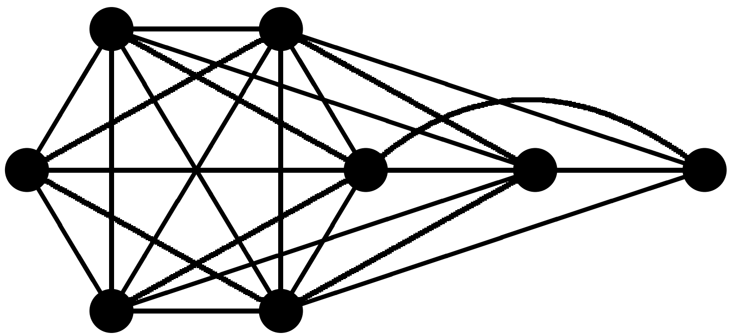
Powers of a graph



G



G^2



G^3

Colouring powers of a graph

easy facts

- $d \geq 2 \quad \Longrightarrow \quad \chi(G^d) \geq \Delta(G) + 1$
- and $\quad \chi(G^d) \leq 1 + \sum_{i=0}^{d-1} \Delta(G)\,(\Delta(G) - 1)^i$
- for connected graphs, we have equality of the upper bound only if
  - any d : odd cycles C_{2d+1}
  - $d = 2$: C_5 and two or three more graphs (including Petersen graph)

The square of k-degenerate graphs

fairly easy

- G k-degenerate

  $\Longrightarrow$ G^2 is $\big((2k-1)\,\Delta(G)\big)$-degenerate

so

- G planar $\Longrightarrow$ $\chi(G^2) \le 9\,\Delta(G)+1$

The square of planar graphs

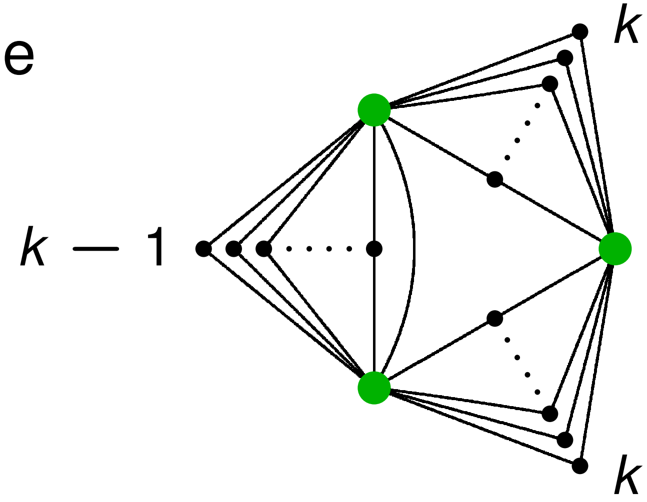
Conjecture (Wegner, 1977)

- G planar

$$\Longrightarrow \quad \chi(G^2) \le \begin{cases} 7, & \text{if } \Delta(G) = 3 \\ \Delta(G) + 5, & \text{if } 4 \le \Delta(G) \le 7 \\ \lfloor {}^3/_2\, \Delta(G) \rfloor + 1, & \text{if } \Delta(G) \ge 8 \end{cases}$$

- bounds would be best possible

case $\Delta(G) = 2k \ge 8$:



Towards Wegner's Conjecture

G planar $\Longrightarrow$

- $\chi(G^2) \le 8\,\Delta(G) - 22$ (Jonas, PhD, 1993)
- $\chi(G^2) \le 3\,\Delta(G) + 5$ (Wong, MSc, 1996)
- $\chi(G^2) \le 2\,\Delta(G) + 25$ (vdH & McGuinness, 2003)
- $\chi(G^2) \le {}^9\!/_5\,\Delta(G) + 1$ (for $\Delta(G) \ge 47$) (Borodin, Broersma, Glebov & vdH, 2001)
- $\chi(G^2) \le {}^5\!/_3\,\Delta(G) + 24$ (for $\Delta(G) \ge 241$) (Molloy & Salavatipour, 2005)

Towards Wegner's Conjecture

Theorem (Havet, vdH, McDiarmid & Reed, 2008+)

- G planar $\implies$ $\chi(G^2) \le \left({}^{3}\!/_{2} + \varepsilon\right)\Delta(G)$
  ($\varepsilon \downarrow 0$ for $\Delta(G) \to \infty$)

Theorem (Amini, Esperet & vdH, 2013)

- G embeddable on a fixed surface S
  - $\implies$ $\chi(G^2) \le \left({}^{3}\!/_{2} + \varepsilon\right)\Delta(G)$ ($\Delta(G) \to \infty$)
  - $\implies$ clique number $\omega(G^2) \le {}^{3}\!/_{2}\,\Delta(G) + C$

What about distances larger than 2 ?

easy upper bound

- $$\chi(G^d) \le 1 + \sum_{i=0}^{d-1} \Delta(G)\,(\Delta(G)-1)^i = \Omega(\Delta(G)^d)$$

Theorem (Agnarsson & Halldórsson, 2003)

- G k-degenerate $\Longrightarrow$ $\chi(G^d) \le c_{k,d}\,\Delta(G)^{\lfloor d/2 \rfloor}$

Colouring the cube of planar graphs

- so there is some constant c_3 such that:

  G planar $\Longrightarrow$ $\chi(G^3) \leq c_3\,\Delta(G) + C$

  - but what is the best c_3 ?
    - we only know: $9/2 \leq c_3 \leq 45$

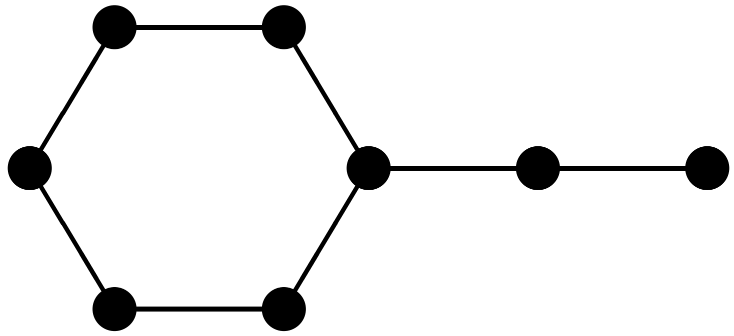
- and what about distances $d > 3$?

A variant with exact distances

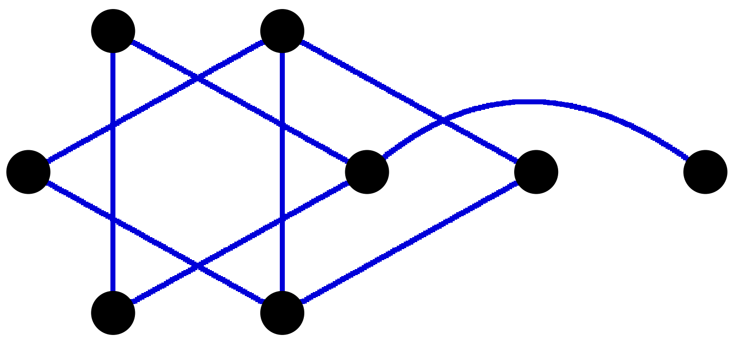
- suppose we only want
  vertices at distance exactly d to have different colours

- can be modelled using the **exact distance graph** $G^{[\sharp d]}$:
  - same vertex set as G
  - edges between vertices with distance exactly d in G

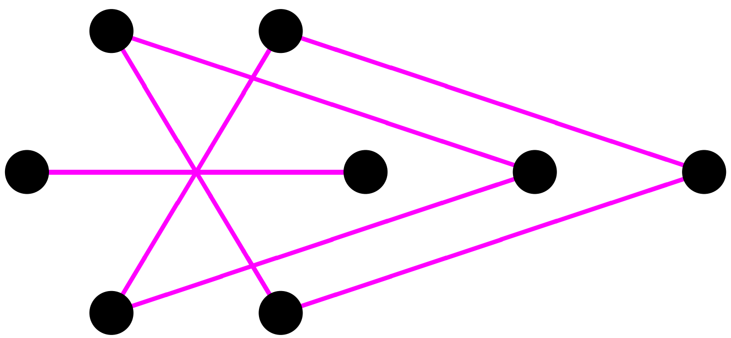
Exact distance graphs



G



$G^{[\sharp 2]}$



$G^{[\sharp 3]}$

Colouring at an exact distance

- for even d, for most classes of graphs
  the chromatic number $\chi(G^{[\sharp d]})$ is not bounded

- but for odd d, the situation is quite different

Theorem (Nešetřil & Ossona de Mendez, 2008)

- $\mathcal{G}$ a class of graphs with bounded expansion,
  d odd

  $\Longrightarrow$ there exists a constant $N_{\mathcal{G},d}$ such that:

  for all $G \in \mathcal{G}$: $\chi(G^{[\sharp d]}) \leq N_{\mathcal{G},d}$

A very, very special case

- the class of planar graphs has bounded expansion

so . . .

A very, very special case

Corollary

- there exists a constant C such that

$$G \text{ planar} \implies \chi(G^{[\sharp 3]}) \le C$$

- - proof of Nešetřil & Ossona de Mendez is long, complicated, and gives little idea what is going on
  - $G^{[\sharp 3]}$ can be very dense for planar G
  - there is no bound on the list-chromatic number of $G^{[\sharp 3]}$
  - until recently, best known bounds on C :

$$6 \le C \le 5 \cdot 2^{10{,}241}$$

A very, very simple result

Theorem (vdH, Kierstead & Quiroz, 2016)

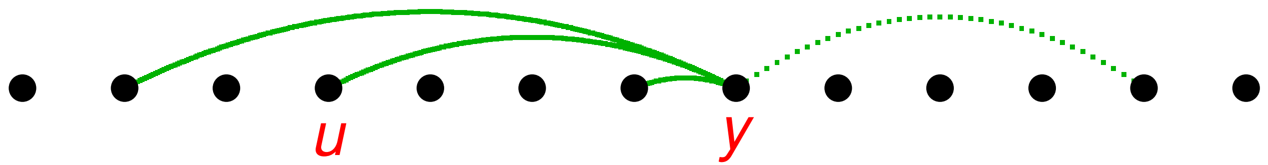
- d odd, then for every graph G :

$$\chi(G^{[\sharp d]}) \leq \mathrm{wcol}_{2d-1}(G)$$

- the **weak d-colouring number $\mathrm{wcol}_d(G)$** is a generalisation of degeneracy

The normal colouring number

- let L be a linear ordering of the vertices of a graph G
- for a vertex $y \in V(G)$,
  let $S(L, y)$ be the set of neighbours u of y with $u <_L y$



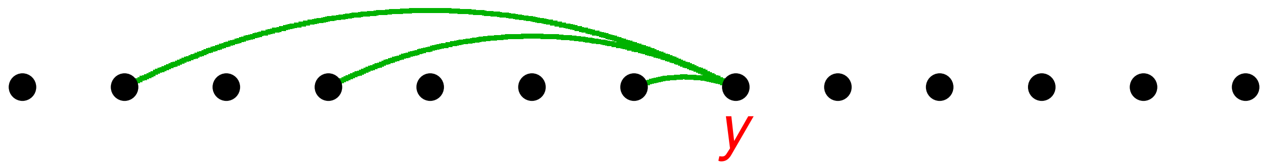
- then the **colouring number** $\mathbf{col}(G)$ is defined as

$$\mathrm{col}(G) = \min_L \max_{y \in V(G)} |S(L, y)| + 1$$

- **note**: G k-degenerate $\Longrightarrow$ $\mathrm{col}(G) \leq k + 1$

Generalising the colouring number

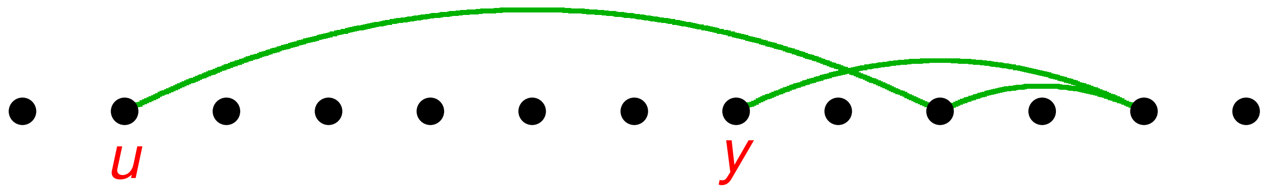
- the set $S(L, y)$ can be defined as
  "vertices $u <_L y$ for which there is a uy-path of length 1"



- what would happen if we allow longer paths ?

The strong colouring number

- a **strong** ***uy*-path** has all internal vertices larger than y



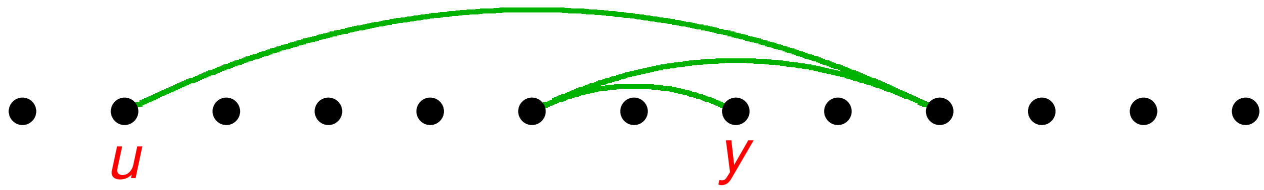
- let $\boldsymbol{S_d(L,y)}$ be the set of vertices $u <_L y$ for which there exists a strong uy-path with length at most d

- then the **strong** ***d*-colouring number** $\mathbf{scol}_d(G)$ is defined as

$$\mathrm{scol}_d(G) = \min_L \max_{y \in V(G)} |S_d(L,y)| + 1$$

The weak colouring number

- a **weak *uy*-path** has all internal vertices larger than u



- let $\boldsymbol{W_d(L,y)}$ be the set of vertices $u <_L y$ for which there exists a weak uy-path of length at most d

- then the **weak *d*-colouring number** $\mathbf{scol}_{\boldsymbol{d}}(\boldsymbol{G})$ is defined as

$$\mathrm{wcol}_d(G) \;=\; \min_L \max_{y\in V(G)} |W_d(L,y)|+1$$

Some basic facts of these generalised colouring numbers

- by definition: $\mathrm{col}(G) = \mathrm{scol}_1(G) = \mathrm{wcol}_1(G)$

- - obviously: $\mathrm{scol}_d(G) \leq \mathrm{wcol}_d(G)$
  - in fact, also: $\mathrm{wcol}_d(G) \leq \big(\mathrm{scol}_d(G)\big)^d$

- hence:
  if one of scol_d, wcol_d, is bounded on some class of graphs,
  then the other one is also bounded on that class

Not so basic fact

- again straightforward from the definition:

  $$\mathrm{scol}_1(G) \leq \mathrm{scol}_2(G) \leq \mathrm{scol}_3(G) \leq \ldots \leq \mathrm{scol}_\infty(G)$$

  (where the " ∞ " means "any length strong path allowed")

- **Property** (Grohe et al., 2014)

  $$\mathrm{scol}_\infty(G) = \mathrm{treewidth}(G) + 1$$

Back to classes with bounded expansion

Definition (Nešetřil & Ossona de Mendez)

- a class of graphs $\mathcal{G}$ has **bounded expansion**

  if there exist constants c_1, c_2, $\ldots$, such that

  for all $G \in \mathcal{G}$ and for all $d = 1, 2, \ldots$ we have:

  - for all minors H of G formed by contracting connected subgraphs with radius at most d :

$$|E(H)| \leq c_d \cdot |V(H)|$$

- generalises classes with “bounded treewidth”, “bounded genus”, “forbidden minors”, “bounded cop number”, etc., etc.

Back to classes with bounded expansion

Equivalent definitions (Zhu, 2009)

a class of graphs $\mathcal{G}$ has **bounded expansion** if

- there exist constants $c'_1,\ c'_2,\ \ldots$ such that
  - for all $G \in \mathcal{G}$ and for all $d = 1, 2, \ldots$:

$$\mathrm{scol}_d(G) \ \leq \ c'_d$$

or

- there exist constants $c''_1,\ c''_2,\ \ldots$ such that
  - for all $G \in \mathcal{G}$ and for all $d = 1, 2, \ldots$:

$$\mathrm{wcol}_d(G) \ \leq \ c''_d$$

Back to colouring exact distance graphs

time to prove that for any graph G :

$$\chi(G^{[\sharp 3]}) \leq \mathrm{wcol}_5(G)$$

- recall the definition of $\mathrm{wcol}_5(G)$:

$$\mathrm{wcol}_5(G) = \min_{L} \max_{y \in V(G)} |W_5(L, y)| + 1$$

- so let's choose an ordering L_5 of the vertices such that

for all y : $|W_5(L_5, y)| + 1 \leq \mathrm{wcol}_5(G)$

Back to colouring exact distance graphs

- so let's choose an ordering L_5 of the vertices such that

  for all y : $|W_5(L_5, y)| + 1 \le \mathrm{wcol}_5(G)$

- stage 1

  going along the ordering L_5, give every vertex y a colour $c(y)$ different from the vertices in $W_5(L_5, y)$

  - requires at most $\mathrm{wcol}_5(G)$ colours

Back to colouring exact distance graphs

- stage 1

  going along the ordering L_5 , give every vertex y a colour $c(y)$ different from the vertices in $W_5(L_5, y)$

- stage 2

  - for a vertex y, define $N[y] = N(y) \cup \{y\}$ and let $\mu(y)$ be the left-most vertex in $N[y]$ (according to the ordering L_5)
  - give every vertex y the colour $C(y) = c(\mu(y))$

- **Claim**: the colouring C is a proper colouring of $G^{[\sharp 3]}$ $\square$

What can we say about $wcol_d(G)$?

Theorem (vdH, Kierstead & Quiroz, 2016)

- d odd, then for every graph G :

$$\chi(G^{[\sharp d]}) \le \mathrm{wcol}_{2d-1}(G)$$

Theorem (vdH, Ossona de Mendez, Quiroz, Rabinovich & Siebertz, 2016)

- bounds on $\mathrm{scol}_d(G)$ and $\mathrm{wcol}_d(G)$ for all kinds of graphs (bounded treewidth, bounded genus, forbidden minor, etc.)
- in particular:
  - G planar $\Longrightarrow$ $\mathrm{wcol}_d(G) \le \binom{d+2}{2} \cdot (2d+1)$

Bounds on $\chi(G^{[\sharp 3]})$ for planar graphs

Corollary (vdH, Kierstead & Quiroz, 2016)

- G planar $\implies$ $\chi(G^{[\sharp 3]}) \le \mathrm{wcol}_5(G) \le 231$

- by being a bit more careful, we can prove:

  G planar $\implies$ $\chi(G^{[\sharp 3]}) \le 143$

- we also constructed a planar H with $\chi(H^{[\sharp 3]}) = 7$

Some final bits and bobs

since a couple of weeks we know

Theorem

(Bousquet, Esperet, Harutyunyan, de Joannis de Verclos, Pastor)

- $\max \{ \chi(G^{[\sharp d]}) \mid G$ planar $\} \longrightarrow \infty$ if $d \longrightarrow \infty$

Some final bits and bobs

Theorem (vdH, Kierstead & Quiroz, 2016)

- d odd, then for every graph G :

$$\chi(G^{[\sharp d]}) \;\leq\; \mathrm{wcol}_{2d-1}(G)$$

- d even, then for every graph G :

$$\chi(G^{[\sharp d]}) \;\leq\; \mathrm{wcol}_{2d}(G) \cdot \Delta(G)$$

Some final bits and bobs

Theorem (vdH, Kierstead & Quiroz, 2016)

- d even, then for every graph G:

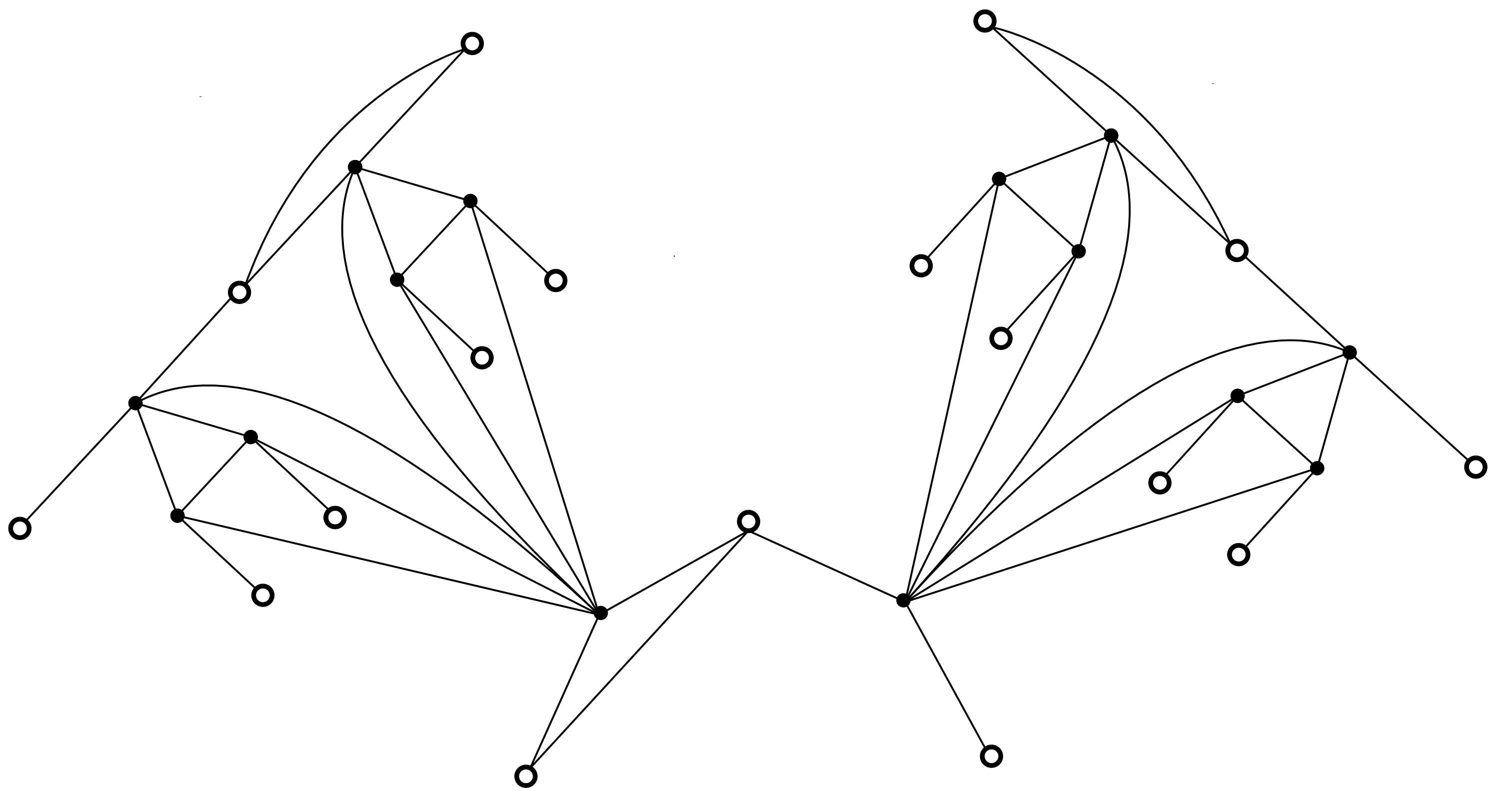
$$\chi(G^{[\sharp d]}) \le \mathrm{wcol}_{2d}(G) \cdot \Delta(G)$$

Corollary

- $\mathcal{G}$ a class of graphs with bounded expansion,
  d even

  $\Longrightarrow$ there exists a constant $N'_{\mathcal{G},d}$ such that:

  for all $G \in \mathcal{G}$: $\chi(G^{[\sharp d]}) \le N'_{\mathcal{G},d} \cdot \Delta(G)$



Thanks for your attention!