Improper Colourings
inspired by Hadwiger's Conjecture

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Graph minors

- a graph *H* is a **minor** of a graph *G*,
  if *H* can be obtained from *G* by a series of:
  - vertex deletions;
  - edge deletions;
  - edge contractions:

Graph minors – more intuitive

■ a graph H is a **minor** of a graph G if:

for $V(H) = \{v_1, \ldots, v_k\}$, there exist
connected, disjoint subgraphs $H_1, \ldots, H_k$ of G such that:

  ■ if $v_i v_j \in E(H)$, then there is
    at least one edge in G between H_i and H_j

H

G
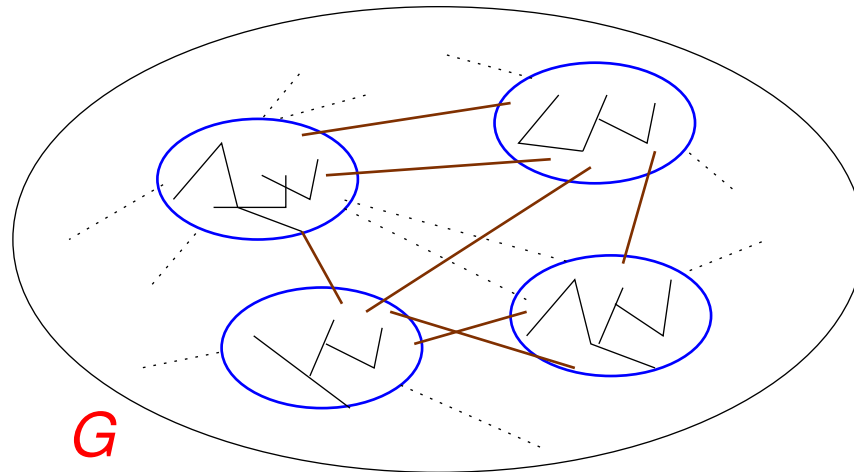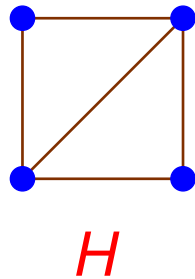
Graph colouring

- a **colouring** of a graph means colouring the vertices

- **proper colouring**: adjacent vertices have different colours

- recurring question in graph theory:

  what structural properties of a graph

  - allow proper colourings with few colours ?
  - force all proper colourings to use many colours ?

Hadwiger's Conjecture

Conjecture (Hadwiger, 1943)

- a graph G needs at least t colours for a proper colouring

  $\implies$ G has the complete graph K_t as a minor

the contrapositive is probably more intuitive:

- G has no K_t-minor

  $\overset{?}{\implies}$ G has a proper $(t-1)$-colouring

Hadwiger's Conjecture – what is known

■ no K_t-minor $\overset{?}{\Longrightarrow}$ G has a proper $(t-1)$-colouring

known

■ nothing to prove for $t = 1, 2$

■ easy for $t = 3$
    (no K_3-minor $\Rightarrow$ no cycles $\Rightarrow$ forest $\Rightarrow$ 2-colourable)

■ not too hard for $t = 4$ (Hadwiger, 1943; Dirac, 1952)

■ case $t = 5$ is equivalent to the Four Colour Theorem

(Wagner, 1937)

■ true for $t = 6$ (Robertson, Seymour & Thomas, 1993)

How many colours do we need ?

Theorem   (Kostochka, 1984; Thomason, 1984)

- G has no K_t -minor

  $\implies$ G has a vertex with degree at most $c\, t \sqrt{\log t}$

Corollary

- G has no K_t -minor

  $\implies$ G has a proper colouring with $c\, t \sqrt{\log t}$ colours

Improper colourings

■ what if we weaken the requirement on the colouring?

  ■ in a proper colouring: the collection of vertices with the same colour is just a collection of isolated vertices

■ we could be happy with:

  the collection of vertices with the same colour
  is just a subgraph with a "simple" structure

  ■ **monochromatic subgraph**:
    subgraph formed by vertices with the same colour

Improper colourings – small monochromatic degree

Theorem (Edwards, Kang, Kim, Oum & Seymour, 2015)

- ■ G has no K_t -minor $\implies$

  G can be coloured with $t - 1$ colours such that each monochromatic subgraph has degree at most $c' \, t^2 \log t$

- ■ the bound $t - 1$ on the number of colours is best possible: for each $t \geq 3$ and S, there exists a graph G such that

  - ■ G has no K_t -minor
  - ■ every $(t - 2)$ -colouring of G has a monochromatic subgraph with degree at least S

Improper colourings – small monochromatic degree

Theorem (Edwards, Kang, Kim, Oum & Seymour, 2015)

- G has no K_t-minor $\implies$

  G can be coloured with $t-1$ colours such that each monochromatic subgraph has degree at most $c' \, t^2 \log t$

- the dependence on no K_t-minor is best possible:

Theorem (Ossona de Mendez, Oum & Wood, 2016)

- H a graph on t vertices, different from K_t :

  - G has no H-minor $\implies$

    G can be coloured with $t-2$ colours such that each monochromatic subgraph has degree at most $f(t)$

Improper colourings – small monochromatic degree

Theorem (Edwards, Kang, Kim, Oum & Seymour, 2015)

■ G has no K_t -minor $\implies$

   G can be coloured with $t - 1$ colours such that each
   monochromatic subgraph has degree at most $c'\, t^2 \log t$

Theorem (vdH & Wood, 2017)

■ G has no K_t -minor $\implies$

   G can be coloured with $t - 1$ colours such that each
   monochromatic subgraph has degree at most $t - 2$

no K_t -minor $\overset{?}{\implies}$ proper $(t - 1)$ -colouring

Improper colourings – small monochromatic components

Theorem (Kawarabayashi & Mohar, 2007)

- G has no K_t-minor $\implies$

  G can be coloured with $\lceil 15\frac{1}{2} t \rceil$ colours such that each monochromatic components has at most $f_1(t)$ vertices

- improved to

  - $\lceil 3\frac{1}{2} t - 1\frac{1}{2} \rceil$ colours; $f_2(t)$ vertices   (Wood, 2010; incl.)
  - $3(t-1)$ colours; $f_3(t)$ vertices   (Liu & Oum, 2015)
  - $2(t-1)$ colours; $f_4(t)$ vertices   (Norin, 2015; unpubl.)

- all use Robertson & Seymour Graph Minor Structure Thm., or worse …

no K_t-minor $\overset{?}{\implies}$ proper $(t-1)$-colouring

Improper colourings – small monochromatic components

Theorem   (vdH & Wood, 2017)

- ■ G has no K_t -minor $\Longrightarrow$

  G can be coloured with $2(t-1)$ colours such that each
  monochromatic component has at most $\left\lceil \frac{1}{2}(t-2) \right\rceil$ vertices

note

- ■ G has no K_t -minor $\Longrightarrow$

  at least $t-1$ colours are needed to guarantee
  monochromatic components of bounded order

  (same examples as for small monochromatic degree)

A simple decomposition theorem for K_t-minor-free graphs

Theorem   (vdH & Wood, 2017)

- G has no K_t-minor $\implies$

  G has a partition into subgraphs $H_1, \ldots, H_\ell$ such that

  - each H_i is adjacent to at most $t-2$
    of the subgraphs $H_1, \ldots, H_{i-1}$

  - each H_i has maximum degree at most $t-2$

    - each H_i can be coloured with 2 colours
      such that each monochromatic component
      has at most $\left\lceil \frac{1}{2}(t-2) \right\rceil$ vertices

Our main results on improper colourings

Theorem   (vdH & Wood, 2017)

■ G has no K_t -minor $\implies$

   G can be coloured with $t - 1$ colours such that each
   monochromatic subgraph has degree at most $t - 2$

Theorem   (vdH & Wood, 2017)

■ G has no K_t -minor $\implies$

   G can be coloured with $2(t - 1)$ colours such that each
   monochromatic component has at most $\left\lceil \frac{1}{2}(t - 2) \right\rceil$ vertices

The "rough" structure we actually prove

■ G any graph $\implies$

we can construct (in many ways) a partition of G

into induced subgraphs $H_1, \ldots, H_\ell$ such that:

■ ▪ each H_i is connected

▪ each H_i is adjacent to k subgraphs $H_{i_1}, \ldots, H_{i_k}$
from $H_1, \ldots, H_{i-1}$

▪ for each H_i, the adjacent subgraphs $H_{i_1}, \ldots, H_{i_k}$
are pairwise adjacent as well
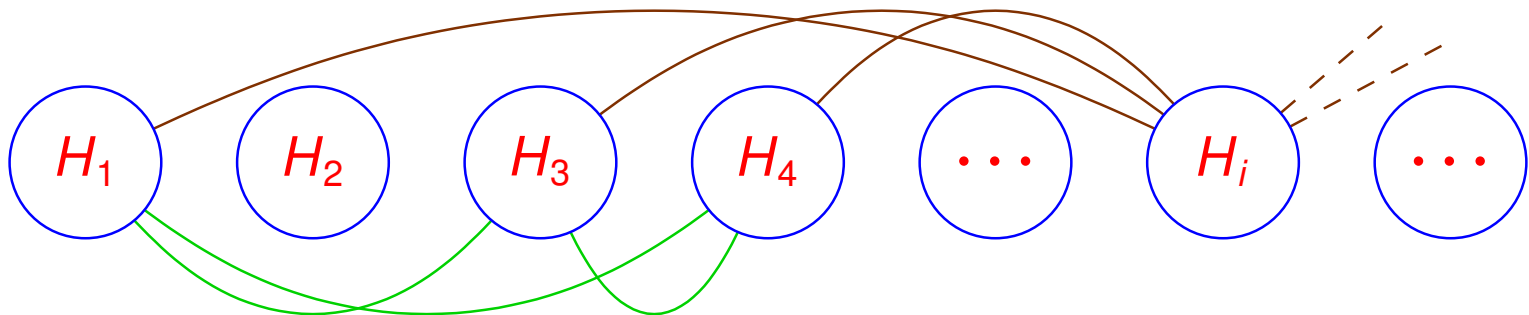
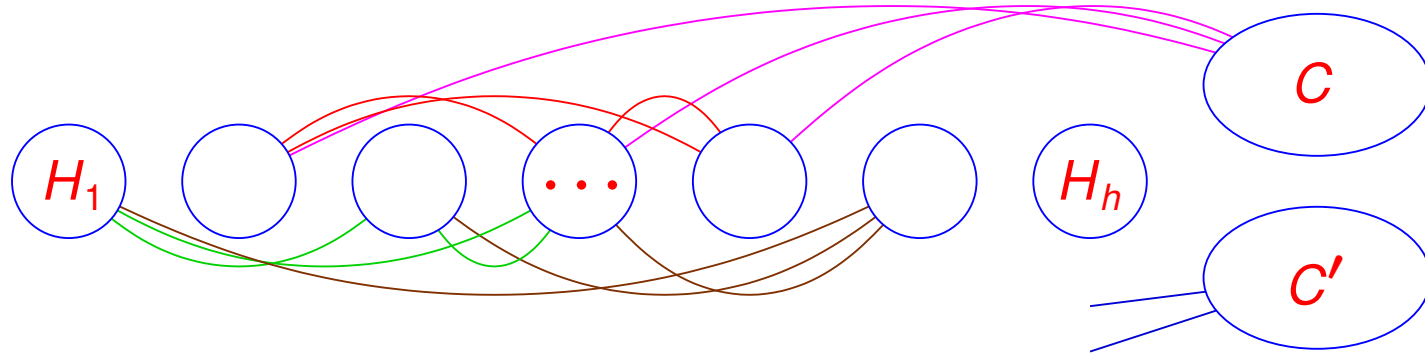no K_t-minor $\overset{?}{\implies}$ proper $(t-1)$-colouring

The "rough" structure – proof

- we will construct the H_i one by one such that once $H_1, \ldots, H_h$ is constructed:

  - each H_i, $i \leq h$, satisfies the requirements
  - each component C of $G - \big(V(H_1) \cup \cdots \cup V(H_h)\big)$ satisfies:

    if C is adjacent to $H_{i_1}, \ldots, H_{i_k}$ from $H_1, \ldots, H_{i-1}$, then $H_{i_1}, \ldots, H_{i_k}$ are pairwise adjacent as well
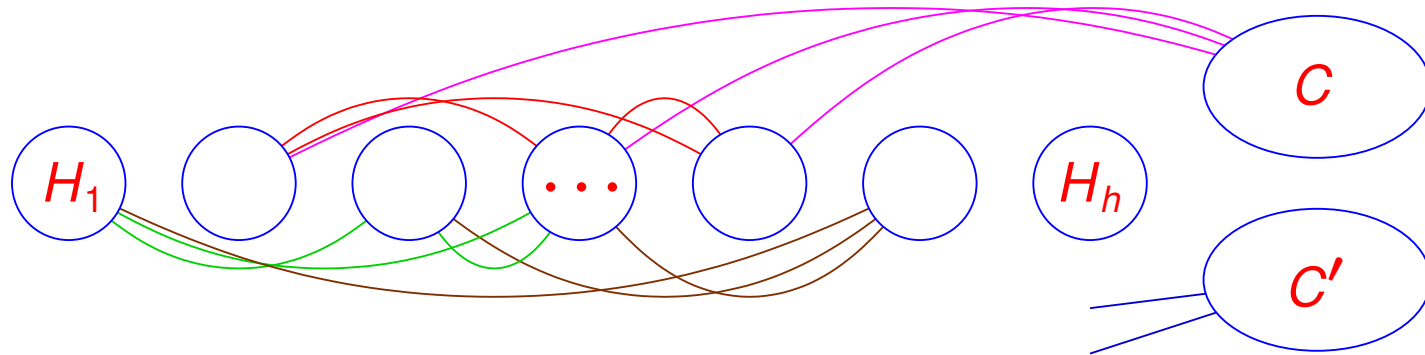
The "rough" structure – proof

- **start** with H_1 any **connected**, **induced subgraph** of G

  (good choice for later: $V(H_1) = \{v\}$ for some $v \in V(G)$)
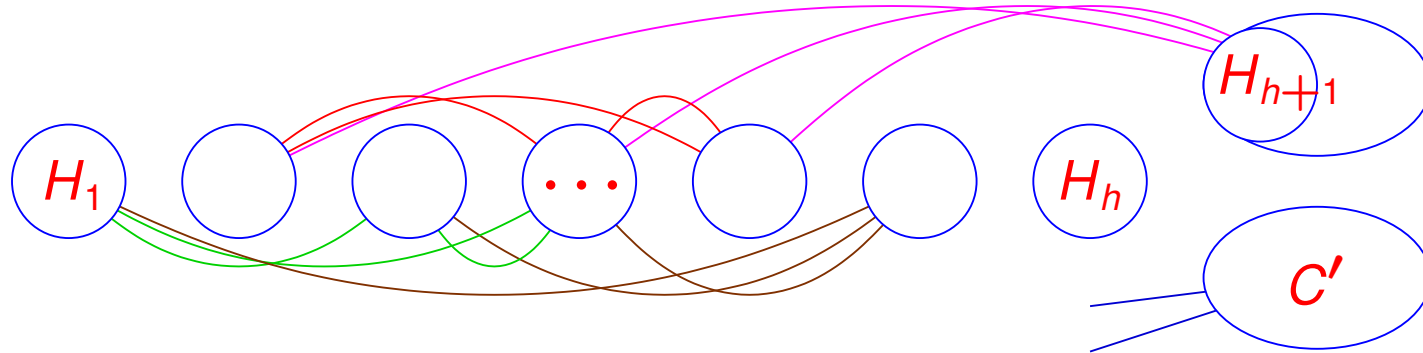
  - all requirements are trivially satisfied

The "rough" structure – proof

- suppose $H_1, \ldots, H_h$ are already constructed
  and C is some component of $G - \big(V(H_1) \cup \cdots \cup V(H_h) \big)$

  - so C is adjacent to $H_{i_1}, \ldots, H_{i_k}$,
    which are also pairwise adjacent

- for each H_{i_ℓ}, choose $a_{i_\ell} \in V(C)$ adjacent to H_{i_ℓ}

- now choose H_{h+1} a connected, induced subgraph of C
 containing all $a_{i_1}, \ldots, a_{i_k}$

The "rough" structure – proof

- H_{h+1} is still adjacent to $H_{i_1}, \ldots, H_{i_k}$

- a component of $G - \big(V(H_1) \cup \cdots \cup V(H_h) \cup V(H_{h+1})\big)$

  - is either a component of $G - \big(V(H_1) \cup \cdots \cup V(H_h)\big)$
    and hence still satisfies the requirements

  - or it is a component of $C - V(H_{h+1})$,
    hence it is adjacent to H_{h+1} and some of $H_{i_1}, \ldots, H_{i_k}$,
    which are all pairwise adjacent

The "rough" structure for K_t-minor-free graphs

- we can construct (in many ways) a partition of any graph G into induced, connected subgraphs $H_1, \ldots, H_\ell$ such that:

  - each H_i is adjacent to k subgraphs $H_{i_1}, \ldots, H_{i_k}$ from $H_1, \ldots, H_{i-1}$, which are pairwise adjacent as well
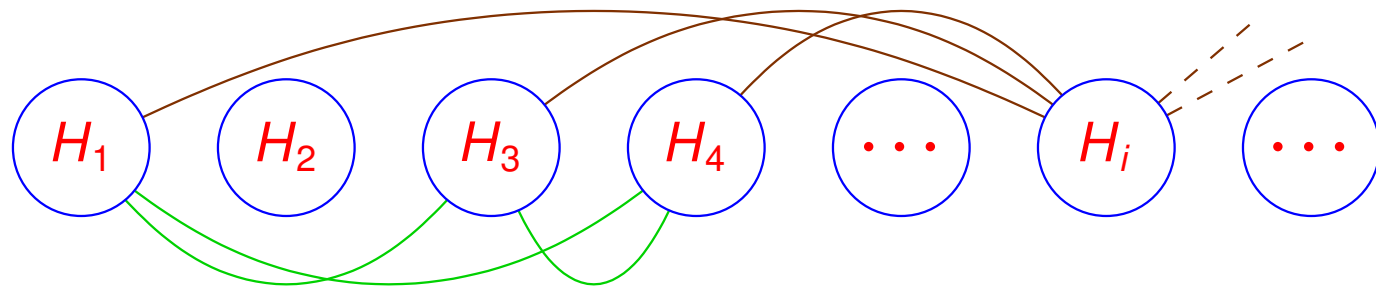
- G has no K_t-minor $\implies$

  for each H_i we must have $k \leq t - 2$

The "fine" structure – inside the H_i

- so how can we choose the H_i

  so that they have the right properties:

  small degree & 2-colourable with small mono. components

- each H_i was chosen as some induced subgraph of some

  connected subgraph C, such that:

  - H_i is connected

  - H_i contains some set $A = \{a_{i_1}, \ldots, a_{i_k}\}$

    (where $|A| \leq t - 2$ if G has no K_t-minor)

- idea:

  choose H_i the smallest subgraph with those properties

The "fine" structure – inside the H_i

Lemma

- C a connected graph, $A \subseteq V(C)$

  H a minimal, induced, connected subgraph of C,

  such that H contains A

- then H satisfies:

  - every vertex in H has degree at most $|A|$ in H

  - every vertex not in A is a cut-vertex of H

    (allows to prove that there is a 2-colouring of H with monochromatic components of size at most $\lceil \frac{1}{2}|A| \rceil$)

Our decomposition theorem again

Theorem

- G has no K_t-minor $\implies$

  G has a partition into subgraphs $H_1, \ldots, H_\ell$ such that

  - each H_i is adjacent to at most $t - 2$
    of the subgraphs $H_1, \ldots, H_{i-1}$

  - each H_i has maximum degree at most $t - 2$

  - - each H_i has maximum degree at most $t - 2$

    - each H_i can be coloured with 2 colours
      such that each monochromatic component
      has at most $\left\lceil \frac{1}{2}(t - 2) \right\rceil$ vertices

With only a little bit extra work ...

Theorem

- G has no K_t-minor $\implies$

  G has a partition into connected subgraphs $H_1, \ldots, H_\ell$ such that

  - contracting all H_i to single vertices gives a chordal graph with treewidth at most $t - 2$

  - each H_i has treewidth at most $t - 3$

  - H_i adjacent to H_h, $h < i$ $\implies$

    each vertex in H_i has at most $2t - 6$ neighbours in H_h

A decomposition theorem for $K_{3,s}$-minor-free graphs

Theorem (vdH & Wood, 2017)

- G has no $K_{3,s}$-minor $\implies$

  G has a partition into subgraphs $H_1, \ldots, H_\ell$ such that

  - each H_i is adjacent to at most 2
    of the subgraphs $H_1, \ldots, H_{i-1}$

  - - each H_i has maximum degree at most $4s$

    - each H_i can be coloured with 2 colours
   such that each monochromatic component
      has at most $2s$ vertices

Improper colourings of $K_{3,s}$-minor-free graphs

Corollary

- G has no $K_{3,s}$-minor $\implies$

  - G can be coloured with 3 colours such that each monochromatic subgraph has degree at most $4s$

  - G can be coloured with 6 colours such that each monochromatic component has at most $2s$ vertices

- to guarantee monochromatic components of bounded size, we need at least 4 colours

Variants – Topological minors

- a graph H is a **topological minor** of a graph G if:

  for $V(H) = \{v_1, \ldots, v_k\}$,
  there are different $u_1, \ldots, u_k$ in $V(G)$ such that:

  - if $v_i v_j \in E(H)$, then there is a u_i, u_j-path P_{ij} in G
  - apart from their end vertices, the paths P_{ij} are disjoint

H

G
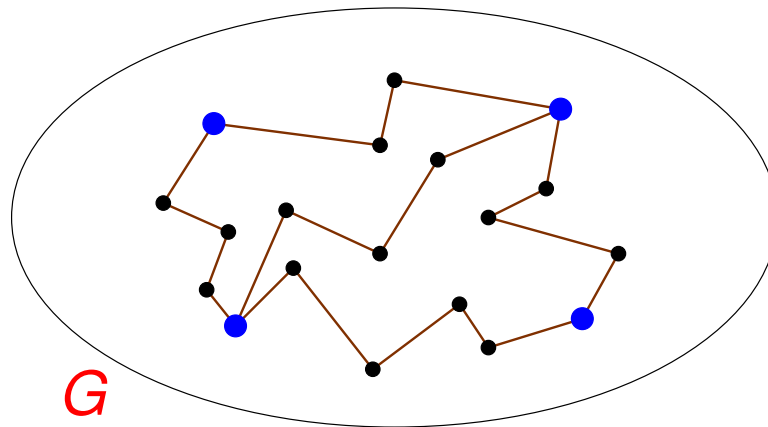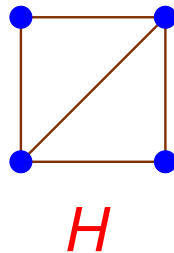
- H a topological minor of G $\implies$ H a minor of G

Hajós' Conjecture

Conjecture   (Hajós, 1961)

- G has no K_t -topological-minor

  $\overset{?}{\Longrightarrow}$ G has a proper $(t-1)$ -colouring

- true for $t \leq 4$ (Dirac, 1952)

- false for $t \geq 7$ (Catlin, 1979)

Theorem   (Edwards, Kang, Kim, Oum & Seymour, 2015)

- G has no K_t -topological-minor $\implies$
  G can be coloured with $t-1$ colours such that each
  monochromatic subgraph has degree at most $c\,t^4$

Variants – Immersions

- a graph H is an **immersion** in a graph G if:

  for $V(H) = \{v_1, \ldots, v_k\}$,
  there are different $u_1, \ldots, u_k$ in $V(G)$ such that:

  - if $v_i v_j \in E(H)$, then there is a u_i, u_j-path P_{ij} in G
  - the paths P_{ij} are edge-disjoint

H

G
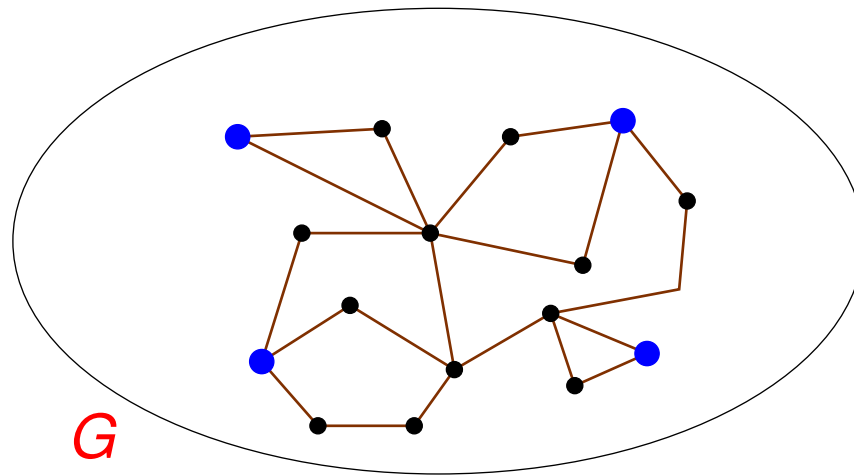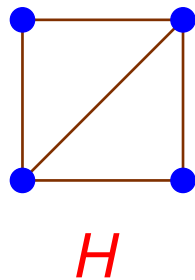
Variants – Immersions

- a graph H is an **immersion** in a graph G if:

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  there are different $u_1, \ldots, u_k$ in $V(G)$ such that:

  - if $v_i v_j \in E(H)$, then there is a u_i, u_j-path P_{ij} in G
  - the paths P_{ij} are edge-disjoint

- H is a topological minor of G $\implies$

  - H is also a minor and an immersion in G
  - but in general not the other way round

- in general there is no relation between minor and immersion

Another conjecture

Conjecture (Lescure & Meyniel, 1989; Abu-Khzam & Langston, 2003)

- ■ G does not contain a K_t -immersion

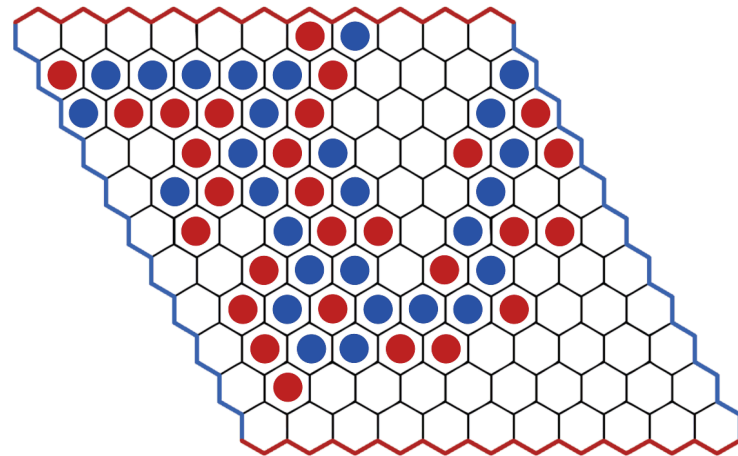 $\overset{?}{\Longrightarrow}$ G has a proper $(t-1)$ -colouring
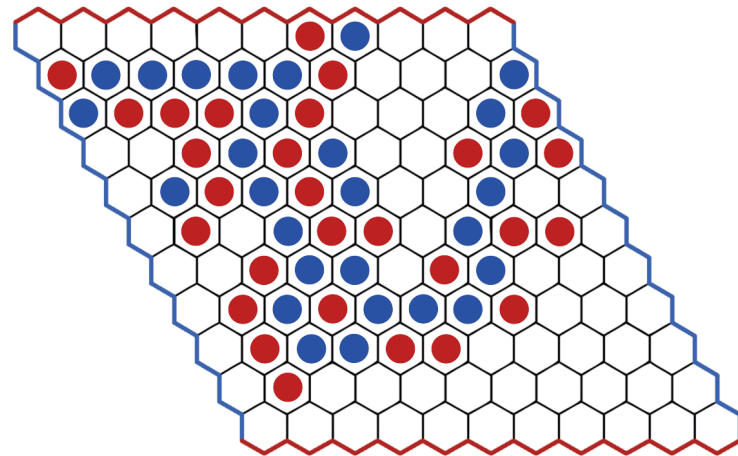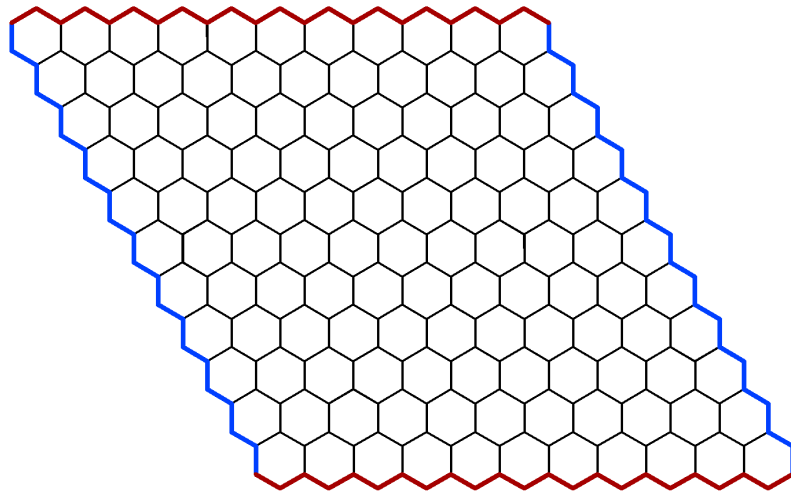
Theorem (vdH & Wood, 2017)

- ■ G does not contain a K_t -immersion $\Longrightarrow$

  G can be coloured with 2 colours such that each
  monochromatic subgraph has degree at most $f(t)$

- ■ to guarantee monochromatic components of bounded size,
  we need at least 3 colours (for $t \geq 8$)

Hex and 2-colouring
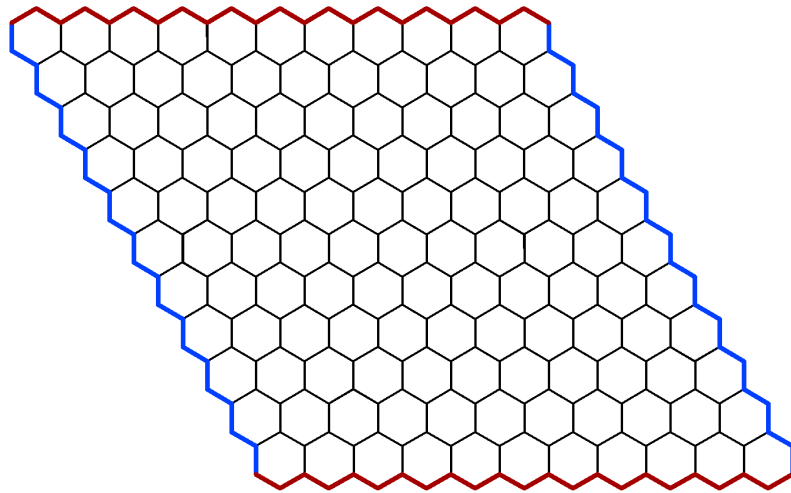
That's all folks!   –   Thanks for listening.