Improper Colourings
inspired by Hadwiger’s Conjecture

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Graph minors

- A graph $H$ is a **minor** of a graph $G$, if $H$ can be obtained from $G$ by a series of:
  - vertex deletions;
  - edge deletions;
  - edge contractions:
Graph minors – more intuitive

- A graph $H$ is a minor of a graph $G$ if:
  - For $V(H) = \{v_1, \ldots, v_k\}$, there exist connected, disjoint subgraphs $H_1, \ldots, H_k$ of $G$ such that:
    - If $v_i v_j \in E(H)$, then there is at least one edge in $G$ between $H_i$ and $H_j$.
Graph colouring

- a **colouring** of a graph means **colouring** the vertices

- **proper colouring**: adjacent vertices have **different colours**

- recurring question in graph theory:
  what **structural properties** of a graph
  - allow **proper colourings** with **few colours**?
  - force all proper colourings to use **many colours**?
Hadwiger’s Conjecture

**Conjecture** (Hadwiger, 1943)

- a graph $G$ needs at least $t$ colours for a proper colouring
  \[ \implies G \text{ has the complete graph } K_t \text{ as a minor} \]

The contrapositive is probably more intuitive:

- $G$ has no $K_t$-minor

\[ \implies G \text{ has a proper } (t - 1) \text{-colouring} \]
Hadwiger’s Conjecture – what is known

- No $K_t$-minor $\Rightarrow$ $G$ has a proper $(t - 1)$-colouring

known

- Nothing to prove for $t = 1, 2$
- Easy for $t = 3$
- Not too hard for $t = 4$  (Hadwiger, 1943; Dirac, 1952)
- Case $t = 5$ is equivalent to the Four Colour Theorem  (Wagner, 1937)
- True for $t = 6$  (Robertson, Seymour & Thomas, 1993)
How many colours do we need?

**Theorem** (Kostochka, 1984; Thomason, 1984)

- $G$ has no $K_t$-minor
  \[ \implies G \text{ has a vertex with degree at most } c t \sqrt{\log t} \]

**Corollary**

- $G$ has no $K_t$-minor
  \[ \implies G \text{ has a proper colouring with } c t \sqrt{\log t} \text{ colours} \]
Improper colourings

- what if we **weaken** the requirement on the **colouring**?
  - in a **proper colouring**:
    - the collection of **vertices with the same colour** is just a collection of **isolated vertices**
  - we could be happy with:
    - the collection of **vertices with the same colour** is just a subgraph with a "**simple**" structure

- **monochromatic subgraph**:
  - subgraph formed by **vertices with the same colour**

\[ \text{no } K_t \text{-minor} \quad \Rightarrow \quad \text{proper } (t - 1)\text{-colouring} \]
Im proper colourings – small monochromatic degree

Theorem (Edwards, Kang, Kim, Oum & Seymour, 2015)

- $G$ has no $K_t$-minor $\implies$
  
  $G$ can be coloured with $t - 1$ colours such that each monochromatic subgraph has degree at most $c't^2 \log t$

- the bound $t - 1$ on the number of colours is best possible:
  
  - there exists a class of graphs without $K_t$-minor, but where you can’t bound the degree of monochromatic subgraphs when using $t - 2$ colours only

no $K_t$-minor $\implies$ proper $(t - 1)$-colouring
Theorem (Edwards, Kang, Kim, Oum & Seymour, 2015)

- $G$ has no $K_t$-minor $\implies$
  
  $G$ can be coloured with $t - 1$ colours such that each monochromatic subgraph has degree at most $c' t^2 \log t$

Theorem (vdH & Wood, 2017)

- $G$ has no $K_t$-minor $\implies$
  
  $G$ can be coloured with $t - 1$ colours such that each monochromatic subgraph has degree at most $t - 2$
Improper colourings – small monochromatic components

**Theorem**  (Kawarabayashi & Mohar, 2007)

- $G$ has no $K_t$-minor $\implies$
  
  $G$ can be coloured with $\lceil 15 \frac{1}{2} t \rceil$ colours such that each monochromatic component has at most $f_1(t)$ vertices

- improved to
  
  - $\lceil 3 \frac{1}{2} t - 1 \frac{1}{2} \rceil$ colours; $f_2(t)$ vertices  \hspace{1cm} (Wood, 2010 (?))
  - $3(t - 1)$ colours; $f_3(t)$ vertices  \hspace{1cm} (Liu & Oum, 2015)
  - $2(t - 1)$ colours; $f_4(t)$ vertices  \hspace{1cm} (Norin, 2015; unpubl.)

- all use Robertson & Seymour Graph Minor Structure Thm., or worse . . .

- no $K_t$-minor $\implies$ proper $(t - 1)$-colouring
Improper colourings – small monochromatic components

**Theorem** (vdH & Wood, 2017)

- $G$ has no $K_t$-minor $\implies$
  
  $G$ can be coloured with $2(t - 1)$ colours such that each monochromatic component has at most $\lceil \frac{1}{2}(t - 2) \rceil$ vertices

**note**

- $G$ has no $K_t$-minor $\implies$
  
  at least $t - 1$ colours are needed to guarantee monochromatic components of bounded size
  
  (same examples as for small monochromatic degree)
A simple decomposition theorem for $K_t$-minor-free graphs

**Theorem** (vdH & Wood, 2017)

- $G$ has no $K_t$-minor $\implies$ $G$ has a partition into subgraphs $H_1, \ldots, H_\ell$ such that
- **global structure:** each $H_i$ is adjacent to at most $t - 2$ of the earlier subgraphs $H_1, \ldots, H_{i-1}$
A simple decomposition theorem for $K_t$-minor-free graphs

**Theorem** (vdH & Wood, 2017)

- $G$ has no $K_t$-minor $\implies$ $G$ has a partition into subgraphs $H_1, \ldots, H_\ell$ such that
  - **global structure:** each $H_i$ is adjacent to at most $t - 2$ of the earlier subgraphs $H_1, \ldots, H_{i-1}$
  - **local structure:**
    - each $H_i$ has maximum degree at most $t - 2$
    - each $H_i$ can be coloured with 2 colours such that each monochromatic component of $H_i$ has at most $\left\lceil \frac{1}{2} (t - 2) \right\rceil$ vertices

no $K_t$-minor $\implies$ proper $(t - 1)$-colouring
The global structure we actually prove

- \( G \) any graph \( \implies \)
  - we can construct (in many ways) a partition of \( G \) into induced subgraphs \( H_1, \ldots, H_\ell \) such that:
    - each \( H_i \) is connected
    - each \( H_i \) is adjacent to \( k \) subgraphs \( H_{i_1}, \ldots, H_{i_k} \)
      - the earlier subgraphs \( H_1, \ldots, H_{i-1} \)
    - for each \( H_i \), the adjacent subgraphs \( H_{i_1}, \ldots, H_{i_k} \)
      - are pairwise adjacent as well

\[ \text{no } K_t \text{-minor } \implies \text{ proper } (t - 1) \text{-colouring} \]
The global structure – proof

- we will construct the $H_i$ one by one such that once $H_1, \ldots, H_h$ is constructed:
  - each $H_i, i \leq h$, satisfies the requirements
  - each component $C$ of $G - (V(H_1) \cup \cdots \cup V(H_h))$ satisfies:
    - if $C$ is adjacent to $H_{i_1}, \ldots, H_{i_k}$ from $H_1, \ldots, H_{i-1}$, then $H_{i_1}, \ldots, H_{i_k}$ are pairwise adjacent as well

$\no K_t$-minor $\Rightarrow$ proper $(t-1)$-colouring
The global structure – proof

- start with $H_1$ any connected, induced subgraph of $G$
  (good choice for later: $V(H_1) = \{v\}$ for some $v \in V(G)$)
- all requirements are trivially satisfied
suppose $H_1, \ldots, H_h$ are already constructed and $C$ is some component of $G - (V(H_1) \cup \cdots \cup V(H_h))$

so $C$ is adjacent to $H_{i_1}, \ldots, H_{i_k}$, which are also pairwise adjacent

for each $H_{i_\ell}$, choose $a_{i_\ell} \in V(C)$ adjacent to $H_{i_\ell}$

now choose $H_{h+1}$ a connected, induced subgraph of $C$ containing all $a_{i_1}, \ldots, a_{i_k}$
The global structure – proof

- $H_{h+1}$ is still adjacent to $H_{i_1}, \ldots, H_{i_k}$
- A component of $G - (V(H_1) \cup \cdots \cup V(H_h) \cup V(H_{h+1}))$
  - is either a component of $G - (V(H_1) \cup \cdots \cup V(H_h))$ and hence still satisfies the requirements
  - or it is a component of $C - V(H_{h+1})$, hence it is adjacent to $H_{h+1}$ and some of $H_{i_1}, \ldots, H_{i_k}$, which are all pairwise adjacent

No $K_t$-minor $\Rightarrow$ proper $(t - 1)$-colouring
The global structure for $K_t$-minor-free graphs

- we can construct (in many ways) a partition of any graph $G$ into induced, connected subgraphs $H_1, \ldots, H_\ell$ such that:
  - each $H_i$ is adjacent to $k$ subgraphs $H_{i_1}, \ldots, H_{i_k}$ from $H_1, \ldots, H_{i-1}$, which are pairwise adjacent as well

  \[ H_1 \quad H_2 \quad H_3 \quad H_4 \quad \cdots \quad H_i \quad \cdots \]

- $G$ has no $K_t$-minor $\implies$ for each $H_i$ we must have $k \leq t - 2$
The local structure – inside the $H_i$

- so how can we choose the $H_i$ so that they satisfy:
  - small degree and
  - 2-colourable with small mono. components

- each $H_i$ was chosen as some induced subgraph of some connected subgraph $C$, such that:
  - $H_i$ is connected
  - $H_i$ contains some set $A = \{a_{i_1}, \ldots, a_{i_k}\}$

- idea:
  - choose $H_i$ the smallest subgraph with those properties

no $K_t$-minor $\Rightarrow$ proper $(t-1)$-colouring
The local structure – inside the $H_i$

Lemma

- $C$ a connected graph, $A \subseteq V(C)$
  - $H$ a minimal, induced, connected subgraph of $C$, such that $H$ contains all of $A$

- then $H$ satisfies:
  - every vertex in $H$ has degree at most $|A|$ in $H$
  - every vertex not in $A$ is a cut-vertex of $H$
    - easy corollary: there is a 2-colouring of $H$ with monochromatic components of size at most $\left\lceil \frac{1}{2} |A| \right\rceil$

no $K_t$-minor $\Rightarrow$ proper $(t - 1)$-colouring
Our decomposition theorem again

**Theorem** (vdH & Wood, 2017)

- $G$ has no $K_t$-minor $\implies$ $G$ has a partition into subgraphs $H_1, \ldots, H_\ell$ such that
  - **global structure:** each $H_i$ is adjacent to at most $t - 2$ of the earlier subgraphs $H_1, \ldots, H_{i-1}$
  - **local structure:**
    - each $H_i$ has maximum degree at most $t - 2$
    - each $H_i$ can be coloured with 2 colours such that each monochromatic component of $H_i$ has at most $\lceil \frac{1}{2} (t - 2) \rceil$ vertices
- some more properties

no $K_t$-minor $\implies$ proper $(t - 1)$-colouring
**Theorem**

- $G$ has no $K_t$-minor $\implies$ $G$ has a partition into connected subgraphs $H_1, \ldots, H_\ell$ such that
  - contracting all $H_i$ to single vertices gives a chordal graph with treewidth at most $t - 2$
  - each $H_i$ has treewidth at most $t - 3$

$$
\begin{align*}
H_1 & \quad H_2 & \quad H_3 & \quad H_4 & \quad \ldots & \quad H_i & \quad \ldots \\
\end{align*}
$$

no $K_t$-minor $\implies$ proper $(t - 1)$-colouring
A similar result for $K_{3,s}$-minor-free graphs

- we can prove a similar decomposition theorem for $K_{3,s}$-minor-free graphs

**Corollary**

- $G$ has no $K_{3,s}$-minor $\implies$
  - $G$ can be coloured with 3 colours such that each monochromatic subgraph has degree at most $4s$
  - $G$ can be coloured with 6 colours such that each monochromatic component has at most $2s$ vertices
That’s all folks! – Thanks for listening.